ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **9** (2020), no.9, 6507–6512
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.9.8

ON FRACTIONAL Q-CALCULUS OF THE R-FUNCTION

CRISTINA GAMMENG[1], U. K. SAHA, AND S. MAITY

ABSTRACT. The F-function and its generalization the R-function are of paramount significance in the fractional calculus. In this paper we establish 2 theorems that interconnects the R-function and the Riemann-Liouville fractional q-integral and q-derivative operators. As special case, we get the fractional q-integral and q-derivative of the F-function.

1. INTRODUCTION

The quantum calculus designated as the calculus devoid of limits, replaces the classical derivative with the difference operator to assist the sets of non-differentiable functions. The fractional q-calculus is the expansion of the regular fractional calculus in the q-theory. It has been employed in various areas of physics, mathematics and engineerings. Al-Salam [5] initiated the conception of q-fractional calculus. Subsequently Al-Salam [4],[5] and Agarwal [1] studied cer- tain q-fractional derivatives and integrals.

The fractional q-integral of Riemann-Liouville type is is given as

$$(I_{q,h}^{\varphi}g)(x)=\frac{1}{\Gamma_q(\varphi)}\int_h^x (x-qz)^{\varphi-1}g(z)d_qz,(\varphi\in R^+)\,. \tag{1.1}$$

---

[1] *corresponding author*

2010 *Mathematics Subject Classification.* 26A33,33E12.

Key words and phrases. R-function, Riemann-Liouville fractional q-integral, Riemann-Liouville fractional q-derivative.

Choosing the lower limit of integration $h = 0$, (1.1) takes the form

$$(I_q^{\varphi} g)(x) = \frac{1}{\Gamma_q(\varphi)} \int_0^x (x - qz)^{\varphi - 1} g(z) d_q z, (\varphi \in R^+)\,. \tag{1.2}$$

The fractional q-derivative can be defined as:

$$(D_q^{\varphi} g)(x) = (I_q^{-\varphi} g)(x) = \frac{1}{\Gamma_q(-\varphi)} \int_0^x (x - qz)^{-\varphi - 1} g(z) d_q z, \varphi < 0. \tag{1.3}$$

Of late, many new developments have been made in the field of fractional q- calculus engaging these q-derivatives and integrals by numerous researchers, see [2,3,6,7,8,11,12]. It was of immense benefit to discover a generalized function which when differintegrated fractionally by whatsoever order, returned itself. A function as such could enormously simplify the evaluation of fractional order differential equations.

The R-function [9] is defined as

$$R_{\rho,\sigma}(a, c, x) \equiv \sum_{m=0}^{\infty} \frac{a^m (x - c)^{(m+1)\rho - 1 - \sigma}}{\Gamma((m+1)\rho - \sigma)}, x > c \geq 0, \rho \geq 0, Re(\rho - \sigma) > 0\,.$$

Hartley and Lorenzo formulated a function that would straight away influence the result of the fractional order fundamental linear differential equation and named it as the F-function.

The F-function [9] is defined as

$$F_{\rho}(a, x) \equiv \sum_{m=0}^{\infty} \frac{a^m (x)^{(m+1)\rho - 1}}{\Gamma((m+1)\rho)}, \rho > 0\,. \tag{1.4}$$

They also showed that the F-function satisfied that which, the Oldham and Spanier (1974) referred to as the eigen function property. Earlier Robotnov (1969,1980) [10] had examined this function in detailed with respect to the hered- itary integrals for utilization in solid mechanics. F-function and its generalization the R-function are of extreme importance in finding solutions of fundamental linear differential equation.

2. MAIN RESULT

In this segment, we present the q-integral and q-derivative formula associated with the R-function. Also, as special case we get the fractional q-integral and q-derivative of the F-function.

