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LAPLACIAN ENERGY'S ECCENTRICITY VERSION OF A FUZZY GRAPH

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ABSTRACT. The addition of absolute values of the eigen values of the adjacency matrix of $\tilde{G}$ is equal to energy of a fuzzy graph $\tilde{G}$ and the sum of absolute value of the difference between the eigen values of the Laplacian matrix of $\tilde{G}$ and average degree of the vertices $\tilde{G}$ ii identical to the Laplacian energy of a fuzzy graph $\tilde{G}$. The biggest distance from v to any other vertex u of $\tilde{G}$ is known as the eccentricity of a vertex v and is signified by $\varepsilon_G(v)$ is. The total eccentricity of a fuzzy graph is denoted by $\xi\left(\tilde{G}\right)$ and is equal to sum of eccentricities of all the vertices of the fuzzy graph. In this paper, we investigate the eccentricity version of Laplacian energy of a fuzzy graph $\tilde{G}$.

1. INTRODUCTION

Let $\tilde{G}$ be a simple fuzzy graph with n vertices and m edges. Let the vertex and edge sets of $\tilde{G}$ are denoted by σ and μ respectively. The degree of a vertex v is denoted by $d_{\tilde{G}}(v)$ is the sum of the membership values of edges incident with v i.e.,

$$d(v) = \sum_{u,v \in E} \mu(u,v) \ .$$

Consider any two vertices $u, v \in \sigma\left(\tilde{G}\right)$, the distance among u and v is signified by $d_{\tilde{G}}(u,v)$ and is assumed by the length of the shortest path among them

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i.e.,

$$d\left(\sigma\left(u_i\right), \sigma\left(v_j\right)\right) = \min \sum \mu\left(u_i, v_j\right).$$

We represent the sum of distances among $v \in V\left(\tilde{G}\right)$ and all additional vertices in $\tilde{G}$ by $d\left(x/_{y}\right)$ i.e $d\left(x/_{y}\right) = \sum_{v \in V(G)} d_G\left(x, v\right)$.

The eccentricity of a vertex$\sigma\left(v\right)$, represented by $\xi_G\left(\sigma\left(\tilde{G}\right)\right)$, is the biggest distance from $\sigma\left(v\right)$to some additional vertex $\sigma\left(u\right)$ of G. The total eccentricity of a fuzzy graph is denoted by $\xi\left(\tilde{G}\right)$ and is equal to sum of eccentricities of all the vertices of the fuzzy graph $\tilde{G}$.

Let the adjacency matrix be $A = [a_{ij}]$ of fuzzy graph $\tilde{G}$ and let $\lambda_1, \lambda_2, ..., \lambda_n$ are eigen values of A i.e the eigen values of the fuzzy graph $\tilde{G}$. The energy of a fuzzy graph is introduced by Anajali N, Mathew S [1] in 2013 and demarcated as the addition of the absolute standards of its eigen values and is denoted by $E\left(\tilde{G}\right)$. Then

$$E\left(\tilde{G}\right) = \sum_{i=1}^{n} |\lambda_i|$$

A large number of results on the fuzzy graph have been reported, see [8]. Beyond theory of fuzzy graph energy, other energy have been proposed and improved by different researchers. Let the diagonal matrix be $D\left(\tilde{G}\right) = [d_{ij}]$ related with the fuzzy graph $\tilde{G}$, where $d_{ij} = d_G\left(\sigma\left(v_i\right)\right)$ and $d_{ij} = 0$ if $i \neq j$.

Again $L\left(\tilde{G}\right) = D\left(\tilde{G}\right) - A\left(\tilde{G}\right)$ is called the laplacian energy of fuzzy graph $\tilde{G}$ is defined as [8]

$$LE\left(\tilde{G}\right) = \sum_{i=1}^{n} \left|\lambda_i - \frac{2\sum_{1 \le i \le j \le n} \mu\left(v_i, v_j\right)}{n}\right|.$$

Various study on laplacian energy of fuzzy graphs were reported in the literature [1, 3, 8].

