

# **ERROR ESTIMATES FOR FINITE ELEMENT APPROXIMATIONS OF NONLINEAR PARABOLIC PROBLEMS IN NONCONVEX POLYGONAL DOMAINS**

TAMAL PRAMANICK<sup>1</sup> AND SHANTIRAM MAHATA

**ABSTRACT.** In this exposition, we consider the nonlinear parabolic problem with homogeneous Dirichlet boundary condition in a plane nonconvex polygonal domain. Due to the reentrant corner on the boundary, the singularity occurs in the finite element solutions near the reentrant corner (cf. Grisvard [1]). As a result, the rate of convergence which is optimal order in a convex polygonal domain is reduced for the case of nonconvex polygonal domain. We analyze the convergence properties in the  $L^2$ -norm for both the spatially semidiscrete and fully discrete methods.

## 1. INTRODUCTION

In this paper we focus our attention to the nonlinear parabolic initial-boundary value problems in domains with nonsmooth boundaries. Consider the nonlinear parabolic problem, for  $u = u(x, t)$ ,

$$\begin{aligned}
 (1.1) \quad & u_t - \nabla \cdot (a(u) \nabla u) = f(u) && \text{in } \Omega, \, t \in J, \\
 & u = 0 && \text{on } \partial\Omega, \, t \in J, \\
 & \text{with } u(\cdot, 0) = v && \text{in } \Omega,
 \end{aligned}$$

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<sup>1</sup>corresponding author

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where  $J = (0, T]$ ,  $T > 0$ , be a finite interval in time and  $\Omega$  be a nonconvex polygonal domain in  $\mathbb{R}^2$ , with boundary  $\partial\Omega$ . Also, define the smooth functions  $a$  and  $f$  on  $\mathbb{R}$  such that

$$(1.2) \quad 0 < \mu \leq a(u) \leq M, \quad |a'(u)| + |f'(u)| \leq B, \quad \text{for } u \in \mathbb{R}.$$

We assume that the above problem admits a unique solution.

Parabolic partial differential equations (PDEs) in nonconvex polygonal domains appear in many applications such as heat conduction in chip design, environment modeling, porous media flow, modeling of complex technical engines and many others (cf. [2, 3]). There has been several considerable research to solve nonlinear parabolic PDEs using finite element method (FEM) in convex domains, see [4, 5]. In [6, 7], Chatzipantelidis *et al.* have been made an effort to investigate the FEM for problems in nonconvex polygonal domains which are mainly focused on linear models. The regularity of the solution of elliptic problems in a nonconvex domain  $\Omega$  can be found in [8, 9].

For simplicity, we assume that  $\omega$  is exactly one interior angle which is reentrant such that  $\pi < \omega < 2\pi$ . We set  $\beta = \pi/\omega$ , and hence  $1/2 < \beta < 1$ . In particular, for the case of L-shaped domain,  $\omega = 3\pi/2$  and  $\beta = 2/3$ . Assume that  $O$  is the associated vertex at the origin and  $(r, \theta)$  denotes the polar coordinates describing the domain near  $O$ , with  $0 < \theta < \omega$ . The singularity in the solution will arise at the corner  $O$  with a leading term near  $O$  of the form

$$\kappa(f)r^\beta \sin(\beta\theta),$$

where  $\kappa(f) \neq 0$  in general, even when  $f$  is smooth. For further details on the singular functions and the singular solutions, we refer to [1, 6, 7]. We show that the order of convergence in the  $L^2$ -norm for both the spatially semidiscrete and fully discrete method is reduced from  $\mathcal{O}(h^2)$  (the order of convergence for the convex domain) to  $\mathcal{O}(h^{2\beta})$ , due to the presence of singularity in the solution of (1.1) at the reentrant corner. All notations used throughout this paper are followed from [10]. To the best of authors' knowledge there are no literature available concerning FEM for nonlinear parabolic problems in nonconvex domains.

The paper is organized as follows. In the next section we define the finite element space corresponding to the triangulation of the domain  $\Omega$  and describe the elliptic projection which is used in the error estimates. Section 3 is devoted

to a priori error estimates for the spatially semidiscrete scheme. In Section 4, we derive a priori error estimates for the fully discrete backward Euler method. Finally, some concluding remarks are presented in Section 5.

