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LACUNARY STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES OF ORDER $\bar{\alpha}$ IN PROBABILISTIC NORMED SPACES

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ABSTRACT. In this paper, the idea of lacunary statistical convergence of order $\bar{\alpha}$ $(0 < \bar{\alpha} \leq 1)$ for double sequences ($S^{\bar{\alpha}}_{\theta_{rs}}-$convergent) in probabilistic normed spaces has been elaborated. We have obtained the relation of usual convergence of order $\bar{\alpha}$ $(0 < \bar{\alpha} \leq 1)$ and $S^{\bar{\alpha}}_{\theta_{rs}}-$convergent in these spaces. We have given examples to show that $S^{\bar{\alpha}}_{\theta_{rs}}-$convergent is more generalized than the usual convergence in these spaces.

1. INTRODUCTION

The notion of statistical convergence was initiated by Fast [5] and has motivated many researchers to work. One of the most important generalization was initiated by Fridy and Orhan [7] named as lacunary statistical convergence in which they studied the relation of $N_\theta-$summability and $(C,1)-$summability. Further, Patterson and Savaş [17] theorize the same concept for double sequences in which they take double lacunary sequence $\theta = \theta_{rs} = (m_r, n_s)$ such that $m_0, n_0 = 0$, $h_r = m_r - m_{r-1} \longrightarrow \infty$ as $r \longrightarrow \infty$ and $h_s = n_s - n_{s-1} \longrightarrow \infty$ as $s \longrightarrow \infty$ and the intervals determined by θ_{rs} will be denoted by $I_r = (m_{r-1}, m_r]$ and $I_s = (n_{s-1}, n_s]$ and also investigated by many researchers. The double sequence can be stated as function f from $N \times N$ to N such that

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$$f(mn) = a_{mn} \text{ where } m, n \in N\,.$$

Patterson and Savaş [17] specified lacunary statistical convergence for double sequences.

Definition 1.1. *[17] Let $\theta = (\theta_{rs})$ be the double lacunary sequence then $y = (y_{mn})$ is termed as lacunary statistically convergent if $\forall\ \epsilon > 0$ we have*

$$\lim_{r,s} \frac{1}{h_r h_s} |\{(m,n) \in I_r \times I_s : |y_{mn} - p| \geq \epsilon\}| = 0.$$

where I_r and I_s be the interval defined as $I_r = (m_{r-1}, m_r]$ and $I_s = (n_{s-1}, n_s]$ and h_r and h_s are the increasing sequences.

Another important generalization of statistical convergence is statistical convergence of double sequences of order $\bar{\alpha}$ where $\bar{\alpha}$ represents the pair (α_1, α_2) such that $0 < \alpha_1 \leq 1$ and $0 < \alpha_2 \leq 1$ was initiated by Çolak and Altin [3]. They work on the hypothesis of $\bar{\alpha}-$double density.

Definition 1.2. *[3] Let A be the subset of $N \times N$ then $\bar{\alpha}-$double density can be stated as*

$$\delta^2_{\bar{\alpha}}(A) = \lim_{m,n} \frac{A(m,n)}{m^{\alpha_1} n^{\alpha_2}}$$

where $A(m,n)$ be the number of (m_1, n_1) in A such that $m_1 \leq m$ and $n_1 \leq n$.

Definition 1.3. *[3] A double sequence $y = (y_{m_1 n_1})$ is termed as statistically convergent of order $\bar{\alpha}$ $(0 < \bar{\alpha} \leq 1)$ if for any $\epsilon > 0$*

$$\lim_{m,n} \frac{1}{m^{\alpha_1} n^{\alpha_2}} |\{(m_1, n_1) : m_1 \leq m, n_1 \leq n : |y_{m_1 n_1} - p| \geq \epsilon\}| = 0.$$

It can be written as $S^2_{\bar{\alpha}} - \lim y_{m_1 n_1} = p$.

