ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **9** (2020), no.9, 7407–7416
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.9.87

WHICH PARTS OF GENERALIZED EXPONENTIAL DISTRIBUTION CONTAIN MORE INFORMATION ABOUT PARAMETERS

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ABSTRACT. In this paper, we consider Fisher information matrix in the rth order statistic of generalized exponential distribution. Numerical tabulation of the matrix are also provided for different values of r and parameter. Then the asymptotic Fisher information in the rth order statistic is evaluated, from which we can derived the percentiles which contain least and most information about parameters.

1. INTRODUCTION

Suppose $X_1, \cdots, X_n$ are independent and identically distributed (iid) random variables. The density function of generalized exponential random variables with shape parameter α and scale parameter λ is denoted by

$$f(x;\alpha,\lambda) = \alpha\lambda\, e^{-\lambda x}\big(1-e^{-\lambda x}\big)^{(\alpha-1)}\ , \tag{1.1}$$

for $x>0,\ \alpha>0$ and $\lambda>0$.

The Fisher information plays an important role in the statistical inference in connection with estimation, sufficiency and properties of variance of estimators. It is related to the covariance matrix of the estimate of $\underline{\vartheta}$, being its inverse under

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2010 *Mathematics Subject Classification.* 33C90, 62E99.
Key words and phrases. Beta function, Euler psi function, Fisher information, Generalized exponential distribution, Order statistics, Quantile.

certain conditions. It is well known that Fisher information serves as a valuable tool for derivation of variance in the asymptotic distribution of maximum likelihood estimators (MLE).

Let $X_i (i = 1, \cdots, n)$ be a sample from $F_{\underline{\vartheta}}$, where $\underline{\vartheta} = (\vartheta_1, \vartheta_2, ..., \vartheta_k)'$. The exact Fisher information matrix contained in the rth order statistics $(X_{r:n})$ about $\underline{\vartheta}$ is defined as $\left[I_{r:n}^{ij}\right] = \left[E\left(\frac{\partial \log f_{r:n}(\underline{\vartheta})}{\partial \vartheta_i} \frac{\partial \log f_{r:n}(\underline{\vartheta})}{\partial \vartheta_j}\right)\right]$, for $i, j = 1, ..., k$, where $f_{r:n}(\underline{\vartheta})$ is probability density function (pdf) of $X_{r:n}$.

The problem of obtaining Fisher information in order statistics was described by [7] with the words: "while the recipe for $I_Y(\theta)$ is simple, the details are messy in most cases", where Y is an arbitrary collection of order statistics. Several results have been published in this direction in recent years. Abo-Eleneen and Nagaraja [1] studied the Fisher information in collections of order statistics and their concomitants from bivariate samples. Nadarajah [8] computed information matrices for Laplace and Pareto mixtures. Park and Kim [9] considered the Fisher information in exponential distribution and simplified the Fisher information in any set of order statistics to a sum of single integrals. In other application, such as life testing, the asymptotic Fisher information is used, which which will be defined in section 4.

2. Exact Fisher Information Matrix for the rth Order Statistic

Let $X_1, ..., X_n$ be a sample from (1.1), then the probability density function of $X_{r:n}$ can be written as

$$f_{r:n}(x; \alpha, \lambda) = c_{r,n} \alpha \lambda \, e^{-\lambda x} \left(1 - e^{-\lambda x}\right)^{(\alpha r - 1)} \left(1 - \left(1 - e^{-\lambda x}\right)^{\alpha}\right)^{n-r}, \tag{2.1}$$

where $c_{r,n} = \frac{n!}{(r-1)!(n-r)!}$.

The computation of elements of Fisher information matrix use the Euler psi function and Beta function defined by $\Psi(a) = \frac{\partial \log \Gamma(a)}{\partial a}$ & $B(a, b) = \int_0^1 t^{a-1}(1 - t)^{b-1} dt$. Now, we can compute the elements of the Fisher information matrix by application the Appendix.