## 2. FINITE ELEMENT SOLUTION

In order to introduce the finite element space, let  $\mathcal{T}_h = \{K\}$  be the family of quasiuniform triangulations of  $\Omega$  with  $\max_{K \in \mathcal{T}_h} \text{diam}(K) \leq h$ . The triangulations are quasiuniform in the sense that there is some constant  $c > 0$  such that  $\min_{K \in \mathcal{T}_h} \text{diam}(K) \geq ch$ . The finite element discretizations for an  $L$ -shaped domain are depicted in Figures 1 and 2.

Let  $S_h$  be the finite dimensional space corresponding to the triangulations  $\mathcal{T}_h$  is defined by

$$S_h = \{\chi \in \mathcal{C} : \chi|_K \text{ is linear, } \forall K \in \mathcal{T}_h \text{ and } \chi|_{\partial\Omega} = 0\},$$

where  $\mathcal{C} = \mathcal{C}(\Omega)$  be the space of continuous functions on  $\bar{\Omega}$ . Then the approximation with the finite elements leads to the semidiscrete problem to find  $u_h(t) = u_h(\cdot, t)$ , belonging to  $S_h$  for  $t \in \bar{J}$ , such that

$$(2.1) \quad (u_{h,t}, \chi) + (a(u_h) \nabla u_h, \nabla \chi) = (f(u_h), \chi) \quad \forall \chi \in S_h, \quad t \in J, \\ u_h(0) = v_h,$$

where  $v_h$  is an approximation of  $v$  in  $S_h$ . Following [4], it is easy to notice that the semidiscrete approximation (2.1) has a unique solution which is bounded for  $t \in J$ .

To start the spatially semidiscrete error analysis for the semidiscrete problem (2.1), we first introduce the elliptic projection  $\tilde{u}_h = \tilde{u}_h(t)$  in  $S_h$  of the exact solution  $u$  defined by

$$(2.2) \quad (a(u(t)) \nabla (\tilde{u}_h(t) - u(t)), \nabla \chi) = 0 \quad \forall \chi \in S_h, \quad t \geq 0.$$

Now we need some estimates for the error in this projection and therefore we first derive the following lemma.

**Lemma 2.1.** Assume  $b = b(x)$  be a smooth function in  $\Omega$  with  $0 < \mu \leq b(x) \leq M$  for  $x \in \Omega$ . Consider  $\xi \in H^{1+s}(\Omega) \cap H_0^1(\Omega)$ , and let  $\xi_h$  be defined by

$$(b \nabla (\xi_h - \xi), \nabla \chi) = 0 \quad \forall \chi \in S_h.$$

Then

$$(2.3) \quad \|\nabla(\xi_h - \xi)\| \leq C_1 h^\beta \|\Delta\xi\|_{H^{-1+s}} \quad \text{for } \beta < s \leq 1,$$

and

$$(2.4) \quad \|\xi_h - \xi\| \leq C_2 h^{2\beta} \|\Delta\xi\|_{H^{-1+s}} \quad \text{for } \beta < s \leq 1.$$

The constants  $C_1$  and  $C_2$  depends on  $\mu$  and  $M$  and on the family of triangulations  $\mathcal{T}_h$ . Also  $C_2$  depends on an upper bound for  $\nabla b$ .

*Proof.* For any  $\chi \in S_h$ , we have

$$\begin{aligned} \mu \|\nabla(\xi_h - \xi)\|^2 &\leq (b\nabla(\xi_h - \xi), \nabla(\xi_h - \xi)) \\ &= (b\nabla(\xi_h - \xi), \nabla(\chi - \xi)) \\ &\leq M \|\nabla(\xi_h - \xi)\| \|\nabla(\chi - \xi)\|, \end{aligned}$$

which implies

$$\|\nabla(\xi_h - \xi)\| \leq (M/\mu) \|\nabla(\chi - \xi)\|.$$

Following [6, Lemma 2.5], and with the standard interpolant  $I_h\xi$  of  $\xi$ , we obtain

