Advances in Mathematics: Scientific Journal **9** (2020), no.9, 7471–7476
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.9.93

BOUNDS ON LOGARITHMIC MEANS

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ABSTRACT. In this paper we show inequalities between the moments of probability distributions whose probability density function takes non-zero values in a given finite real interval $[a, b]$ can be used to study the inequalities involving geometric mean, logarithmic mean and arithmetic means. Also we obtained the refinement of the logarithmic-arithmetic mean inequality and a better lower bound of logarithmic mean.

1. INTRODUCTION

The logarithmic mean of two positive real numbers a and b is the number $L(a, b)$, defined as

$$L(a,b) = \frac{b-a}{\log b - \log a},\ a \neq b,$$

with the understanding that

$$L(a,a) = \lim_{b \to a} L(a,b) = a.$$

The logarithmic mean always falls between the geometric and arithmetic means; i.e.,

$$G(a,b) \leq L(a,b) \leq A(a,b), \tag{1.1}$$

where

$$G(a,b) = \sqrt{ab}$$

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2010 *Mathematics Subject Classification.* 60E05, 00A05.

Key words and phrases. Moments, Logarithmic mean, Probability distribution.

and

$$A(a,b) = \frac{a+b}{2}.$$

The interestsing in (1.1) is that it provides a refinement of the geometric mean-arithmetic mean inequality [1–3]. When $a = b$, all the three means in (1.1) are equal to a. There have been many inequalities involving power mean and logarithmic and their applications [4, 11].

The rth order moment of a continuous probability distribution with probability density function $\Phi(x)$ is defined as

$$\mu_r^{'} = \int_a^b x^r \Phi(x)\, dx.$$

The rth order moment about mean is defined as

$$\mu_r = \int_a^b \left(x - \mu_1^{'}\right)^r \Phi(x)\, dx.$$

It is clear that $\mu_1 = 0$, $\mu_1^{'}$ is the mean and μ_2 is the variance of the distribution [9, 10].

2. Main Results

Theorem 2.1. *For two positive real numbers a and b*

$$\sqrt{ab} \leq \frac{b-a}{\log b - \log a} \leq \frac{a+b}{2}.$$

Proof. We have from [5–8] that:

$$\mu_1' \geq H,$$

or

$$\int_a^b x\Phi(x)\, dx \geq \left(\int_a^b \frac{1}{x}\Phi(x)\, dx\right)^{-1}. \tag{2.1}$$

Let

$$\Phi(x) = \frac{1}{b-a}. \tag{2.2}$$

Combining (2.1) and (2.2), we get

$$\frac{b-a}{\log b-\log a}\leq\frac{a+b}{2}.$$

Further, we have

$$H\geq\left(\mu'_{-2}\right)^{-\frac{1}{2}}$$

or

$$\left(\int_a^b\frac{1}{x}\Phi\left(x\right)dx\right)^{-1}\geq\left(\int_a^b\frac{1}{x^2}\Phi\left(x\right)dx\right)^{-\frac{1}{2}}. \tag{2.3}$$

Combining (2.2) and (2.3), we get that

$$\frac{b-a}{\log b-\log a}\geq\sqrt{ab}. \tag{2.4}$$

□

Theorem 2.2. *A refinement of the logarithmic-arithmetic mean inequality is*

$$L(a,b)\leq A(a,b)-\frac{A^2(a,b)-G^2(a,b)}{3A(a,b)}, \tag{2.5}$$

or equivalently

$$L(a,b)\leq\frac{a+b}{2}-\frac{\left(b-a\right)^2}{6\left(a+b\right)}.$$

Proof. We have from [12]

$$\mu_4^{'}\geq\frac{\left(\mu_3^{'}-\mu_1^{'}\mu_2^{'}\right)^2}{\mu_2^{'}-\mu_1^{'2}}+\mu_2^{'2}. \tag{2.6}$$