For $r < n - 1$, it is easy by using lemmas in Appendix to verify that

$$
\begin{aligned}
I_{r:n}^{11} &= E(-\frac{\partial^2 \log f_{r:n}}{\partial\alpha^2}) = \frac{1}{\alpha^2} + \frac{(n-r)c_{r,n}}{\alpha^2} \cdot B(r+1, n-r-1) \\
&\quad \cdot \Big\{\Psi'(r+1) - \Psi'(n) + \big[\Psi(r+1) - \Psi(n)\big]^2\Big\}, \\
I_{r:n}^{12} &= E(-\frac{\partial^2 \log f_{r:n}}{\partial\alpha\partial\lambda}) = \frac{\alpha r c_{r,n}}{\lambda} \sum_{j=0}^{n-r} C_{n-r,j}(-1)^j \cdot B(\alpha r + \alpha j - 1, 2) \\
&\quad \cdot \big[\Psi(2) - \Psi(\alpha r + \alpha j + 1)\big] - \frac{\alpha(n-r)c_{r,n}}{\lambda} \cdot \Big\{ \sum_{j=0}^{n-r-1} C_{n-r-1,j}(-1)^j \\
&\quad \cdot B(\alpha r + \alpha j + \alpha - 1, 2) \times \big[\Psi(2) - \Psi(\alpha r + \alpha j + \alpha + 1)\big] \\
&\quad + \alpha \sum_{j=0}^{n-r-1} C_{n-r-1,j}(-1)^j B(\alpha r + \alpha j + \alpha - 1, 2)\Big(\big[\Psi(\alpha r + \alpha j + \alpha - 1) \\
&\quad - \Psi(\alpha r + \alpha j + \alpha + 1)\big]\big[\Psi(2) - \Psi(\alpha r + \alpha j + \alpha + 1)\big] \\
&\quad - \Psi'(\alpha r + \alpha j + \alpha + 1)\Big) + \alpha \sum_{j=0}^{n-r-2} C_{n-r-2,j}(-1)^j B(\alpha r + \alpha j + 2\alpha - 1, 2) \\
&\quad \cdot \Big(\big[\Psi(\alpha r + \alpha j + 2\alpha - 1) - \Psi(\alpha r + \alpha j + 2\alpha + 1)\big] \\
&\quad \cdot \big[\Psi(2) - \Psi(\alpha r + \alpha j + 2\alpha + 1)\big] - \Psi'(\alpha r + \alpha j + 2\alpha + 1)\Big)\Big\}, \\
I_{r:n}^{22} &= E(-\frac{\partial^2 \log I_{r:n}}{\partial\lambda^2}) \\
&= \frac{1}{\lambda^2} + \frac{\alpha(\alpha r - 1)c_{r,n}}{\lambda^2} \sum_{j=0}^{n-r} C_{n-r,j}(-1)^j \; B(\alpha r + \alpha j - 2, 2)\Big(\Psi'(2) \\
&\quad - \Psi'(\alpha r + \alpha j) + \big[\Psi(2) - \Psi(\alpha r + \alpha j)\big]^2\Big) \\
&\quad - \frac{\alpha^2(n-r)c_{r,n}}{\lambda^2}\Big\{ \sum_{j=0}^{n-r-1} C_{n-r-1,j}(-1)^j B(\alpha r + \alpha j + \alpha - 1, 2) \\
&\quad \times \Big(\Psi'(2) - \Psi'(\alpha r + \alpha j + \alpha + 1) + \big[\Psi(2) - \Psi(\alpha r + \alpha j + \alpha + 1)\big]^2\Big) \\
&\quad - (\alpha - 1) \sum_{j=0}^{n-r-1} C_{n-r-1,j}(-1)^j B(\alpha r + \alpha j + \alpha - 2, 3) \\
&\quad \times \Big(\Psi'(3) - \Psi'(\alpha r + \alpha j + \alpha + 1) + \big[\Psi(3) - \Psi(\alpha r + \alpha j + \alpha + 1)\big]^2\Big)
\end{aligned}
$$

$$
\begin{aligned}
&- \alpha \sum_{j=0}^{n-r-2} C_{n-r-2,j}(-1)^j B(\alpha r + \alpha j + 2\alpha - 2, 3) \\
&\times \Big(\Psi'(3) - \Psi'(\alpha r + \alpha j + 2\alpha + 1) + \big[\Psi(3) - \Psi(\alpha r + \alpha j + 2\alpha + 1)\big]^2 \Big) \Big\},
\end{aligned}
$$

