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INTEGER CORDIAL LABELING FOR CERTAIN FAMILIES OF GRAPHS

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ABSTRACT. An integer cordial labeling of a graph $G(p,q)$ is an injective map $f : V \rightarrow [-\frac{p}{2}...\frac{p}{2}]^*$ or $[-\lfloor\frac{p}{2}\rfloor...\lfloor\frac{p}{2}\rfloor]$ as p is even or odd, which induces an edge labeling $f^* : E \rightarrow \{0,1\}$ defined by $f^*(uv) = \begin{cases} 1, f(u)+f(v) \geq 0 \\ 0, \text{ otherwise} \end{cases}$ such that the number of edges labeled with 1 and the number of edges labeled with 0 differ at most by 1. If a graph has integer cordial labeling, then it is called an integer cordial graph. In this paper, we have proved that the Sierpinski Sieve graph, the graph obtained by joining two friendship graphs by a path of arbitrary length, (n,k)- kite graph and Prism graph admits integer cordial labeling.

1. INTRODUCTION

The assignment of labels, such as integers or symbols to edges and/or vertices of a graph under certain conditions is called graph labeling. It has wide applications in mathematics as well as in other fields such as coding theory, X-ray crystallography and communications network addressing. For a dynamic survey on various graph labeling problems we refer to Gallian [3]. Let $G = (V,E)$ be a graph. A function $f : V(G) \rightarrow \{0,1\}$ is called the binary vertex labeling of G if for an edge $e = uv$, the induced function $f^* : E(G) \rightarrow \{0,1\}$ is given by

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$f^*(e) = |f(u) - f(v)|$. Here $f(v)$ is called the label of the vertex v under f and $v_f(0), v_f(1)$ are the number of vertices of G having labels 0 and 1 respectively under f and $e_f(0), e_f(1)$ are the number of edges of G having labels 0 and 1 respectively under f^*.

2. Preliminaries

Definition 2.1. *A graph G is said to have a cordial labeling if there exists a binary vertex labeling for G in which the difference of the number of vertices with label 1 and label 0 should differ by at most 1 and also the difference between the number of edges with label 1 and label 0 should differ by at most 1. A graph G is called cordial if it admits cordial labeling [2].*

I.Cahit [1] introduced the concept of cordial labeling as a weaker version of graceful and harmonious labeling.

Definition 2.2. *A graph $G(p,q)$ is said to have an integer cordial labeling if there exists an injective map $f : V \to [-\frac{p}{2}...\frac{p}{2}]^*$ or $[-\lfloor\frac{p}{2}\rfloor...\lfloor\frac{p}{2}\rfloor]$ as p is even or odd, which induces an edge labeling, $f^* : E \to \{0,1\}$ defined by $f^*(uv) = \begin{cases} 1, f(u)+f(v) \geq 0 \\ 0, \text{ otherwise} \end{cases}$ such that the number of edges labeled with 1 and the number of edges labeled with 0 differ by atmost 1. If a graph G admits integer cordial labeling, then the graph is called an integer cordial graph [4].*

In general, $[-x,\dots,x] = \{y\backslash y$ is an integer and$|y| \leq x\}$ and $[-x,x]^* = \{y\backslash y$ is an integer and $|y| \leq x - \{0\}\}$.

In [4], Nicholas et al. introduced the concept of integer cordial labeling of graphs and proved that some standard graphs such as Path P_n, Star graph $K_{1,n}$, Wheel graph W_n; $n > 3$, Cycle C_n, Helm graph H_n, Closed helm graph CH_n are integer cordial. K_n is not integer cordial, however $K_{n,n}$ is integer cordial iff n is even and $K_{n,n}\backslash M$ is integer cordial for any n, where M is perfect matching of $K_{n,n}$.

Definition 2.3. *The friendship graph F_n is formed by joining n copies of the cycle C_3 with a common vertex [1]. See figure 2.*

Definition 2.4. *The (n,k)- kite graph is obtained by joining a cycle C_n and a path P_k, [1]. See Figure 3.*

3. MAIN RESULTS

Theorem 3.1. *The Sierpinski Sieve graph S_n is integer cordial.*

Proof. The proof is obvious when $n = 1$. We know that S_n is symmetric about the central line. Label all the vertices on the left half of S_n by $v_1, v_2, .., v_m$ and the right half of S_n by $v_{m+1}, v_{m+2}, ..., v_{2m}$ except the exactly middle vertices. Label the exactly middle vertices of S_n as $w_1, w_1^{'}, w_2, w_2^{'}, ..., w_k, w_k^{'}$ (label it from top to bottom in a successive manner). Name the antipodal vertex of $u^{'}$ as x.