$$\|\nabla(\xi_h - \xi)\| \leq C_1 h^\beta \|\Delta\xi\|_{H^{-1+s}} \quad \text{for } \beta < s \leq 1,$$

which proofs (2.3). In order to show (2.4) we use the duality argument. For this purpose, we consider the problem

$$(2.5) \quad -\nabla \cdot (b\nabla\psi) \equiv -b\Delta\psi - \nabla b \cdot \nabla\psi = \varphi \quad \text{in } \Omega, \quad \psi = 0 \quad \text{on } \partial\Omega,$$

and since,  $\|\psi\| \leq C\|\nabla\psi\|$  for  $\psi \in H_0^1$ , we have

$$\mu \|\nabla\psi\|^2 \leq (b\nabla\psi, \nabla\psi) = (\varphi, \psi) \leq \|\varphi\| \|\psi\| \leq C\|\varphi\| \|\nabla\psi\|,$$

which gives  $\|\nabla\psi\| \leq C\|\varphi\|$ . Therefore, using the elliptic regularity estimate (see, e.g. [6, Lemma 2.5], or Bacuta *et al.* [8]) and for boundedness of  $\nabla b$  together with equation (2.5), we get

$$(2.6) \quad \|\Delta\psi\|_{H^{-1+s}} \leq C\|\varphi + \nabla b \cdot \nabla\psi\| \leq C\|\varphi\|.$$

Hence, with  $\chi = I_h\psi$ ,

$$\begin{aligned}
 (\xi_h - \xi, \varphi) &= (b\nabla(\xi_h - \xi), \nabla\psi) \\
 &= (b\nabla(\xi_h - \xi), \nabla(\psi - \chi)) \\
 &\leq M\|\nabla(\xi_h - \xi)\|\|\nabla(\psi - \chi)\| \\
 &\leq (Ch^\beta\|\Delta\xi\|_{H^{-1+s}})(Ch^\beta\|\Delta\psi\|_{H^{-1+s}}) \\
 &\leq C_2h^{2\beta}\|\Delta\xi\|_{H^{-1+s}}\|\varphi\|,
 \end{aligned}$$

and this completes the proof of the lemma.  $\square$

### 3. SPATIALLY DISCRETE ERROR ANALYSIS

In this section we have concerned on some error estimates for the spatially semidiscrete finite element approximation (2.1) of the parabolic problem (1.1). For this purpose, we split the error term using the so called elliptic projection  $\tilde{u}_h$  defined in (2.2) as a sum of two terms,

$$(3.1) \quad u_h - u = (u_h - \tilde{u}_h) + (\tilde{u}_h - u) = \theta + \rho.$$

Hence, for estimates the error we need to first estimate the terms  $\rho$  and  $\rho_t$ , which is given in the following lemma.

**Lemma 3.1.** *Let  $\rho$  be defined by (3.1) and  $C(u)$  is independent of  $t \in J$ . Then under consideration the appropriate regularity assumptions on  $u$ , we have for  $t \in J$ ,  $\beta < s \leq 1$ ,*

$$\|\rho(t)\| + h^\beta\|\nabla\rho(t)\| \leq C(u)h^{2\beta} \quad \text{and} \quad \|\rho_t(t)\| + h^\beta\|\nabla\rho_t(t)\| \leq C(u)h^{2\beta}.$$

*Proof.* Since  $\nabla a(u) = a'(u)\nabla u$ , the first estimate easily follows from Lemma 2.1 with  $b(x) = a(u(x, t))$ .