In this present study inspired by the work in [7], we investigate the laplacian energy of fuzzy graph's eccentricity version denoted by $LE_\varepsilon\left(\tilde{G}\right)$, we define the laplacian eccentricity matrix as $L_\epsilon\left(\tilde{G}\right) = \epsilon\left(\tilde{G}\right) - A\left(\tilde{G}\right)$ where$\varepsilon\left(\tilde{G}\right) = [e_{ij}]$ is $n \times n$ diagonal matrix of fuzzy graph $\tilde{G}$ with $e_{ij} = \varepsilon_G\left(\sigma\left(v_i\right)\right)$ and $e_{ij} = 0$ if $i \neq j$. $\varepsilon_G\left(\sigma\left(v_i\right)\right)$ with eccentricity of the vertex $\sigma\left(v_i\right), i = 1, 2, \cdots, n$.

Let the eigen values of the matrix$L_\varepsilon\left(\tilde{G}\right)$be $\mu'_1, \mu'_2, ..., \mu'_n$. The eccentricity version of laplacian energy of the fuzzy graph $\tilde{G}$ is demarcated as

$$EL_\epsilon\left(\tilde{G}\right) = \sum_{i=1}^{n}\left|\mu' - \frac{\zeta\left(\tilde{G}\right)}{n}\right| .$$

Recall that $\xi\left(\tilde{G}\right)\Big/n$ is the average vertex eccentricity. In this paper, we first calculate some basic proportion and then establish some lower and upper bounds for $LE_\varepsilon\left(\tilde{G}\right)$.

2. Main results

We already know that, the ordinary and laplacian energy of fuzzy graph eigen values conform the succeeding associations

$$\sum_{i=1}^{n}\lambda_i = 0; \sum_{i=1}^{n}\lambda_i^2 = 2m; \sum_{i=1}^{n}\mu_i = 2m; \sum_{i=1}^{n}\mu_i^2 = 2m + \sum_{i=1}^{n}d\left(v_i\right)^2 .$$

As the laplacian spectrum is denoted by $\mu'_1, \mu'_2, \cdots, \mu'_n$. Let $v'_i = \left|\mu'_i - \frac{\xi(\tilde{G})}{n}\right|$.

In the following, we investigate some basic properties of μ'_i and v'_i.

Lemma 2.1. *The eigen values $\mu'_1, \mu'_2, \cdots, \mu'_n$ satisfies the following relations*

(i) $\sum_{i=1}^{n}\mu'_i = \xi\left(\tilde{G}\right)$.
(ii) $\sum_{i=1}^{n}\mu_i'^2 = E_1\left(\tilde{G}\right) + 2m.$

Proof.

(i) Subsequently the trace of a square matrix is equivalent to the addition of the eigen standards

$$\sum_{i=1}^{n}\mu'_i = tr\left(EL_\varepsilon\left(\tilde{G}\right)\right) = \sum_{i=1}^{n}\left[\varepsilon\left(v_i\right) - a_{ij}\right] = \xi\left(\tilde{G}\right).$$

(ii) Again we have

$$\begin{aligned}\sum_{i=1}^{n}\mu_i'^2 &= tr\left[\left(\varepsilon\left(\tilde{G}\right)-A\left(\tilde{G}\right)\right)\left(\varepsilon\left(\tilde{G}\right)-A\left(\tilde{G}\right)\right)^T\right]\\ &= tr\left[\left(\varepsilon\left(\tilde{G}\right)-A\left(\tilde{G}\right)\right)\left(\varepsilon\left(\tilde{G}\right)^T-A\left(\tilde{G}\right)^T\right)\right]\\ &= tr\left[\left(\varepsilon\left(\tilde{G}\right)\varepsilon\left(\tilde{G}\right)^T-A\left(\tilde{G}\right)\varepsilon\left(\tilde{G}\right)^T-A\left(\tilde{G}\right)^T\varepsilon\left(\tilde{G}\right)-A\left(\tilde{G}\right)A\left(\tilde{G}\right)^T\right)\right]\\ &= \sum_{i=1}^{n}\varepsilon\left(v_i\right)^2+\sum_{i=1}^{n}\lambda_i^2 \; = E_1\left(\tilde{G}\right)+2m.\end{aligned}$$