Mohiuddine and Savaş [13] had researched on the conceit of lacunary statistical convergence double sequences for probabilistic normed spaces (PN spaces). Some basic terminologies given as follows :

Definition 1.4. *[13] A continuous mapping $\circledast : [0,1] \times [0,1] \to [0,1]$ is termed as t-norm if it is abelian monoid with unit one and $u \circledast v \leq t \circledast w$ then $u \leq t$ and $v \leq w$ for all $u, v, t, w \in [0,1]$.*

Definition 1.5. *[13] Let $F : R \to R_0^+$ be the function then the function F is termed as distribution function if it is non-decreasing and left continuous with $inf_{t\in R}F(t) = 0$ and $sup_{t\in R}F(t) = 1$. It is denoted by D.*

Definition 1.6. *[13] Let Y be the real vector space and $\upsilon : Y \to D$ where D denotes the set of all distribution function then $(Y, \upsilon, \circledast)$ is termed as probabilistic normed space if following postulates holds.*

(i) $\upsilon_y(0) = 0$.
(ii) $\upsilon_y(r) = 1 \; \forall \; r > 0$ *iff* $y = 0$.
(ii) $\upsilon_{cy}(r) = \upsilon_y(\frac{r}{|c|}) \; \forall \; c \in R - \{0\}$.
(iv) $\upsilon_{y+z}(r+s) \geq \upsilon_y(r) + \upsilon_z(s) \; \forall \; y, z \in Y$ *and* $r, s \in R_0^+$.

Definition 1.7. *[13] Let $(Y, \upsilon, \circledast)$ be a PN space then $y = (y_{mn})$ is termed as lacunary statistically convergent in PN space Y if for any $\lambda \in [0,1]$ we have*

$$\lim_{r,s} \frac{1}{h_r h_s} |\{(m,n) \in I_r \times I_s : |\upsilon_{y_{mn}-p}| \leq 1 - \lambda\}| = 0.$$

The hypothesis of statistical convergence, double sequences, PN spaces and order $\bar{\alpha}$ $(0 < \bar{\alpha} \leq 1)$ was an active area of research by many researchers [6] [14] [19] [15] [9] [12] [20] [21] [8] [16] [11] [2] [4] [10], [1] [18].

2. $S_{\theta_{rs}}^{\bar{\alpha}}-$CONVERGENT IN PROBABILISTIC NORMED SPACES

Throughout the paper, we consider $\bar{\alpha}$ as $0 < \bar{\alpha} \leq 1$ and $\theta = (\theta_{rs})$ be a double lacunary sequence, otherwise specified.

In this section, we initiate the idea of $S_{\theta_{rs}}^{\bar{\alpha}}-$convergent and $S_{\theta_{rs}}^{\bar{\alpha}}-$Cauchy in PN spaces.

Definition 2.1. *Let $(Y, \upsilon, \circledast)$ be a PN space. Then $y = (y_{mn})$ is termed as $S_{\theta_{rs}}^{\bar{\alpha}}-$convergent to p in PN space Y, if for every $\epsilon > 0$ and $\lambda \in (0,1)$*

$$\delta_{\theta_{rs}}^{\bar{\alpha}}\{(m,n) \in I_r \times I_s : \upsilon_{y_{mn}-p}(\epsilon) \leq 1 - \lambda\} = 0$$

i.e.,

$$\frac{1}{h_r^{\alpha_1} h_s^{\alpha_2}} |\{(m,n) \in I_r \times I_s : \upsilon_{y_{mn}-p}(\epsilon) \leq 1 - \lambda\}| = 0.$$

Theorem 2.1. *Suppose $(Y, \upsilon, \circledast)$ is a PN space. If $y = (y_{mn})$ is $S^{\bar{\alpha}}_{\theta_{rs}}-$convergent in PN space Y, then the limit is unique.*

Proof. Suppose that $S^{\bar{\alpha}}_{\theta_{rs}} - \lim y = p_1$ and $S^{\bar{\alpha}}_{\theta_{rs}} - \lim y = p_2$. For given $\epsilon > 0$, $t > 0$. Take $\lambda \in (0,1)$ such that

$$(1-\lambda) \circledast (1-\lambda) \geq 1-t.$$

Define,

$$A(\lambda, \epsilon) = \{(m,n) \in I_r \times I_s : \upsilon_{y_{mn}-p_1}(\frac{\epsilon}{2}) \leq 1-\lambda\},$$

$$B(\lambda, \epsilon) = \{(m,n) \in I_r \times I_s : \upsilon_{y_{mn}-p_2}(\frac{\epsilon}{2}) \leq 1-\lambda\}.$$