For the second estimate, differentiating (2.2) we have

$$(a(u)\nabla\rho_t, \nabla\chi) + (a(u)_t\nabla\rho, \nabla\chi) = 0 \quad \forall \chi \in S_h.$$

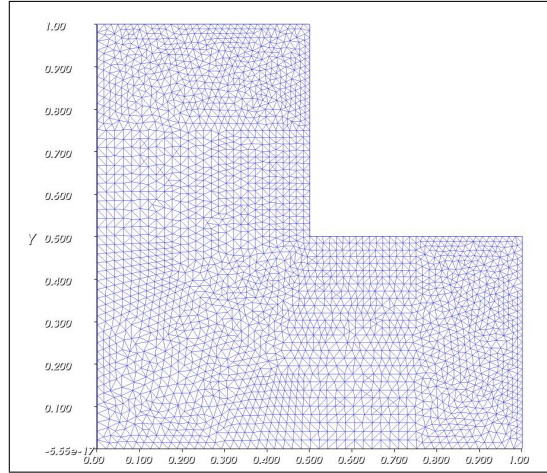


FIGURE 1. Finite element discretizations for the  $L$ -shaped domain,  $\#$  triangles = 5178 and  $\#$  dof = 2702.

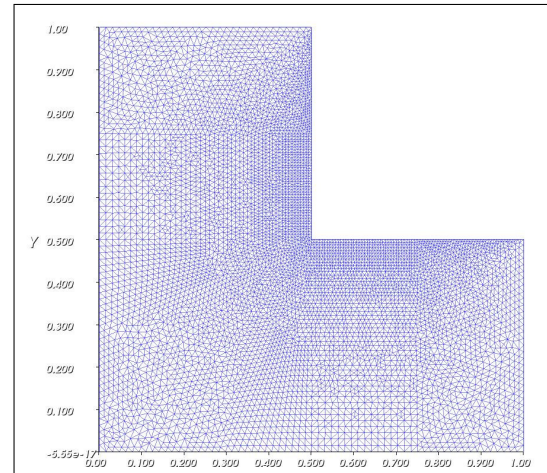


FIGURE 2. Further refinement made towards the nonconvex corner for the  $L$ -shaped domain,  $\#$  triangles = 8646 and  $\#$  dof = 4468.

So, for uniform boundedness of  $a(u)$  and  $a(u)_t$ ,

$$\begin{aligned}
 \mu \|\nabla \rho_t\|^2 &\leq (a(u) \nabla \rho_t, \nabla \rho_t) \\
 &= (a(u) \nabla \rho_t, \nabla(\chi - u_t)) + (a(u) \nabla \rho_t, \nabla(\tilde{u}_{h,t} - \chi)) \\
 &= (a(u) \nabla \rho_t, \nabla(\chi - u_t)) + (a(u)_t \nabla \rho, \nabla(\chi - \tilde{u}_{h,t})) \\
 &\leq C(\|\nabla \rho_t\| \|\nabla(\chi - u_t)\| + \|\nabla \rho\| \|\nabla(\chi - \tilde{u}_{h,t})\|) \\
 &\leq C(\|\nabla \rho_t\| \|\nabla(\chi - u_t)\| + \|\nabla \rho\| (\|\nabla(\chi - u_t)\| + \|\nabla \rho_t\|)),
 \end{aligned}$$

following [4], with  $\chi = I_h u_t$  and using Lemma 2.1, this yields

$$\mu \|\nabla \rho_t\|^2 \leq \frac{\mu}{2} \|\nabla \rho_t\|^2 + C \|\nabla \rho\|^2 + C(u) h^{2\beta},$$

together with the previous estimate of  $\nabla \rho$  already shown, we have  $\|\nabla \rho_t\| \leq C(u) h^\beta$ .

Now for the estimate of  $\rho_t$ , we use the duality argument as in the proof of Lemma 2.1. With  $b = a(u)$  and  $\psi$  defined as in (2.5), we have

$$(3.2) \quad (\rho_t, \varphi) = (a(u) \nabla \rho_t, \nabla \psi) = (a(u) \nabla \rho_t, \nabla(\psi - \chi)) - (a(u)_t \nabla \rho, \nabla \chi).$$