□

Lemma 2.2. *The eigen values $v_1', v_2', \cdots, v_n'$ fulfils the succeeding associations*

(i) $\sum_{i=1}^{n} v_i' = 0.$

(ii) $\sum_{i=1}^{n} v_i'^2 = E_1\left(\tilde{G}\right) - \frac{\xi\left(\tilde{G}\right)}{n} + 2m.$

Proof. (i)

$$\begin{aligned}\sum_{i=1}^{n} v_i' &= \sum_{i=1}^{n}\left[\mu_i' - \frac{\xi\left(\tilde{G}\right)}{n}\right]\\ &= \sum_{i=1}^{n}\mu_i - \xi\left(\tilde{G}\right) = 0.\end{aligned}$$

(ii)

$$\begin{aligned}\sum_{i=1}^{n} v_i'^2 &= \sum_{i=1}^{n}\left[\mu_i' - \frac{\xi\left(\tilde{G}\right)}{n}\right]^2\\ &= \sum_{i=1}^{n}\left[\mu_i'^2 - \frac{\xi\left(\tilde{G}\right)^2}{n^2} - 2\mu_i'\frac{\xi\left(\tilde{G}\right)}{n}\right]\\ &= E_1\left(\tilde{G}\right) + 2m + \frac{\xi\left(\tilde{G}\right)^2}{n} - 2\frac{\xi\left(\tilde{G}\right)^2}{n}\\ &= E_1\left(\tilde{G}\right) - \frac{\xi\left(\tilde{G}\right)^2}{n} + 2m.\end{aligned}$$

□

Lemma 2.3. *The eigen values v'_1, v'_2, $\cdots$, v'_n fulfils the succeeding associations*

$$\left|\sum_{i<j} v'_i v'_j\right| = \frac{1}{2}\left|E_1\left(\tilde{G}\right) - \frac{\xi\left(\tilde{G}\right)^2}{n} - 2m\right|.$$

Proof. Since $\sum_{i=1}^{n} v'_i = 0$. So we can write

$$\begin{aligned}&\textstyle\sum_{i=1}^{n} v_i'^2 = -2\sum_{i<j} v'_i v'_j\\ &2\left|\textstyle\sum_{i\leq j} v'_i v'_j\right| = \textstyle\sum_{i=1}^{n} v_i'^2 = E_1\left(\tilde{G}\right) - \frac{\xi\left(\tilde{G}\right)^2}{n} + 2m \ .\end{aligned}$$

Hence the desired result. □

Next we investigate some upper and lower bounds of eccentricity version of laplacian energy of a fuzzy graph $\tilde{G}$.

Theorem 2.1. *Let a associated fuzzy graph be $\tilde{G}$ of n^{th} order and m^{th} size , then*

$$EL_\varepsilon\left(\tilde{G}\right) \geq 2\sqrt{m + \frac{1}{2}\left(E_1\left(\tilde{G}\right) - \frac{\xi\left(\tilde{G}\right)^2}{n}\right)}.$$

Proof. We have from the definition

$$EL_\varepsilon\left(G\right) = \sum_{i=1}^{n} \left|v'_i\right|,$$

so we can write

$$\begin{aligned}EL_\varepsilon\left(\tilde{G}\right)^2 &= \textstyle\sum_{i=1}^{n} v_i'^2 + 2\sum_{i<j}\left|v'_i v'_j\right|\\ &\geq \left(E_1\left(\tilde{G}\right) - \frac{\xi\left(\tilde{G}\right)^2}{n} + 2m\right) + 2\textstyle\sum_{i<j}\left|v'_i v'_j\right| \ .\end{aligned}$$

By using Lemma 2.3, the result follows as

$$\begin{aligned}EL_\varepsilon\left(\tilde{G}\right)^2 &\geq \left(E_1\left(\tilde{G}\right) - \frac{\xi\left(\tilde{G}\right)^2}{n} + 2m\right) + 2\tfrac{1}{2}\left(E_1\left(\tilde{G}\right) - \frac{\xi\left(\tilde{G}\right)^2}{n} + 2m\right)\\ &\geq 2\left(E_1\left(\tilde{G}\right) - \frac{\xi\left(\tilde{G}\right)^2}{n} + 2m\right)\end{aligned}$$

$$EL_{\varepsilon}\left(\tilde{G}\right) \geq 2\sqrt{m + \frac{1}{2}\left(E_1\left(\tilde{G}\right) - \frac{\xi\left(\tilde{G}\right)^2}{n}\right)}.$$