As $S^{\bar{\alpha}}_{\theta_{rs}} - \lim y = p_1$ and $S^{\bar{\alpha}}_{\theta_{rs}} - \lim y = p_2$, therefore $\delta^{\bar{\alpha}}_{\theta_{rs}}(A(\lambda,\epsilon)) = 0$ and $\delta^{\bar{\alpha}}_{\theta_{rs}}(B(\lambda,\epsilon)) = 0$.
$i.e.$,

$$\frac{1}{h_r^{\alpha_1} h_s^{\alpha_2}} |\{(m,n) \in I_r \times I_s : \upsilon_{y_{mn}-p_1}(\frac{\epsilon}{2}) \leq 1-\lambda\}| = 0,$$

and

$$\frac{1}{h_r^{\alpha_1} h_s^{\alpha_2}} |\{(m,n) \in I_r \times I_s : \upsilon_{y_{mn}-p_2}(\frac{\epsilon}{2}) \leq 1-\lambda\}| = 0.$$

Suppose

$$C(\lambda,\epsilon) = A(\lambda,\epsilon) \cap B(\lambda,\epsilon).$$

It gives $\delta^{\bar{\alpha}}_{\theta_{rs}}(C(\lambda,\epsilon)) = 0$, which implies $\delta^{\bar{\alpha}}_{\theta_{rs}}(C^c(\lambda,\epsilon)) = 1$.
Suppose $(m,n) \in (C^c(\lambda,\epsilon))$, then

$$\upsilon_{p_1-p_2}(\epsilon) = \upsilon_{p_1+y_{mn}-y_{mn}-p_2}(\epsilon) = \upsilon_{y_{mn}-p_1+y_{mn}-p_2}(\frac{\epsilon}{2}+\frac{\epsilon}{2}) \geq \upsilon_{y_{mn}-p_1}(\frac{\epsilon}{2}) \circledast \upsilon_{y_{mn}-p_2}(\frac{\epsilon}{2})$$

$$\geq (1-\lambda) \circledast (1-\lambda) \geq 1-t.$$

Since $t > 0$ was arbitrary, So we have,

$$\upsilon_{p_1-p_2}(\epsilon) = 1,$$

which implies that

$$p_1 = p_2.$$

□

Theorem 2.2. *Let (y_{mn}) and (z_{mn}) be the double sequence in PN space Y then*

(1) If $S_{\theta_{rs}}^{\bar{\alpha}} - \lim y_{mn} = p_1$ *and* $S_{\theta_{rs}}^{\bar{\alpha}} - \lim z_{mn} = p_2$ *then* $S_{\theta_{rs}}^{\bar{\alpha}} - \lim(y_{mn} + z_{mn}) = p_1 + p_2$.

(2) If $S_{\theta_{rs}}^{\bar{\alpha}} - \lim y_{mn} = p_1$ *and* $a \in R$ *then* $S_{\theta_{rs}}^{\bar{\alpha}} - \lim(a y_{mn}) = a p_1$.

Proof. (1) Suppose that $S_{\theta_{rs}}^{\bar{\alpha}} - \lim y_{mn} = p_1$ and $S_{\theta_{rs}}^{\bar{\alpha}} - \lim z_{mn} = p_2$. For given $\epsilon > 0$ and $t > 0$, take $\lambda \in (0,1)$ such that

$$(1-\lambda) \circledast (1-\lambda) > (1-t).$$

Define,

$$A(\lambda, \epsilon) = \{(m,n) \in I_r \times I_s : \upsilon_{y_{mn}-p_1}(\frac{\epsilon}{2}) \leq 1 - \lambda\}$$

and

$$B(\lambda, \epsilon) = \{(m,n) \in I_r \times I_s : \upsilon_{z_{mn}-p_2}(\frac{\epsilon}{2}) \leq 1 - \lambda\}.$$