Theorem 2.1. *Let* $x > c \geq 0, \rho \geq 0, Re(\rho - \sigma) > 0$ *and* I_q^{φ} *be the fractional q-integral operator, then:*

$$I_q^{\varphi}[R_{\rho,\sigma}(a,c,x)] = \sum_{m=0}^{\infty} \frac{a^m}{\Gamma((m+1)\rho - \sigma)} I_q^{\varphi}(x-c)^{(m+1)\rho-1-\sigma} .$$

Proof.

$$\begin{aligned} \Omega &\equiv I_q^{\varphi}[R_{\rho,\sigma}(a,c,x)] \\ &= \frac{1}{\Gamma_q(\varphi)} \int_0^x (x-qz)^{\varphi-1} R_{\rho,\sigma}(a,c,z) d_q z \\ &= \frac{1}{\Gamma_q(\varphi)} \int_0^x (x-qz)^{\varphi-1} \sum_{m=0}^{\infty} \frac{a^m (z-c)^{(m+1)\rho-1-\sigma}}{\Gamma((m+1)\rho-\sigma)} d_q z \, . \end{aligned}$$

Shifting the integration and summation order, we get:

$$\begin{aligned} &= \sum_{m=0}^{\infty} \frac{a^m}{\Gamma((m+1)\rho-\sigma)} \frac{1}{\Gamma_q(\varphi)} \int_0^x (x-qz)^{\varphi-1} (z-c)^{(m+1)\rho-1-\sigma} d_q z \\ &= \sum_{m=0}^{\infty} \frac{a^m}{\Gamma((m+1)\rho-\sigma)} I_q^{\varphi}(x-c)^{(m+1)\rho-1-\sigma} . \end{aligned}$$

Corollary 2.1. *For* $c = 0, \sigma = 0$*, there holds the formula*

$$I_q^{\varphi}[R_{\rho}(a,0,x)] = \sum_{m=0}^{\infty} \frac{a^m}{\Gamma((m+1)\rho)} I_q^{\varphi}(x)^{(m+1)\rho-1}$$

is the fractional q-integral of F-function, that can be calculated by using (1.2) and (1.4).

Corollary 2.2. *For* $c = 0, \sigma = 0, m = 1$*, there holds the formula*

$$I_q^{\varphi}[R_{\rho}(a,x)] = \frac{a}{\Gamma(2\rho)} I_q^{\varphi}(x)^{2\rho-1} .$$

Theorem 2.2. *Let* $x > c \geq 0, \rho \geq 0, Re(\rho - \sigma) > 0$ *and* D_q^{φ} *be the fractional q-derivative, then:*

$$D_q^{\varphi}[R_{\rho,\sigma}(a,c,x)] = \sum_{m=0}^{\infty} \frac{a^m}{\Gamma((m+1)\rho - \sigma)} D_q^{\varphi}(x-c)^{(m+1)\rho-1-\sigma} .$$

Proof.

$$\begin{aligned}\Omega &\equiv D_q^{\varphi}[R_{\rho,\sigma}(a,c,x)]\\ &= \frac{1}{\Gamma_q(-\varphi)}\int_0^x (x-qz)^{-\varphi-1}R_{\rho,\sigma}(a,c,z)d_qz\\ &= \frac{1}{\Gamma_q(-\varphi)}\int_0^x (x-qz)^{-\varphi-1}\sum_{m=0}^{\infty}\frac{a^m(z-c)^{(m+1)\rho-1-\sigma}}{\Gamma((m+1)\rho-\sigma)}d_qz\,.\end{aligned}$$

Shifting the integration and summation order, we get:

$$\begin{aligned}&= \sum_{m=0}^{\infty}\frac{a^m}{\Gamma((m+1)\rho-\sigma)}\frac{1}{\Gamma_q(-\varphi)}\int_0^x (x-qz)^{-\varphi-1}(z-c)^{(m+1)\rho-1-\sigma}d_qz\\ &= \sum_{m=0}^{\infty}\frac{a^m}{\Gamma((m+1)\rho-\sigma)}D_q^{\varphi}(x-c)^{(m+1)\rho-1-\sigma}\,.\end{aligned}$$

□

Corollary 2.3. *For $c=0,\sigma=0$, theorem 2.4 reduces to*

$$D_q^{\varphi}[R_\rho(a,0,x)] = \sum_{m=0}^{\infty}\frac{a^m}{\Gamma((m+1)\rho)}D_q^{\varphi}(x)^{(m+1)\rho-1}\,,$$

which is the fractional q-derivative of F-function, that can be obtained by solving (1.3) and (1.4).