Hence the proof. □

Theorem 2.2. *Let a associated fuzzy graph be $\tilde{G}$ of n^{th} order and m^{th} size; and v_1' and v_n' are maximum and minimum absolute values of v_i' 's then*

$$EL_{\varepsilon}\left(\tilde{G}\right) \geq \sqrt{nE_1\left(\tilde{G}\right) - \xi\left(\tilde{G}\right)^2 + 2mn - \frac{n^2}{4}\left(v_1' - v_n'\right)^2}.$$

Proof. Let $c_i$and d_i,$1 \leq i \leq n$ are non-negative real numbers, then using the ozek's inequality [5]. We have

$$\sum_{i=1}^{n} c_i^2 \sum_{i=1}^{n} d_i^2 - \left(\sum_{i=1}^{n} c_i d_i\right) \leq \frac{n^2}{4}\left(M_1 M_2 - m_1 m_2\right)^2 ,$$

where $M_1 = \max(c_i)$, $m_1 = \min(c_i)$ and $M_2 = \max(d_i)$, $m_2 = \min(d_i)$.

Let $c_i = |v_i'|$ and $d_i = 1$then from the above inequality we have

$$\sum_{i=1}^{n} v_i'^2 \sum_{i=1}^{n} 1^2 - \left(\sum_{i=1}^{n} |v_i'|\right)^2 \leq \frac{n^2}{4}\left(v_1' - v_n'\right)^2$$

$$EL_S\left(G\right)^2 \geq n\sum_{i=1}^{n} |v_i'|^2 - \frac{n^2}{4}\left(v_1' - v_n'\right)^2$$

$$EL_S\left(G\right)^2 \geq n\left(E_1\left(G\right) - \frac{\zeta\left(G\right)^2}{n} + 2m\right) - \frac{n^2}{4}\left(v_1' - v_n'\right)^2$$

$$EL_{\varepsilon}\left(\tilde{G}\right) \geq \sqrt{nE_1\left(\tilde{G}\right) - \xi\left(\tilde{G}\right)^2 + 2mn - \frac{n^2}{4}\left(v_1' - v_n'\right)^2}$$

$$EL_{\varepsilon}\left(\tilde{G}\right) \geq 2\sqrt{m + \frac{1}{2}\left(E_1\left(\tilde{G}\right) - \frac{\xi\left(\tilde{G}\right)^2}{n}\right)}.$$

Hence the desired proof. □

Theorem 2.3. *Let a associated fuzzy graph be $\tilde{G}$ of n^{th} order and m^{th} size and v_1' and v_n' are maximum and minimum absolute values of v_i' s then*

$$EL_\varepsilon\left(\tilde{G}\right) \geq \frac{1}{(v_1' + v_n')}\left(E_1\left(\tilde{G}\right) - \frac{\xi\left(\tilde{G}\right)^2}{n} + 2m + nv_1'v_n'\right)$$

Proof. Let $c_i$and d_i,$1 \leq i \leq n$ are being positive real numbers, then using the Diaz-Metcalf inequality [2]. We have

$$\sum_{i=1}^{n} d_i^2 + mM\sum_{i=1}^{n} c_i^2 \leq (m+M)\left(\sum_{i=1}^{n} c_i d_i\right),$$

where $mc_i \leq d_i \leq Mc_i$. Let $c_i = 1$ and $d_i = |v_i'|$then from the above inequality we have

$$\sum_{i=1}^{n} |v_i'|^2 + v_1'v_n'\sum_{i=1}^{n} 1^2 \leq (v_1' + v_n')^2\left(\sum_{i=1}^{n} |v_i'|\right)$$

$$E_1\left(\tilde{G}\right) - \frac{\xi\left(\tilde{G}\right)^2}{n} + 2m + nv_1'v_n' \leq (v_1' + v_n')\, EL_\varepsilon\left(\tilde{G}\right)$$

$$EL_\varepsilon\left(\tilde{G}\right) \geq \frac{1}{(v_1' + v_n')}\left(E_1\left(\tilde{G}\right) - \frac{\xi\left(\tilde{G}\right)^2}{n} + 2m + nv_1'v_n'\right).$$