As $S_{\theta_{rs}}^{\bar{\alpha}} - \lim y_{mn} = p_1$ and $S_{\theta_{rs}}^{\bar{\alpha}} - \lim z_{mn} = p_2$, therefore $\delta_{\theta_{rs}}^{\bar{\alpha}}(A(\lambda, \epsilon)) = 0$ and $\delta_{\theta_{rs}}^{\bar{\alpha}}(B(\lambda, \epsilon)) = 0$.
Let $(m,n) \notin A(\lambda, \epsilon) \cup B(\lambda, \epsilon)$ then

$$\upsilon_{y_{mn}+z_{mn}-p_1-p_2}(\epsilon) \geq \upsilon_{y_{mn}-p_1}(\frac{\epsilon}{2}) \circledast \upsilon_{z_{mn}-p_2}(\frac{\epsilon}{2}) > (1-\lambda) \circledast (1-\lambda) > 1-t$$

which implies that

$$\{(m,n) \in I_r \times I_s : \upsilon_{y_{mn}+z_{mn}-p_1-p_2}(\epsilon) \leq 1-t\} \subseteq A(\lambda, \epsilon) \cup B(\lambda, \epsilon),$$

$i.e.$,

$$\delta_{\theta_{rs}}^{\bar{\alpha}}\{(m,n) \in I_r \times I_s : \upsilon_{y_{mn}+z_{mn}-p_1-p_2}(\epsilon) \leq 1-t\} \subseteq \delta_{\theta_{rs}}^{\bar{\alpha}}(A(\lambda, \epsilon)) \cup \delta_{\theta_{rs}}^{\bar{\alpha}}(B(\lambda, \epsilon)).$$

Hence

$$S_{\theta_{rs}}^{\bar{\alpha}} - \lim(y_{mn} + z_{mn}) = p_1 + p_2.$$

(2) Suppose that $S_{\theta_{rs}}^{\bar{\alpha}} - \lim y_{mn} = p_1$. Let $a \neq 0$. For each $\epsilon > 0$ and $\lambda \in (0,1)$. Define,

$$A(\lambda, \epsilon) = \{(m,n) \in I_r \times I_s : \upsilon_{y_{mn}-p_1}(\epsilon) \leq 1 - \lambda\}$$

As $S_{\theta_{rs}}^{\bar{\alpha}} - \lim y_{mn} = p_1$, therefore $\delta_{\theta_{rs}}^{\bar{\alpha}}(A(\lambda, \epsilon)) = 0$.
Let $(m,n) \notin A(\lambda, \epsilon)$ then

$$\upsilon_{a(y_{mn}-p_1)}(\epsilon) = \upsilon_{y_{mn}-p_1}(\frac{\epsilon}{|a|}) \geq \upsilon_{y_{mn}-p_1}(\epsilon) \circledast \upsilon_o(\frac{\epsilon}{|a|} - \epsilon) = \upsilon_{y_{mn}-p_1}(\epsilon) > 1 - \lambda$$

which implies that

$$\{(m,n) \in I_r \times I_s : \upsilon_{a(y_{mn}-p_1)}(\epsilon) \leq 1-\lambda\} \subseteq A(\lambda,\epsilon),$$

$i.e.,$

$$\delta^{\bar{\alpha}}_{\theta_{rs}}\{(m,n) \in I_r \times I_s : \upsilon_{a(y_{mn}-p_1)}(\epsilon) \leq 1-\lambda\} \leq \delta^{\bar{\alpha}}_{\theta_{rs}}(A(\lambda,\epsilon)) = 0.$$

Let $a = 0$. For every $\epsilon > 0$ and $\lambda \in (0,1)$

$$\upsilon_{0.y_{mn}-0.p_1}(\epsilon) = \upsilon_0(\epsilon) = 1 > 1-\lambda.$$

This shows that $S^{\bar{\alpha}}_{\theta_{rs}} - \lim(ay_{mn}) = ap_1$. □

Theorem 2.3. *Let $(Y, \upsilon, \circledast)$ be a PN space. If $y = (y_{mn})$ is $\upsilon-$convergent to p then it is $S^{\bar{\alpha}}_{\theta_{rs}}-$convergent to p. But not conversely.*

Proof. Suppose that $\upsilon - \lim y = p$. Then for every $\epsilon > 0$ and $\lambda \in (0,1)$, $\exists$ a pair (M, N) such that $\forall$ $m \geq M$ and $n \geq N$, we have

$$\upsilon_{y_{mn}-p}(\epsilon) > 1-\lambda.$$

So the set $\{(m,n) \in I_r \times I_s : \upsilon_{y_{mn}-p}(\epsilon) \leq 1-\lambda\}$ has $\delta^{\bar{\alpha}}_{\theta_{rs}}-$density zero and hence

$$\frac{1}{h_r^{\alpha_1} h_s^{\alpha_2}} |\{(m,n) \in I_r \times I_s : \upsilon_{y_{mn}-p}(\epsilon) \leq 1-\lambda\}| = 0,$$

$i.e.,$ $S^{\bar{\alpha}}_{\theta_{rs}} - \lim y = p$. □

Next to show that the converse of above result does not hold in general which can be illustrated by the following Example.