Case 1: n- even.

Let $n = 2$. Let u be the top most vertex and let $u^{'}$ be the exactly opposite lower most vertex. Let v_1, v_2, v_3 and v_4 be the vertices on the left half side and right half side of S_2 respectively. We define $f : V \to [-\frac{p}{2}...\frac{p}{2}]^*$ as follows: $f(u) = 1; f(u^{'}) = -1; f(v_1) = 2; f(v_2) = 3; f(v_3) = -2$ and $f(v_4) = -3$. Then five edges will receive the label 1 and four edges will receive the label 0. Hence $|e_f(1) - e_f(0)| = 1$. Suppose $n \geq 4$. We define $f : V \to [\lfloor\frac{-p}{2}\rfloor...\lfloor\frac{p}{2}\rfloor]^*$ as follows: $f(u) = 1; f(u^{'}) = -1; f(v_i) = (i + 1), 1 \leq i \leq m; f(v_{m+i}) = -(i + 1), 1 \leq i \leq m; f(w_i) = m + 1 + i, 1 \leq i \leq k; f(w_i^{'}) = -(m + 1 + i), 1 \leq i \leq k$. Then there are $\lfloor\frac{3^n}{2}\rfloor + 1$ edges which receive the value 1 and $\lfloor\frac{3^n}{2}\rfloor$ edges which receive the value 0. Hence $|e_f(1) - e_f(0)| = 1$.

Case 2: n- odd.

We define $f : V \to [-\lfloor\frac{p}{2}\rfloor...\lfloor\frac{p}{2}\rfloor]$ as follows: $f(u) = 1$; $f(u^{'}) = -1; f(x) = 0; f(v_i) = (i+1), 1 \leq i \leq m; f(v_{m+i}) = -(i+1), 1 \leq i \leq m; f(w_i) = m+1+i, 1 \leq i \leq k; f(w_i^{'}) = -(m+1+i), 1 \leq i \leq k$. Then there are $\lfloor\frac{3^n}{2}\rfloor+1$ edges which receive the value 1 and $\lfloor\frac{3^n}{2}\rfloor$ edges which receive the value 0. Hence $|e_f(1) - e_f(0)| = 1$. Therefore, the Sierpinski Sieve graph S_n is integer cordial. See Figure 1. □

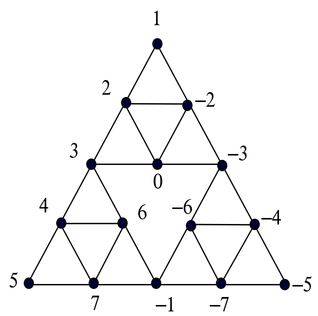

FIGURE 1. Integer cordial labeling of S_3

Theorem 3.2. *The graph G obtained by joining two friendship graphs F_n and F_m by an arbitrary path is integer cordial.*

Proof. Let u and $u^{'}$ be the centre vertex of F_n and F_m respectively. Label the remaining vertices of F_n as $v_1, v_2, ..., v_{2n}$ in the anticlockwise direction and the remaining vertices of F_m as $v_1^{'}, v_2^{'}, ..., v_{2m}^{'}$ in the clockwise direction. Let $u_1, u_2, ..., u_t$ be the vertices of the path P_t of length $t-1$ with $u_1 = v_1$ and $u_t = v_1^{'}$. Let w be the centre vertex of P_t (if it exists).

Case 1: Let F_n and F_m be joined by a path of length one.
We define $f : V \to [-\frac{p}{2}...\frac{p}{2}]^*$ as follows: $f(u) = 1; f(u^{'}) = -1; f(v_i) = i+1, 1 \leq i \leq n; f(v_{n+i}) = -(i+1), 1 \leq i \leq n; f(v_j^{'}) = n+1+j, 1 \leq j \leq m; f(v_{m+j}^{'}) = -(n+1+j), 1 \leq j \leq m$. Since n vertices of F_n and m vertices of F_m are labeled with positive integers and the remaining with negative integers, we see that $\frac{3n}{2} + \frac{3m}{2}$ edges receives the value 1 and $\frac{3n}{2} + \frac{3m}{2}$ edges receives the value 0. Also the two friendship graphs F_n and F_m are connected by a path of length one, which implies that the edge receives the label either 1 or 0. Therefore in all possible cases we have $|e_f(1) - e_f(0)| \leq 1$.