Since  $a(u)$  is bounded by (1.2), hence using (2.2), the second term of the right hand side of (3.2) gives

$$(a(u)_t \nabla \rho, \nabla \chi) = \frac{a(u)_t}{a(u)} (a(u) \nabla \rho, \nabla \chi) = 0,$$

and therefore, we have

$$(\rho_t, \varphi) = (a(u) \nabla \rho_t, \nabla(\psi - \chi)),$$

choosing  $\chi = I_h \psi$ , together with (2.6) and with the estimates for  $\nabla \rho_t$ , we obtain

$$|(\rho_t, \varphi)| \leq C \|\nabla \rho_t\| h^\beta \|\Delta \psi\|_{H^{-1+s}} \leq C(u) h^{2\beta} \|\varphi\|,$$

which gives,  $\|\rho_t\| \leq C(u) h^{2\beta}$ . This completes the proof.  $\square$

We are now ready to prove the following estimate in  $L^2$  for the error between the solutions of the spatially semidiscrete problem (2.1) and the continuous problem (1.1).

**Theorem 3.1.** *Let  $u_h$  and  $u$  be the solutions of (2.1) and (1.1), respectively. Then under the assumption of (1.2), we have*

$$\|u_h(t) - u(t)\| \leq C \|v_h - v\| + C(u) h^{2\beta} \quad \text{for } \beta < s \leq 1, t \in \bar{J}.$$

*Proof.* We first write the error term as in (3.1), and since  $\rho$  is bounded in view of Lemma 3.1, so it remains to estimate  $\theta$ . For  $\chi \in S_h$  and using (2.2) yields

$$\begin{aligned} & (\theta_t, \chi) + (a(u_h) \nabla \theta, \nabla \chi) \\ &= (u_{h,t}, \chi) + (a(u_h) \nabla u_h, \nabla \chi) - (\tilde{u}_{h,t}, \chi) - (a(u_h) \nabla \tilde{u}_h, \nabla \chi) \\ &= (f(u_h), \chi) - (\tilde{u}_{h,t} - u_t, \chi) - (u_t, \chi) - (a(u) \nabla \tilde{u}_h, \nabla \chi) + ((a(u) - a(u_h)) \nabla \tilde{u}_h, \nabla \chi) \\ &= (f(u_h), \chi) - (\rho_t, \chi) - (u_t, \chi) - (a(u) \nabla u, \nabla \chi) + ((a(u) - a(u_h)) \nabla \tilde{u}_h, \nabla \chi), \end{aligned}$$

and thus

$$(3.3) \quad (\theta_t, \chi) + (a(u_h)\nabla\theta, \nabla\chi) = (f(u_h) - f(u), \chi) + ((a(u) - a(u_h))\nabla\tilde{u}_h, \nabla\chi) - (\rho_t, \chi).$$

Now, using (1.2) and (2.2),

$$\mu\|\nabla\tilde{u}_h\|^2 = \mu(\nabla\tilde{u}_h, \nabla\tilde{u}_h) \leq (a(u)\nabla\tilde{u}_h, \nabla\tilde{u}_h) = (a(u)\nabla u, \nabla\tilde{u}_h) \leq M(\nabla u, \nabla\tilde{u}_h),$$

which leads to  $\|\nabla\tilde{u}_h\| \leq (M/\mu)\|\nabla u\| = C\|\nabla u\|$ , and this yields

$$(3.4) \quad \|\nabla\tilde{u}_h(t)\| \leq C(u).$$

Therefore with  $\chi = \theta$  in (3.3), together with using (1.2) and (3.4), we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\theta\|^2 + \mu\|\nabla\theta\|^2 &\leq C\|u_h - u\|(\|\theta\| + \|\nabla\theta\|) + \|\rho_t\|\|\theta\| \\ &\leq \mu\|\nabla\theta\|^2 + C(\|\theta\|^2 + \|\rho\|^2 + \|\rho_t\|^2), \end{aligned}$$

after integration this leads to

$$\|\theta(t)\|^2 \leq \|\theta(0)\|^2 + C \int_0^t (\|\theta\|^2 + \|\rho\|^2 + \|\rho_t\|^2) ds.$$