where $C_{n,m} = \frac{n!}{m!(n-m)!}$. For $r = n-1$ and $r = n$, we have

$$
\begin{aligned}
I_{n-1:n}^{11} &= \frac{1}{\alpha^2}\Big\{ 1 + 2n(n-1)\sum_{j=0}^{\infty}\frac{B(\alpha n + \alpha j, 1)}{(n+j)^2} \Big\}, \\
I_{n-1:n}^{12} &= \frac{\alpha n(n-1)^2}{\lambda}\Big\{ B(\alpha n - \alpha - 1, 2)\big[\Psi(2) - \Psi(\alpha n - \alpha + 1)\big] \\
&\quad - B(\alpha n - 1, 2)\big[\Psi(2) - \Psi(\alpha n + 1)\big]\Big\} \\
&\quad - \frac{\alpha n(n-1)}{\lambda}\Big\{ B(\alpha n - 1, 2)\big[\Psi(2) - \Psi(\alpha n + 1)\big] - \alpha B(\alpha n - 1, 2) \\
&\quad \cdot \Big(\big[\Psi(2) - \Psi(\alpha n + 1)\big]\big[\Psi(\alpha n - 1) - \Psi(\alpha n + 1)\big] - \Psi'(\alpha n + 1) \Big) \\
&\quad + \alpha \sum_{j=0}^{\infty} B(\alpha n + \alpha j + \alpha - 1, 2)\Big(\big[\Psi(2) - \Psi(\alpha n + \alpha j + \alpha + 1)\big] \\
&\quad \cdot \big[\Psi(\alpha n + \alpha j + \alpha - 1) - \Psi(\alpha n + \alpha j + \alpha + 1)\big] - \Psi'(\alpha n + \alpha j + \alpha + 1) \Big) \Big\}, \\
I_{n-1:n}^{22} &= \frac{1}{\lambda^2} + \frac{\alpha n(n-1)(\alpha n - \alpha - 1)}{\lambda^2}\Big\{ B(\alpha n - \alpha - 2, 2)\Big(\Psi'(2) \\
&\quad - \Psi'(\alpha n - \alpha) + \big[\Psi(2) - \Psi(\alpha n - \alpha)\big]^2 \Big) \\
&\quad - B(\alpha n - 2, 2)\Big(\Psi'(2) - \Psi'(\alpha n) + \big[\Psi(2) - \Psi(\alpha n)\big]^2 \Big) \Big\} \\
&\quad - \frac{\alpha^2 n(n-1)}{\lambda^2}\Big\{ B(\alpha n - 1, 2)\Big(\Psi'(2) - \Psi'(\alpha n + 1) + \big[\Psi(2) - \Psi(\alpha n + 1)\big]^2 \Big) \\
&\quad - (\alpha - 1)B(\alpha n - 2, 3)\Big(\Psi'(3) - \Psi'(\alpha n + 1) + \big[\Psi(3) - \Psi(\alpha n + 1)\big]^2 \Big) \\
&\quad - \alpha \sum_{j=0}^{\infty} B(\alpha n + \alpha j + \alpha - 2, 3)\Big(\Psi'(3) - \Psi'(\alpha n + \alpha j + \alpha + 1)
\end{aligned}
$$

$$+\left[\Psi(3)-\Psi(\alpha n+\alpha j+\alpha+1)\right]^2\Big)\Big\},$$

$$I_{n:n}^{11}=\frac{1}{\alpha^2},$$
$$I_{n:n}^{12}=\frac{\alpha n^2}{\lambda}B(\alpha n-1,2)\left[\Psi(2)-\Psi(\alpha n+1)\right],$$
$$I_{n:n}^{22}=\frac{1}{\lambda^2}+\frac{\alpha n(\alpha n-1)}{\lambda^2}B(\alpha n-2,2)\Big(\Psi'(2)-\Psi'(\alpha n)+\left[\Psi(2)-\Psi(\alpha n)\right]^2\Big).$$

3. TABLES

In this section, we compute the Fisher information matrix derived in section 2. We used manipulation package MAPLE 9.5 in order to compute of Psi(.) function involving in the formulas. The table 1 gives the numerical values of the elements of Fisher information matrix for $\alpha=4$, $\lambda=2$, $n=30$ and $r=\{1,2,...,n\}$.