Example 1. *Let us consider the space of real numbers (R,$|.|$) under usual norm. Let $\upsilon_y(\epsilon)$ =1-$e^{(\frac{-\epsilon}{|y|})}$. Define the sequence $y = (y_{mn})$ by*

$$y_{mn} = \begin{cases} mn & : m_r - [(h_r)^{(\frac{2}{3})}] + 1 \leq m \leq m_r, n_s - [(h_s)^{(\frac{2}{3})}] + 1 \leq n \leq n_s; \\ 0 & : otherwise. \end{cases}$$

For $\epsilon > 0$ and $\lambda \in (0,1)$, Consider

$$\begin{aligned} A(\lambda,\epsilon) &= \{(m,n) \in I_r \times I_s : \upsilon_{y_{mn}}(\epsilon) \leq 1-\lambda\} \\ &= \{(m,n) \in I_r \times I_s : 1 - e^{\frac{-\epsilon}{|y_{mn}|}} \leq 1-\lambda\}, \\ &= \{(m,n) \in I_r \times I_s : e^{\frac{-\epsilon}{|y_{mn}|}} \geq \lambda\}, \end{aligned}$$

$$= \{(m,n) \in I_r \times I_s : \frac{1}{e^{\frac{-\epsilon}{|y_{mn}|}}} \leq \frac{1}{\lambda}\},$$

$$= \{(m,n) \in I_r \times I_s : e^{\frac{\epsilon}{|y_{mn}|}} \leq \frac{1}{\lambda}\},$$

$$= \{(m,n) \in I_r \times I_s : \frac{\epsilon}{|y_{mn}|} \leq log(\frac{1}{\lambda})\},$$

$$= \{(m,n) \in I_r \times I_s : |y_{mn}| \geq \frac{\epsilon}{log(\frac{1}{\lambda})} > 0\},$$

$$= \{(m,n) \in I_r \times I_s : y_{mn} = mn\},$$

$$= \{(m,n) \in I_r \times I_s : m_r - [(h_r)^{(\frac{2}{3})}] + 1 \leq m \leq m_r, \quad n_s - [(h_s)^{(\frac{2}{3})}] + 1 \leq n \leq n_s\}.$$

So,

$$\frac{1}{h_r^{\alpha_1} h_s^{\alpha_2}} |A(\lambda, \epsilon)| \leq \frac{1}{h_r^{\alpha_1} h_s^{\alpha_2}} |\{(m,n) \in I_r \times I_s : m_r - [(h_r)^{(\frac{2}{3})}] + 1 \leq m \leq m_r, \quad n_s - [(h_s)^{(\frac{2}{3})}] + 1 \leq n \leq n_s\}|$$

$$\frac{1}{h_r^{\alpha_1} h_s^{\alpha_2}} |A(\lambda, \epsilon)| \leq \frac{(h_r)^{\frac{2}{3}} (h_s)^{\frac{2}{3}}}{h_r^{\alpha_1} h_s^{\alpha_2}},$$

Thus for $\bar{\alpha} > \frac{2}{3}$*,* $\lim_{r,s} \frac{1}{h_r^{\alpha_1} h_s^{\alpha_2}} |A(\lambda, \epsilon)| = 0$ $i.e.,$ $y = (y_{mn})$ *is* $S_{\theta_{rs}}^{\bar{\alpha}}-$*convergent, but not* $\upsilon-$*convergent since*

$$\upsilon_{y_{mn}}(\epsilon) = \begin{cases} 1 - e^{\frac{-\epsilon}{|y_{mn}|}} & : m_r - [(h_r)^{(\frac{2}{3})}] + 1 \leq m \leq m_r, n_s - [(h_s)^{(\frac{2}{3})}] + 1 \leq n \leq n_s; \\ 1 & : otherwise. \end{cases} \leq 1$$