Case 2: Let F_n and F_m be joined by a path of odd length. We define $f : V \to [-\frac{p}{2}...\frac{p}{2}]^*$ as follows: $f(u) = 1; f(u^{'}) = -1; f(v_i) = i+1, 1 \leq i \leq n; f(v_{n+i}) = -(i+1), 1 \leq i \leq n; f(v_j^{'}) = n+1+j, 1 \leq j \leq m; f(v_{m+j}^{'}) = -(n+1+j), 1 \leq j \leq m; f(u_k) = n+m+k, 1 < k \leq \frac{t}{2}; f(u_{\frac{t}{2}+k}) = -(n+m+k), 1 \leq k < t$. For the friendship graphs F_n and F_m, the discussion remains the same as in Case 1. Hence it is sufficient to discuss the case when the path is of odd length. Here, $\frac{t-2}{2}$ vertices of P_t are labeled with positive integers and the remaining vertices are labeled with negative integers. Thus $\lfloor \frac{t-1}{2} \rfloor + 1$ edges receive the value 1 and $\lfloor \frac{t-1}{2} \rfloor$ edges receive the value 0. Therefore, in all cases we have $|e_f(1) - e_f(0)| = 1$.

Case 3: Let F_n and F_m be joined by a path of even length. We define $f : V \to [-\lfloor \frac{p}{2} \rfloor ... \lfloor \frac{p}{2} \rfloor]$ as follows: $f(u) = 1; f(u^{'}) = -1; f(v_i) = i+1, 1 \leq i \leq n; f(v_{n+i}) = -(i+1), 1 \leq i \leq n; f(v_j^{'}) = n+1+j, 1 \leq j \leq m; f(v_{m+j}^{'}) = -(n+1+j), 1 \leq j \leq m; f(u_k) = n+m+k, 1 < k \leq \frac{t}{2}; f(u_{\frac{t}{2}+k+1}) = -(n+m+k), 1 \leq k < t; f(w) = 0$. For the friendship graphs F_n and F_m, the discussion remains the same as in Case 1. Hence it is sufficient to discuss the case when the path is of even length. Here, the centre vertex of path P_t is labeled with 0, $\frac{t-2}{2}$ vertices are labeled with positive integers and the remaining vertices are labeled with negative integers. We observe that $\frac{t-1}{2}$ edges receive the value 1 and $\frac{t-1}{2}$ edges receive the value 0.

Hence we have, $|e_f(1) - e_f(0)| = 0$. The result holds true when $n = m$. Therefore the graph G obtained by joining two friendship graphs F_n and F_m by an arbitrary path is integer cordial. □

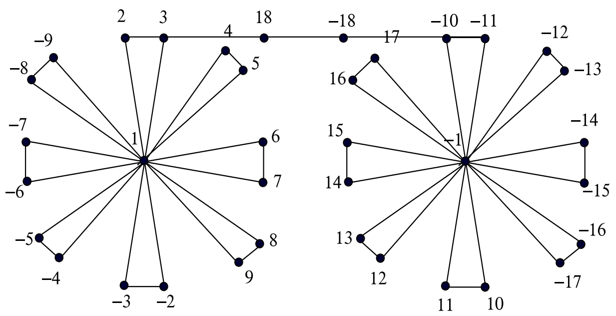

FIGURE 2. Integer cordial labeling of two F_8 joined by a path of length three

Theorem 3.3. *The* (n, k)*- kite graph is integer cordial.*

Proof. **Case 1:** Suppose $n = k$
Let v_1 be the vertex which joins the cycle C_n and the path P_n. The remaining vertices of the C_n are labeled as $v_2, v_3, ...v_n$ either in clockwise or anticlockwise direction. Let $w_1, w_2, ..., w_n$ be the vertices of the path P_n. We define $f : V \to [-\frac{p}{2}...\frac{p}{2}]^*$ as follows: $f(v_i) = i, 1 \leq i \leq n; f(w_i) = -(n+1-i), 1 \leq i \leq n$. Here we notice that n edges of C_n will receive the value 1 and n edges of P_n receive the value 0. Therefore, $|e_f(1) - e_f(0)| = 0$.
Case 2: Suppose $n \neq k$ and $n < k$.
Let u be the centre vertex if it exists. Label the remaining vertices as $v_1, v_2, ..., v_{2m}$ in a successive manner starting from the vertex of the cycle followed by the vertex of the path.
Case 2.1: Suppose the number of vertices be even.
We define $f : V \to [-\frac{p}{2}...\frac{p}{2}]^*$ as follows: $f(v_i) = m + 1 - i, 1 \leq i \leq m; f(v_{m+i}) = -(m+1-i), 1 \leq i \leq m$. We notice that m edges receive the value 1 and m edges receive the value 0. Therefore, $|e_f(1) - e_f(0)| = 0$.
Case 2.2: Suppose the number of vertices be odd.
We define $f : V \to [-\lfloor\frac{p}{2}\rfloor...\lfloor\frac{p}{2}\rfloor]$ as follows: $f(u) = 0; f(v_i) = m + 1 - i, 1 \leq i \leq m; f(v_{m+i}) = -(m+1-i), 1 \leq i \leq m$. Here, $m+1$ edges receive the value 1 and m edges receive the value 0. Therefore, $|e_f(1) - e_f(0)| = 1$.