TABLE 1. The elements of Fisher information matrix for $\alpha=4$, $\lambda=2$ and $n=30$

r	$I_{r:n}^{11}$	$I_{r:n}^{12}$	$I_{r:n}^{22}$	r	$I_{r:n}^{11}$	$I_{r:n}^{12}$	$I_{r:n}^{22}$
1	0.69078787	-1.17372783	2.14531236	16	0.90288331	-3.62297314	14.87177646
2	0.95427478	-1.87557286	3.86636347	17	0.84827720	-3.53577099	15.07971368
3	1.09656449	-2.37465231	5.34252022	18	0.79214663	-3.43111221	15.21148131
4	1.17715469	-2.75069882	6.64381106	19	0.73474616	-3.30925516	15.26250431
5	1.22038573	-3.04097860	7.80744350	20	0.67628860	-3.17023588	15.22674089
6	1.23873510	-3.26658508	8.85620067	21	0.61695303	-3.01385659	15.09630518
7	1.23948212	-3.44084183	9.80518326	22	0.55689112	-2.83965828	14.86090599
8	1.22724790	-3.57278426	10.66487030	23	0.49623201	-2.64687122	14.50699357
9	1.20514659	-3.66884822	11.44270540	24	0.43508619	-2.43433219	14.01641971
10	1.17537240	-3.73378119	12.14399250	25	0.37354863	-2.20034658	13.36423436
11	1.13952569	-3.77117551	12.77243590	26	0.31170124	-1.94245118	12.51483449
12	1.09880626	-3.78379552	13.33046609	27	0.24961495	-1.65697951	11.41465732
13	1.05413408	-3.77379076	13.81944858	28	0.18735136	-1.33818355	9.97664543
14	1.00622786	-3.74283787	14.23979801	29	0.12496287	-0.97536097	8.02820919
15	0.95565799	-3.69223588	14.59103214	30	0.06250000	-0.55069768	5.24597398

4. ASYMPTOTIC FISHER INFORMATION IN A SINGLE ORDER STATISTIC

From table 1, one can see that $X_{(30:30)}$ and $X_{(7:30)}$ contain the least and the most information about the shape parameter, and, $X_{(1:30)}$ and $X_{(19:30)}$ contain the least and the most information about the scale parameter, respectively.

Now, we derive order statistics that have most information about the shape and the scale parameters for large values of n.

Definition 4.1. *(Zheng, [19]) Assume* $\frac{r_i}{n} \to p_i$ *(for* $i = 1, 2, ..., k$*), as* $n \to \infty$*, where* $0 \le p_1 < p_2 < ... < p_k \le 1$*. The asymptotic Fisher information about* θ *contained in* k *sample quantiles* $(X_{r_1:n}, X_{r_2:n}, ...X_{r_k:n})$ *, denoted by* $I_{p_1 p_2 \cdots p_k}(\vartheta)$*, is defined as*

$$I_{p_1 p_2 \cdots p_k}(\vartheta) = \lim_{n\to\infty} \frac{1}{n} I_{r_1 r_2 \cdots r_k : n}(\vartheta), \tag{4.1}$$

which can be written as

$$I_{p_1 p_2 \cdots p_k}(\vartheta) = \sum_{i=0}^{k} \frac{1}{p_{i+1} - p_i} \left\{ \int_{\xi_{p_i}}^{\xi_{p_{i+1}}} \frac{\partial}{\partial \vartheta} f(x; \vartheta) dx \right\}^2 , \tag{4.2}$$

where $p_0 = 0,\ p_{k+1} = 1$*, and* $\xi_p = F^{-1}(p; \vartheta)$*.*

So, the asymptotic Fisher information in a single order statistic can be obtained rapidly by setting $k = 1$ in (4.2). Thus, we get

$$I_p(\vartheta) \ = \ \frac{1}{p(1-p)} \left\{ \int_{-\infty}^{\xi_p} \frac{\partial}{\partial \vartheta} f(x; \vartheta) \right\}^2 . \tag{4.3}$$

where $0 \le p \le 1$.