Now, using Gronwall's lemma we obtain

$$(3.5) \quad \|\theta(t)\|^2 \leq C\|\theta(0)\|^2 + C \int_0^t (\|\rho\|^2 + \|\rho_t\|^2) ds,$$

where  $C$  now depends on  $T$ . We have

$$(3.6) \quad \|\theta(0)\| \leq \|v_h - v\| + \|\tilde{u}_h(0) - v\| \leq \|v_h - v\| + Ch^{2\beta},$$

where  $C = C(v)$ . With this and together with Lemma 3.1 we obtain from (3.5)

$$\|\theta(t)\| \leq C\|v_h - v\| + C(u)h^{2\beta},$$

and this completes the proof of the theorem. □



## 4. FULLY DISCRETE ERROR ANALYSIS

Now we shall turn to the fully discrete scheme for the discretization with respect to time variable of the spatially semidiscrete problem (2.1). We introduce the *backward Euler* method and derive some a priori error estimates. Consider a partitioning of the time interval  $\bar{J} = [0, T]$  as

$$\bar{J} = \{0\} \cup J_1 \cup J_2 \cup \cdots \cup J_N$$

with subintervals  $J_i = (t_{i-1}, t_i]$  of size  $k$  and time points

$$0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T,$$

where  $k$  denote the constant time step. Consider the backward Euler quotient  $\bar{\partial}U^n = (U^n - U^{n-1})/k$ , where  $U^n$  the approximation in  $S_h$  of  $u^n$ , with  $u^n = u(t_n)$ . Then the backward Euler method is given by

$$(4.1) \quad (\bar{\partial}U^n, \chi) + (a(U^n) \nabla U^n, \nabla \chi) = (f(U^n), \chi) \quad \forall \chi \in S_h, \quad 1 \leq n \leq N,$$

with  $U^0 = v_h$ .

The existence and uniqueness of the solution of (4.1) are easily follow from the argument of [4]. Next, we are ready to prove the following estimate in  $L^2$  for the error between the solutions of the fully discrete problem (4.1) and the continuous problem (1.1).

**Theorem 4.1.** *Let  $U^n$  and  $u$  be the solutions of (4.1) and (1.1), respectively. Then under the assumption (1.2) we have, for small  $k$ ,*

$$\|U^n - u(t_n)\| \leq C\|v_h - v\| + C(u)(h^{2\beta} + k),$$

for  $0 \leq n \leq N$ ,  $\beta < s \leq 1$ .

*Proof.* With  $u^n = u(t_n)$  and  $\tilde{U}^n = \tilde{u}_h(t_n)$ , we first split the error term in two parts  $\theta$  and  $\rho$  as follows

$$(4.2) \quad U^n - u^n = (U^n - \tilde{U}^n) + (\tilde{U}^n - u^n) = \theta^n + \rho^n,$$

where  $\tilde{u}_h(t_n)$  be the elliptic projection of  $u^n$  given in (2.2). Since,  $\rho^n$  is bounded in view of Lemma 3.1, we only need to estimate  $\theta^n$ . For  $\chi \in S_h$ , using (2.2) and

the weak form of the continuous problem we have

$$\begin{aligned}
& (\bar{\partial}\theta^n, \chi) + (a(U^n)\nabla\theta^n, \nabla\chi) \\
&= (\bar{\partial}U^n, \chi) - (\bar{\partial}\tilde{U}^n, \chi) + (a(U^n)\nabla U^n, \nabla\chi) - (a(U^n)\nabla\tilde{U}^n, \nabla\chi) \\
&= (f(U^n), \chi) - (u_t^n, \chi) - (\bar{\partial}\tilde{U}^n - u_t^n, \chi) - (a(u^n)\nabla\tilde{U}^n, \nabla\chi) - ((a(U^n) - a(u^n))\nabla\tilde{U}^n, \nabla\chi) \\
&= (f(U^n), \chi) - (u_t^n, \chi) - (\bar{\partial}\tilde{U}^n - u_t^n, \chi) - (a(u^n)\nabla u^n, \nabla\chi) - ((a(U^n) - a(u^n))\nabla\tilde{U}^n, \nabla\chi),
\end{aligned}$$

and hence

$$\begin{aligned}
(\bar{\partial}\theta^n, \chi) + (a(U^n)\nabla\theta^n, \nabla\chi) &= (f(U^n) - f(u^n), \chi) - (\bar{\partial}\rho^n, \chi) - (\bar{\partial}u^n - u_t^n, \chi) \\
&\quad - ((a(U^n) - a(u^n))\nabla\tilde{U}^n, \nabla\chi).
\end{aligned}$$