Theorem 2.4. *Let* $(Y, \upsilon, \circledast)$ *be a PN space and* $0 < \bar{\alpha} \leq \bar{\beta} \leq 1$*. Then* $S_{\theta_{rs}}^{\bar{\alpha}} \subset S_{\theta_{rs}}^{\bar{\beta}}$*.*

Proof. Suppose $y = (y_{mn})$ is $S_{\theta_{rs}}^{\bar{\alpha}}-$convergent in PN space.
Then for given $\epsilon > 0$ and $\lambda \in (0,1)$ and $0 < \bar{\alpha} \leq \bar{\beta} \leq 1$ we have

$$\frac{1}{h_r^{\alpha_1} h_s^{\alpha_2}} |\{(m,n) \in I_r \times I_s : \upsilon_{y_{mn}-p}(\epsilon) \leq 1 - \lambda\}| \leq \frac{1}{h_r^{\beta_1} h_s^{\beta_2}} |\{(m,n) \in I_r \times I_s : \upsilon_{y_{mn}-p}(\epsilon) \leq 1 - \lambda\}|.$$

This gives that $S_{\theta_{rs}}^{\bar{\alpha}} \subseteq S_{\theta_{rs}}^{\bar{\beta}}$. $\square$

Theorem 2.5. *Let $(Y, \upsilon, \circledast)$ be a PN space. If $y = (y_{mn})$ is $S_{\theta_{rs}}^{\bar{\alpha}}-$convergent in PN space Y then $\exists$ a set $A = \{(m_i, n_i) : m_1 < m_2 < \cdots ; n_1 < n_2 < \cdots\}$ such that $\delta_{\theta_{rs}}^{\bar{\alpha}}(A) = 1$ with $\upsilon - \lim_i y_{m_i n_i} = p$.*

Proof. Suppose that $S_{\theta_{rs}}^{\bar{\alpha}} - \lim y = p$. Then for given $\epsilon > 0$ and $k \in N$
Define,

$$A(k, \epsilon) = \{(m, n) \in I_r \times I_s : \upsilon_{y_{mn}-p}(\epsilon) > 1 - \frac{1}{k}\}\,.$$

As $S_{\theta_{rs}}^{\bar{\alpha}} - \lim y = p$, therefore $\delta_{\theta_{rs}}^{\bar{\alpha}}(A^c(k, \epsilon)) = 0$.
So $\forall\ k \in N$, we have

$$A(k + 1, \epsilon) \subset A(k, \epsilon)\,.$$

Let $\{H_{ij}\}_{i,j \in N}$ is the increasing double sequence of non-negative numbers that is $\lim_{i,j} H_{ij} = \infty$.
Choose $(v_1, v_2) \in A(1, \epsilon)$ and choose another pair say $(v_3, v_4) \in A(2, \epsilon)$ such that $(v_3, v_4) > (v_1, v_2)$ we have $\frac{A(2,\epsilon)}{h_r^{\alpha_1} h_s^{\alpha_2}} > H_{22}$. Similarly we get $\frac{A(3,\epsilon)}{h_r^{\alpha_1} h_s^{\alpha_2}} > H_{33}$. Proceeding the same way then we get a double sequence of positive integer $(v_1, v_2) < (v_3, v_4) \cdots < (v_k, v_l) \cdots$ such that

$$\frac{A(k, \epsilon)}{h_r^{\alpha_1} h_s^{\alpha_2}} > H_{kk}\,.$$

Now let $A \subseteq I_r \times I_s$ such that every number of the interval $(h_{r-1}, h_r] \times (h_{s-1}, h_s]$ belongs to A. Further any number of the interval $(h_{r-1}, h_r] \times (h_{s-1}, h_s]$ belongs to A iff it belongs to $A(k, \epsilon)$. So,

$$\frac{A}{h_r^{\alpha_1} h_s^{\alpha_2}} \geq \frac{A(k, \epsilon)}{h_r^{\alpha_1} h_s^{\alpha_2}} > H_{kk}$$

which implies that

$$\delta_{\theta_{rs}}^{\bar{\alpha}}(A) = 1.$$

Let $\lambda \in (0, 1)$ and take $k \in N$ such that $\frac{1}{k} < \lambda$. Then $\exists$ a natural number $r \geq k$ such that for all $(m, n) \geq (v_k, v_l)$

$$\upsilon_{y_{mn}-p}(\epsilon) > 1 - \frac{1}{r} > 1 - \frac{1}{k} > 1 - \lambda\,.$$