Now, we find the asymptotic Fisher information in a single quantile of generalized exponential distribution. From (1.1), we have $F(x; \alpha, \lambda) = \left(1 - e^{-\lambda x}\right)^{\alpha}$ and so $\xi_p = -\log\left(1 - p^{1/\alpha}\right)/\lambda$. By (4.3), $I_p(\alpha)$ and $I_p(\lambda)$ for generalized exponential distribution can be calculated as follows

$$\begin{aligned} I_p(\alpha) &= \frac{1}{p(1-p)} \left\{ \int_{-\infty}^{\xi_p} \frac{\partial}{\partial \alpha} f(x; \alpha, \lambda) dx \right\}^2 = \frac{1}{p(1-p)} \left\{ -f(\xi_p; \alpha, \lambda)\, \frac{\partial}{\partial \alpha} \xi_p \right\}^2 \\ &= \frac{\alpha^2 p\, (\log p)^2}{(1-p)}, \qquad (4.4) \\ I_p(\lambda) &= \frac{1}{p(1-p)} \left\{ \int_{-\infty}^{\xi_p} \frac{\partial}{\partial \lambda} f(x; \alpha, \lambda) dx \right\}^2 = \frac{1}{p(1-p)} \left\{ -f(\xi_p; \alpha, \lambda)\, \frac{\partial}{\partial \lambda} \xi_p \right\}^2 \\ &= \frac{\alpha^2 p\left(p^{-1/\alpha} - 1\right)\, \log^2\left(1 - p^{1/\alpha}\right)}{\lambda^2 (1-p)}. \qquad (4.5) \end{aligned}$$

Figures 1 and 2 show the areas with the least and the most information about the shape and the scale parameters, respectively, which the weak color and

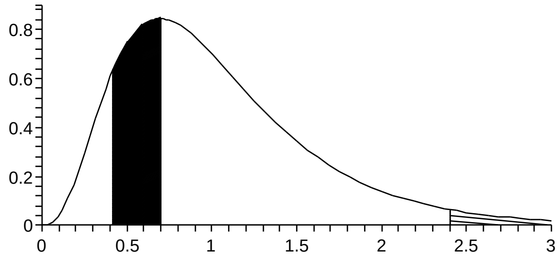

FIGURE 1. The areas with least and most information about shape parameter in generalized exponential distribution with $\alpha = 4$ and $\lambda = 2$.

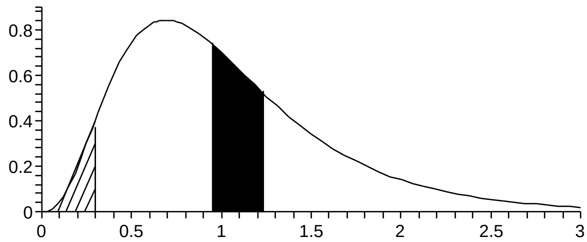

FIGURE 2. The areas with least and most information about scale parameter in generalized exponential distribution with $\alpha = 4$ and $\lambda = 2$.

strong color parts have the least and the most information about the parameters. From figure 4, we can conclude that in skew shape-scale distribution, the shorter tail of density function has the least information about scale parameter.

DISCUSSION

The results of this work can potentially be extended to the case of generalized exponential distributed errors in the poruses of variance comparison of proposed estimators in [2–6, 10–18].

5. APPENDIX

We need the following technical lemmas to calculate the Fisher information matrix for order statistics.

Lemma 5.1. *Suppose X is a random variable with the pdf (2.1),*

$$E\left(\frac{Xe^{-\lambda X}}{1-e^{-\lambda X}}\right) = -\frac{\alpha c_{r,n}}{\lambda}\sum_{j=0}^{n-r} C_{n-r,j}(-1)^j B(\alpha r+\alpha j-1,2) \cdot \big[\Psi(2)-\Psi(\alpha r+\alpha j+1)\big],$$

where $\alpha r > 1$.