Corollary 2.4. *For $c=0,\sigma=0,m=1$, the formula reduces to*

$$D_q^{\varphi}[R_\rho(a,0,x)] = \frac{a}{\Gamma(2\rho)}D_q^{\varphi}(x)^{2\rho-1}\,.$$

3. Conclusion

It is contemplated that the outcomes of the study may find utilization in finding solutions of fundamental linear fractional differential equation and fractional order problems of physical sciences and engineering areas where the R-function and F-function plays a pivotal role.

REFERENCES

[1] R. P. AGARWAL: *Certain fractional q-integrals and q-derivatives,* Proc. Camb. Philos. Soc., **66** (1969), 365-370.

[2] B. AHMAD, J. J. NIETO: *On nonlocal boundary value problems of nonlinear q-difference equations*, Adv. Differ.Equ., **2012** (2012), Article ID: 81, https://doi.org/10.1186/1687-1847-2012-81

[3] B. AHMAD, A. ALSAEDI, S. K. NTOUYAS: *Positive solutions for boundary value problem of nonlinear fractional q-difference equation*, ISRN Math. Anal., **2011** (2011), Article ID: 385459. https://doi.org/10.5402/2011/385459

[4] W. A. AL-SALAM, A. VERMA I. DIMOVSKI, V. KIRYAKOVA: *A fractional Leibniz q-formula,* Pac. J. Math., **60** (1975), 1-9.

[5] W. A. AL-SALAM: *Some fractional q-integrals and q-derivatives*, Proc. Edinburg. Math. Soc., **15** (1966), 135-140.

[6] M. EL-SHAHED, F. M. AL-ASKARL: *Positive solutions for boundary value problem of non-linear fractional q-difference equation,* ISRN Math. Anal., **2011** (2011), Article ID: 385459. https://doi.org/10.5402/2011/385459

[7] R. A. C. FERREIRA: *Positive solutions for a class of boundary value problems with fractional q-differences,* Comput. Math. Appl., **61**(2) (2000), 367- 373. https://doi.org/10.1016/j.camwa.2010.11.012

[8] R. A. C. FERREIRA: *Nontrivial solutions for fractional q-difference boundary value problems*, Electron, J. Qual. Theory Differ. Equ., **70** (2010), 1- 10. https://doi.org/10.14232/ejqtde.2019.1.44

[9] C. F. LORENZO, T. T. HARTLEY: *R-function relationships for application in the fractional calculus,* NASA/TM-2000-210361, E-12410, NAS 1.15:210361, 2000.

[10] Y. N. ROBOTNOV: *Elements of hereditary solid mechanics,* (In English) MIR Pub- lishers, Moscow, 1980.

[11] L. ZHANG,D. BALEANU, G. WANG: *Nonlocal boundary value problem for nonlin- ear impulsive q(k)-integrodifference equation,* Abstr. Appl. Anal., **2014** (2014), Article ID: 478185. https://doi.org/10.1155/2014/478185

[12] W. X. ZHOU, H. Z. LIU: *Existence solutions for boundary value problem of non- linear fractional q- differenceequations,* Adv. Differ. Equ., **2013** (2013), Article ID: 113. https://doi.org/10.1186/1687-1847-2013-113

DEPARTMENT OF BASIC AND APPLIED SCIENCE
NATIONAL INSTITUTE OF TECHNOLOGY
ARUNACHAL PRADSH, INDIA
Email address: cgammeng@gmail.com

DEPARTMENT OF BASIC AND APPLIED SCIENCE
NATIONAL INSTITUTE OF TECHNOLOGY
ARUNACHAL PRADSH, INDIA
Email address: utpal@nitap.ac.in

DEPARTMENT OF BASIC AND APPLIED SCIENCE
NATIONAL INSTITUTE OF TECHNOLOGY
ARUNACHAL PRADSH, INDIA
Email address: susantamaiti@gmail.com