So $y = (y_{mn})$ is $\upsilon - \lim_i y_{m_i n_i} = p$. $\square$

3. $S^{\bar{\alpha}}_{\theta_{rs}}-$CAUCHY IN PROBABILISTIC NORMED SPACES

Definition 3.1. *Let $(Y, \upsilon, \circledast)$ be a PN space. Then $y = (y_{mn})$ is termed as $S^{\bar{\alpha}}_{\theta_{rs}}-$Cauchy in probabilistic normed space Y, if $\forall\ \epsilon > 0$ and $\lambda \in (0,1)$ $\exists$ a pair $(\grave{M}, \grave{N})$ such that $\forall\ m, m_1 \geq \grave{M}$ and $n, n_1 \geq \grave{N}$ we have*

$$\delta^{\bar{\alpha}}_{\theta_{rs}} |\{(m,n) \in I_r \times I_s : \upsilon_{y_{mn}-y_{m_1 n_1}}(\epsilon) \leq 1-\lambda\}| = 0$$

$i.e.,$

$$\frac{1}{h_r^{\alpha_1} h_s^{\alpha_2}} |\{(m,n) \in I_r \times I_s : \upsilon_{y_{mn}-y_{m_1 n_1}}(\epsilon) \leq 1-\lambda\}| = 0.$$

Theorem 3.1. *Let $(Y, \upsilon, \circledast)$ be a PN space. If $y = (y_{mn})$ is $S^{\bar{\alpha}}_{\theta_{rs}}-$convergent then it is $S^{\bar{\alpha}}_{\theta_{rs}}-$Cauchy in PN space Y.*

Proof. Suppose that $S^{\bar{\alpha}}_{\theta_{rs}} - \lim y = p$. For any $\epsilon > 0$ and $\lambda \in (0,1)$. Take $\lambda_1 > 0$

$$(1-\lambda) \circledast (1-\lambda) > (1-\lambda_1).$$

Define,

$$A(\lambda, \epsilon) = \{(m,n) \in I_r \times I_s : \upsilon_{y_{mn}-p}(\frac{\epsilon}{2}) \leq 1-\lambda\}$$

which implies that

$$A^c(\lambda, \epsilon) = \{(m,n) \in I_r \times I_s : \upsilon_{y_{mn}-p}(\frac{\epsilon}{2}) > 1-\lambda\}.$$

As $S^{\bar{\alpha}}_{\theta_{rs}} - \lim y = p$, therefore $\delta^{\bar{\alpha}}_{\theta_{rs}}(A(\lambda, \epsilon)) = 0$ and $\delta^{\bar{\alpha}}_{\theta_{rs}}(A^c(\lambda, \epsilon))$=1.
$i.e.,$

$$\delta^{\bar{\alpha}}_{\theta_{rs}}(A(\lambda, \epsilon)) = \frac{1}{h_r^{\alpha_1} h_s^{\alpha_2}} |\{(m,n) \in I_r \times I_s : \upsilon_{y_{mn}-p}(\frac{\epsilon}{2}) \leq 1-\lambda\}| = 0$$

and

$$\delta^{\bar{\alpha}}_{\theta_{rs}}(A^c(\lambda, \epsilon)) = \frac{1}{h_r^{\alpha_1} h_s^{\alpha_2}} |\{(m,n) \in I_r \times I_s : \upsilon_{y_{mn}-p}(\frac{\epsilon}{2}) > 1-\lambda\}| = 1.$$

Let $(m_1, n_1) \in A^c(\lambda, \epsilon)$. Then $\upsilon_{y_{m_1 n_1}-p}(\frac{\epsilon}{2}) > 1-\lambda$.
Define,

$$B(\lambda_1, \epsilon) = \{(m,n) \in I_r \times I_s : \upsilon_{y_{mn}-y_{m_1 n_1}}(\epsilon) \leq 1-\lambda_1\}.$$