Choosing  $\chi = \theta^n$ , together with (1.2) and the boundedness of  $\nabla\tilde{U}^n$  in (3.4), we obtain

$$\frac{1}{2}\bar{\partial}\|\theta^n\|^2 + \mu\|\nabla\theta^n\|^2 \leq C\|U^n - u^n\|(\|\theta^n\| + \|\nabla\theta^n\|) + (\|\bar{\partial}\rho^n\| + \|\bar{\partial}u^n - u_t^n\|)\|\theta^n\|,$$

using (4.2) this yields

$$\bar{\partial}\|\theta^n\|^2 + \mu\|\nabla\theta^n\|^2 \leq C(\|\theta^n\|^2 + \|\bar{\partial}\rho^n\|^2 + \|\rho^n\|^2 + \|\bar{\partial}u^n - u_t^n\|^2) = C(\|\theta^n\|^2 + w^n),$$

where  $w^n = \|\bar{\partial}\rho^n\|^2 + \|\rho^n\|^2 + \|\bar{\partial}u^n - u_t^n\|^2$ . This reduce to

$$(1 - Ck)\|\theta^n\|^2 \leq \|\theta^{n-1}\|^2 + Ckw^n,$$

which gives

$$\|\theta^n\|^2 \leq (1 + Ck)\|\theta^{n-1}\|^2 + Ckw^n,$$

for small  $k$ . Then by repeated applications, for  $1 \leq n \leq N$ ,

$$(4.3) \quad \|\theta^n\|^2 \leq (1 + Ck)^n\|\theta^0\|^2 + Ck \sum_{j=1}^n (1 + Ck)^{n-j} w^j \leq C\|\theta^0\|^2 + Ck \sum_{j=1}^n w^j.$$

We need to estimate  $w^j$  which includes three terms. Now from Lemma 3.1, we obtain

$$\|\rho^j\| \leq C(u)h^{2\beta},$$

$$\|\bar{\partial}\rho^j\| = \left\| \frac{\rho^j - \rho^{j-1}}{k} \right\| = \|k^{-1} \int_{t_{j-1}}^{t_j} \rho_t ds\| \leq C(u)h^{2\beta},$$

and

$$\begin{aligned}\|\bar{\partial}u^j - u_t^j\| &= \|k^{-1}(u^j - u^{j-1} - ku_t^j)\| \\ &= \|k^{-1} \int_{t_{j-1}}^{t_j} (s - t_{j-1})u_{tt}(s) ds\| \leq C(u)k.\end{aligned}$$

Altogether these estimates, we have

$$w^j = \|\bar{\partial}\rho^j\|^2 + \|\rho^j\|^2 + \|\bar{\partial}u^j - u_t^j\|^2 \leq C(u)(h^{2\beta} + k)^2,$$

with this and the estimate for  $\theta^0$  in (3.6), we obtain from (4.3)

$$\|\theta^n\| \leq C\|v_h - v\| + C(u)(h^{2\beta} + k).$$

Hence the proof is complete.  $\square$

## 5. CONCLUSIONS

In this article, we have presented the finite element method for nonlinear parabolic problems in nonconvex polygonal domains. A priori error bounds in the  $L^2$ -norm has been derived for both spatially semidiscrete and fully discrete methods. The fully discrete scheme is based on the backward Euler method. The derivation gives the convergence rate of order  $\mathcal{O}(h^{2\beta})$  for  $\beta < s \leq 1$  with respect to the space discretization and  $\mathcal{O}(k)$  with respect to the time discretization.

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DEPARTMENT OF MATHEMATICS  
NATIONAL INSTITUTE OF TECHNOLOGY CALICUT  
KERALA - 673601, INDIA  
Email address: tamal@nitc.ac.in

DEPARTMENT OF MATHEMATICS  
INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI  
ASSAM - 781039, INDIA  
Email address: shantiramamahata1@gmail.com