Case 3: Suppose $n \neq k$ and $n > k$.
Case 3.1: Suppose the number of vertices be even.
We define $f : V \to [-\frac{p}{2}...\frac{p}{2}]^*$ as follows: $f(v_i) = i, 1 \leq i \leq m; f(v_{m+i}) = -(m+1-i), 1 \leq i \leq m$. We notice that m edges receive the value 1 and m edges receive the value 0. Therefore, $|e_f(1) - e_f(0)| = 0$.
Case 3.2: Suppose the number of vertices be odd.
We define $f : V \to [-\lfloor \frac{p}{2} \rfloor ... \lfloor \frac{p}{2} \rfloor]$ as follows: $f(u) = 0; f(v_i) = i, 1 \leq i \leq m; f(v_{m+i}) = -(m+1-i), 1 \leq i \leq m$. Here, $m+1$ edges receive the value 1 and m edges receive the value 0. Hence, $|e_f(1) - e_f(0)| = 1$.
From all the above possible cases, we conclude that $|e_f(1) - e_f(0)| \leq 1$. Therefore, (n, k) -kite graph admits integer cordial labeling. □

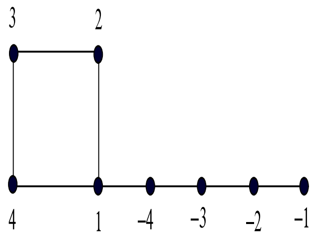

FIGURE 3. Integer cordial labeling of $(4, 4)$- kite graph

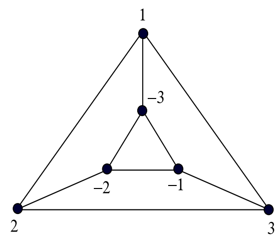

FIGURE 4. Integer cordial labeling of Y_3

Theorem 3.4. *The Prism graph $Y_n, n > 2$ is integer cordial.*

Proof. Label the vertices of the outer cycle as $v_1, v_2, ..., v_n$ and let the vertices of inner cycle be $u_1, u_2, ..., u_n$. We define $f : V \to [-\frac{p}{2}...\frac{p}{2}]^*$ as follows: $f(v_i) = i, 1 \leq i \leq n; f(u_i) = -(n+1-i), 1 \leq i \leq n$.
Case 1: Suppose n is odd.
Here, n edges of the outer cycle receive the label 1 and n edges of the inner cycle receive the label 0. Also, $\lfloor \frac{n}{2} \rfloor + 1$ edges which connect the inner cycle and outer cycle receive the value 1 and $\lfloor \frac{n}{2} \rfloor$ edges which connect the inner cycle and outer cycle receive the value 0. So, here we notice that $n + \lfloor \frac{n}{2} \rfloor + 1$ edges will receive the value 1 and $n + \lfloor \frac{n}{2} \rfloor$ edges receive the value 0. Therefore, in all possible cases, $|e_f(1) - e_f(0)| = 1$. See Figure 4.
Case 2: Suppose n is even.
Here, n edges of the outer cycle receive the value 1 and n edges of the inner cycle

receive the value 0. Also $\frac{n}{2}$ edge which connects the inner cycle and outer cycle receives the value 1 and $\frac{n}{2}$ edges which connects the inner cycle and outer cycle receives the value 0. So, here we notice that $\frac{3n}{2}$ edges will receive the value 1 and $\frac{3n}{2}$ edges receive the value 0. Therefore, in all possible cases, $|e_f(1) - e_f(0)| = 0$. Hence the Prism graph is integer cordial. □

4. Conclusion

In this paper, we have found that Sierpinski Sieve graph, the graph obtained by joining two friendship graphs by a path of arbitrary length, (n, k) -kite graph and the Prism graph are integer cordial. To investigate similar results for interconnection network is an open area of research.

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