Now we have to prove that $B(\lambda_1, \epsilon)$ is a subset of $A(\lambda, \epsilon)$.
Suppose $(m,n) \in B(\lambda_1, \epsilon)$ Then, $\upsilon_{y_{mn}-y_{m_1 n_1}}(\epsilon) \leq 1-\lambda_1$.
Let if possible

$$\upsilon_{y_{mn}-p}(\frac{\epsilon}{2}) > 1-\lambda.$$

Then

$$1-\lambda_1 \geq \upsilon_{y_{mn}-y_{m_1n_1}}(\epsilon) \geq \upsilon_{y_{mn}-p}(\frac{\epsilon}{2}) \circledast \upsilon_{y_{m_1n_1}-p}(\frac{\epsilon}{2}) > (1-\lambda) \circledast (1-\lambda) > (1-\lambda_1),$$

which is not possible. So

$$\upsilon_{y_{mn}-p}(\frac{\epsilon}{2}) \leq 1-\lambda,$$

which implies that $(m,n) \in A(\lambda,\epsilon)$. Hence $B(\lambda_1,\epsilon) \subseteq A(\lambda,\epsilon)$. So, $y=(y_{mn})$ is $S^{\bar{\alpha}}_{\theta_{rs}}-$Cauchy. □

Theorem 3.2. *Let $(Y,\upsilon,\circledast)$ is a PN space. If $y=(y_{mn})$ is $S^{\bar{\alpha}}_{\theta_{rs}}-$Cauchy in PN space Y then for any $\epsilon>0$ and $\lambda \in (0,1)$ $\exists$ a set $A(\lambda,\epsilon) \subset I_r \times I_s$ with $\delta^{\bar{\alpha}}_{\theta_{rs}}(A(\lambda,\epsilon))=0$ such that $\upsilon_{y_{mn}-y_{m_1n_1}} > 1-\lambda$ for any $(m,n),(m_1,n_1) \notin A(\lambda,\epsilon)$.*

Proof. Let $\epsilon>0$, $\lambda>0$ and take $\lambda_1 \in (0,1)$ such that

$$(1-\lambda_1) \circledast (1-\lambda_1) > 1-\lambda.$$

Since the double sequence $y=(y_{mn})$ is $S^{\bar{\alpha}}_{\theta_{rs}}-$Cauchy. So $\exists$ a non-negative integer $\grave{M}$ and $\grave{N}$ such that

$$\frac{1}{h_r^{\alpha_1}h_s^{\alpha_2}}|\{(m,n) \in I_r \times I_s : \upsilon_{y_{mn}-y_{\grave{M}\grave{N}}}(\frac{\epsilon}{2}) \leq 1-\lambda_1\}| = 0.$$

Define,

$$A(\lambda,\epsilon) = \{(m,n) \in I_r \times I_s : \upsilon_{y_{mn}-y_{\grave{M}\grave{N}}}(\frac{\epsilon}{2}) \leq 1-\lambda_1\},$$

which implies that $\delta^{\bar{\alpha}}_{\theta_{rs}}(A(\lambda,\epsilon))=0$.
If (m,n) and $(m_1,n_1) \notin A(\lambda,\epsilon)$ then,
$\upsilon_{y_{mn}-y_{\grave{M}\grave{N}}}(\frac{\epsilon}{2}) > 1-\lambda_1$ and $\upsilon_{y_{m_1n_1}-y_{\grave{M}\grave{N}}}(\frac{\epsilon}{2}) > 1-\lambda_1$. So, we have

$$\upsilon_{y_{mn}-y_{m_1n_1}}(\epsilon) \geq \upsilon_{y_{mn}-y_{\grave{M}\grave{N}}}(\frac{\epsilon}{2}) \circledast \upsilon_{y_{m_1n_1}-y_{\grave{M}\grave{N}}}(\frac{\epsilon}{2}) > (1-\lambda_1) \circledast (1-\lambda_1) > 1-\lambda.$$

which implies that for any $(m,n),(m_1,n_1) \notin A(\lambda,\epsilon)$,

$$\upsilon_{y_{mn}-y_{m_1n_1}}(\epsilon) > 1-\lambda.$$

□

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