Hence the desired proof. □

Theorem 2.4. *Let a connected fuzzy graph be $\tilde{G}$ of n^{th} order and m^{th} size, then*

$$EL_\varepsilon\left(\tilde{G}\right) \leq \sqrt{nE_1\left(\tilde{G}\right) - \xi\left(\tilde{G}\right)^2 + 2mn}.$$

Proof. Using the Cauchy-Schwartz inequality to the vertices $(|v_1'|\,,\,|v_2'|\,,\ldots,|v_n'|)$ and $(1\,,1\,,\ldots,1)$, we have:

$$\sum_{i=1}^{n} |v_i'| \leq \sqrt{n\left(\sum_{i=1}^{n} |v_i'|\right)^2}$$

Then from definition, we have

$$EL_\varepsilon\left(\tilde{G}\right) = \sum_{i=1}^{n} |v_i'| \leq \sqrt{n\left(\sum_{i=1}^{n} |v_i'|\right)^2}$$

$$EL_{\varepsilon}\left(\tilde{G}\right) \leq \sqrt{n\left(E_1\left(\tilde{G}\right) - \frac{\xi\left(\tilde{G}\right)^2}{n} + 2m\right)}$$

$$EL_{\varepsilon}\left(\tilde{G}\right) \leq \sqrt{nE_1\left(\tilde{G}\right) - \xi\left(\tilde{G}\right)^2 + 2mn.}$$

Hence the desired result. □

Theorem 2.5. *Let a associated fuzzy graph be G of n^{th} order and m^{th} size, then*

$$EL_{\varepsilon}\left(\tilde{G}\right) \geq \frac{2\sqrt{v_1'v_n'}}{v_1' + v'}\sqrt{nE_1\left(\tilde{G}\right) - \xi\left(\tilde{G}\right)^2 + 2mn.}$$

Proof. We have, from Polya-szego inequality [6] for non-negative real numbers c_i and d_i, $1 \leq i \leq n.$

$$\sum_{i=1}^{n} c_i^2 \sum_{i=1}^{n} d_i^2 \leq \frac{1}{4}\left(\sqrt{\frac{M_1M_2}{m_1m_2}} + \sqrt{\frac{m_1m_2}{M_1M_2}}\right)^2 \left(\sum_{i=1}^{n} c_i d_i\right)^2,$$

where $M_1 = \max(c_i)$, $m_1 = \min(c_i)$ and $M_2 = \max(d_i)$, $m_2 = \min(d_i)$. Let $c_i = 1$ and $d_i = |v_i'|$ then from the above inequality

$$n\sum_{i=1}^{n} |v_i'|^2 \sum_{i=1}^{n} 1^2 \leq \frac{1}{4}\left(\sqrt{\frac{v_n'}{v_1'}} + \sqrt{\frac{v_1'}{v_n'}}\right)^2 \left(\sum_{i=1}^{n} |v_i'|\right)^2$$

$$n\sum_{i=1}^{n} |v_i'|^2 \leq \frac{1}{4}\left(\sqrt{\frac{v_n'}{v_1'}} + \sqrt{\frac{v_1'}{v_n'}}\right)^2 \left(EL_{\varepsilon}\left(\tilde{G}\right)\right)^2$$

$$EL_{\varepsilon}\left(\tilde{G}\right) \geq \frac{2\sqrt{v_1'v_n'}}{v_1' + v'}\sqrt{nE_1\left(\tilde{G}\right) - \xi\left(\tilde{G}\right)^2 + 2mn.}$$

Hence the desired result. □

3. Conclusion

Here in paper, as we examine different properties and also lower and upper limits of laplacian energy ‘s eccentricity version of a fuzzy graph $\tilde{G}$. We observe that there is great analogy between the original laplacian energy and eccentricity version of laplacian energy whereas also has some distinct differences.

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