Let

$$\Phi\left(x\right)=\frac{4a^4b^4}{b^4-a^4}\frac{1}{x^5}.$$

Then,

$$\mu_1^{'}=\frac{4}{3}\frac{ab\left(a^2+ab+b^2\right)}{\left(a+b\right)\left(a^2+b^2\right)}, \tag{2.7}$$

$$\mu_2^{'}=\frac{2a^2b^2}{a^2+b^2}, \tag{2.8}$$

$$\mu_3^{'}=\frac{4a^3b^3}{\left(a+b\right)\left(a^2+b^2\right)} \tag{2.9}$$

and

$$\mu_4^{'} = \frac{4a^4b^4\left(\log b - \log a\right)}{b^4 - a^4}. \tag{2.10}$$

On substituting values of $\mu_1^{'}, \mu_2^{'}, \mu_3^{'}$ and $\mu_4^{'}$ respectively from (2.7), (2.8), (2.9) and (2.10) in (2.6), we get that

$$\frac{b-a}{\log b - \log a} \leq \frac{a^2 + 4ab + b^2}{3\left(a+b\right)}. \tag{2.11}$$

The inequality (2.5) now follows easily from (2.11).Alternatively, It may be noted here that the inequality:

$$\mu_5^{'} \geq \frac{\mu_3^{'3} - 2\mu_2^{'}\mu_3^{'}\mu_4^{'} + \mu_1^{'}\mu_4^{'2}}{\mu_1^{'}\mu_3^{'} - \mu_2^{'2}}, \tag{2.12}$$

also yields the inequality (2.5) for

$$\Phi\left(x\right) = \frac{5a^5b^5}{b^5 - a^5}\frac{1}{x^6}.$$

In this case, we have

$$\mu_1^{'} = \frac{5}{4}\frac{ab\left(b^4 - a^4\right)}{b^5 - a^5},$$

$$\mu_2^{'} = \frac{5}{3}\frac{a^2b^2\left(b^3 - a^3\right)}{b^5 - a^5}\ ,$$

$$\mu_3^{'} = \frac{5}{2}\frac{a^3b^3\left(b^2 - a^2\right)}{b^5 - a^5},$$

$$\mu_4^{'} = 5\frac{a^4b^4\left(b - a\right)}{b^5 - a^5}$$

and

$$\mu_5^{'} = 5\frac{a^5b^5\left(\log b - \log a\right)}{b^5 - a^5}.$$

□

Theorem 2.3. *We have*

$$L(a,b) \geq \frac{3G^2(a,b)A(a,b)}{A^2(a,b) + 2G^2(a,b)}. \tag{2.13}$$

The inequality (2.13) *provides a refinement of* (2.4) *for* $b \leq \left(7 + 4\sqrt{3}\right)a$.

Proof. We have:

$$\mu_4^{'} \leq (a+b)\,\mu_3^{'} - ab\mu_2^{'} + \frac{\left[\mu_3^{'} - (a+b)\,\mu_2^{'} + ab\mu_1^{'}\right]^2}{\mu_2^{'} - (a+b)\,\mu_1^{'} + ab}. \tag{2.14}$$

On substituting values of $\mu_1^{'}, \mu_2^{'}, \mu_3^{'}$ and $\mu_4^{'}$ respectively from (2.7), (2.8), (2.9) and (2.10) in (2.14), we get

$$\frac{b-a}{\log b - \log a} \geq \frac{6ab\,(a+b)}{a^2 + 10ab + b^2}. \tag{2.15}$$

The inequality (2.13) now follows easily from (2.15). We also note that

$$\frac{6ab\,(a+b)}{a^2 + 10ab + b^2} \geq \sqrt{ab}$$

if and only if

$$(b-a)^2\left(b^2 - 14ab + a^2\right) \leq 0.$$

This gives $b \leq \left(7 + 4\sqrt{3}\right) a$. □

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