Proof. By setting $y = \big(1-e^{-\lambda x}\big)$, we have

$$\begin{aligned} E\left(\frac{Xe^{-\lambda X}}{1-e^{-\lambda X}}\right) &= -\frac{\alpha c_{r,n}}{\lambda}\int_0^1 \log(1-y)(1-y^{\alpha})^(n-r)y^{\alpha r-2}(1-y)dy \\ &= -\frac{\alpha c_{r,n}}{\lambda}\sum_{j=0}^{n-r} C_{n-r,j}(-1)^j \int_0^1 \log(1-y)y^{\alpha r+\alpha j-2}(1-y)dy, \end{aligned}$$

which the last equality follows by using binomial expansion. Now, by using

$$\int_0^1 \log(1-t)\, t^{a-1}(1-t)^{b-1}dt = \frac{\partial}{\partial b}B(a,b), \tag{5.1}$$

the proof is completed. □

Lemma 5.2. *Suppose X is a random variable with the pdf (2.1),*

$$E\left(\frac{X^2e^{-\lambda X}}{\big(1-e^{-\lambda X}\big)^2}\right) = \frac{\alpha c_{r,n}}{\lambda^2}\sum_{j=0}^{n-r} C_{n-r,j}(-1)^j B(\alpha r+\alpha j-2,2) \cdot \Big(\Psi'(2)-\Psi'(\alpha r+\alpha j)+\big[\Psi(2)-\Psi(\alpha r+\alpha j)\big]^2\Big),$$

where $\alpha r > 2$.

Proof. By setting $y = \big(1-e^{-\lambda x}\big)$, we have

$$\begin{aligned} E\left(\frac{X^2e^{-\lambda X}}{\big(1-e^{-\lambda X}\big)^2}\right) &= \frac{\alpha c_{r,n}}{\lambda^2}\int_0^1 \log^2(1-y)(1-y^{\alpha})^(n-r)y^{\alpha r-3}(1-y)dy \\ &= \frac{\alpha c_{r,n}}{\lambda^2}\sum_{j=0}^{n-r} C_{n-r,j}(-1)^j \int_0^1 \log^2(1-y)\; y^{\alpha r+\alpha j-3}(1-y)dy\;, \end{aligned}$$

Now, by using $\int_0^1 \log^2(1-t)\, t^{a-1}(1-t)^{b-1}dt = \frac{\partial^2}{\partial b^2}B(a,b)$, the proof is completed.
□

Lemma 5.3. *Suppose X is a random variable with the pdf (2.1),*

$$E\left(\frac{Xe^{-\lambda X}\big(1-e^{-\lambda X}\big)^{\alpha-1}\log\big(1-e^{-\lambda X}\big)}{1-\big(1-e^{-\lambda X}\big)^{\alpha}}\right)$$

$$=-\frac{\alpha c_{r,n}}{\lambda}\sum_{j=0}^{n-r-1}C_{n-r-1,j}(-1)^jB(\alpha r+\alpha j+\alpha-1,2)\Big(\big[\Psi(2)-\Psi(\alpha r+\alpha j+\alpha+1)\big]$$

$$\times\big[\Psi(\alpha r+\alpha j+\alpha-1)-\Psi(\alpha r+\alpha j+\alpha+1)\big]-\Psi'(\alpha r+\alpha j+\alpha+1)\Big),$$

for $r<n$, where $\alpha r+\alpha>1$.

Proof. By setting $y=\big(1-e^{-\lambda x}\big)$, we have

$$E\left(\frac{Xe^{-\lambda X}\big(1-e^{-\lambda X}\big)^{\alpha-1}\log\big(1-e^{-\lambda X}\big)}{1-\big(1-e^{-\lambda X}\big)^{\alpha}}\right)$$
$$=-\frac{\alpha c_{r,n}}{\lambda}\sum_{j=0}^{n-r-1}C_{n-r-1,j}(-1)^j\int_0^1\log(y)\log(1-y)y^{\alpha r+\alpha-2}(1-y)dy.$$

Now, by using $\int_0^1\log(t)\log(1-t)\ t^{a-1}(1-t)^{b-1}dt=\frac{\partial^2}{\partial a\partial b}B(a,b)$, the proof is completed. $\square$

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