

A NOTE ON PRIME GRAPH OF A SEMIRING

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ABSTRACT. Let S be a semiring (not necessarily commutative). In this paper, we define the prime graph $PG(S)$ of S and study some fundamental properties of this graph. Also we study the interplay between some graph theoretic properties of $PG(S)$ and algebraic properties of the semiring S . We prove that S is a prime semiring if and only if the graph $PG(S)$ is a tree.

1. INTRODUCTION

Semiring theory, which is a generalization of ring theory and the theory of distributive lattices has become an interesting branch for study in last few years. The theory of semiring is applied in various areas of science like combinatorics and graph theory, Euclidean geometry and topology, functional analysis, automata and formal language, mathematical modelling of quantum physics, probability theory etc, which is why it has achieved an importance in recent development of theory. As semirings are generalizations of rings, it is very natural to generalize different concepts of rings to semirings and many classical notions of ring theory have been generalized to semiring.

In recent years, the investigation of graphs related to various algebraic structures like rings, semirings, modules etc. has emerged as an interesting area of research. In 1988, I. Beck first introduced the notion of zero divisor graph of a

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commutative ring in [10]. Later on, in 1999 Anderson et al. [4] slightly modified this concept and defined the widely used definition of zero divisor graph of a commutative ring and the various properties of this graph is extensively studied by many authors in the following years. Many researchers got motivated by the notion of zero divisor graph of a commutative ring and they associated graph with ring in various ways [1–3, 12] different from the one in [4].

The study of zero-divisor graph and some of the other variants of graphs related to ring has been extended to semiring and this area of research has also grown rapidly in recent years. S.E. Atani [5, 6] has studied the zero divisor graph of a commutative semiring and some other graphs related to semiring have been investigated in [8, 9]. In the literature, we get many papers on assigning a graph to different algebraic structures, for example see [7, 14, 15].

In 2010, Bhavanari et al. [11] introduced another interesting way to relate a graph to a ring. They defined the prime graph of a ring R . It is defined as the undirected graph $PG(R)$ with all elements of R as vertices and any two distinct vertices x and y are adjacent if and only if either $xRy = 0$ or $yRx = 0$. In this paper, we generalize the notion of prime graph of a ring to semiring. For a semiring S , we define the prime graph of S as the undirected graph $PG(S)$ with all elements of S as vertices and any two distinct vertices x and y are adjacent if and only if either $xSy = 0$ or $ySx = 0$. We investigate various properties of this graph and also study some interrelation between the graph theoretic properties of $PG(S)$ and algebraic properties of S .

2. PRELIMINARIES

In this section, we provide some definitions and notations that will be used throughout the paper. For the basics of semiring and graph theory we follow the books by Golan [13] and Harary [16] respectively.

Let G be a simple graph. A walk of G is an alternating sequence of vertices and lines $v_0, e_1, v_1, \dots, v_{n-1}, e_n, v_n$, beginning and ending with vertices, in which each line is incident with the two vertices immediately preceding and following it. It is said to be closed if $v_0 = v_n$. A closed walk is said to be a cycle provided all of its n vertices are distinct and $n \geq 3$. A cycle of length 3 is called as a triangle. The graph G is called connected if every pair of distinct vertices are joined by a path. G is said to be disconnected if it is not connected. The graph

G is complete if every pair of distinct vertices are adjacent. The complete graph with n vertices is denoted by K_n . If the vertex set V of the graph G can be partitioned into two subsets V_1 and V_2 in such a way that every line of G joins V_1 with V_2 , then G is called a bipartite graph. A bipartite graph is said to be complete if its vertex set V can be partitioned into two subsets V_1 and V_2 such that each vertex of V_1 is adjacent to each vertex of V_2 . $K_{m,n}$ is the notation used to denote a complete bipartite graph, where m and n are the number of vertices in V_1 and V_2 respectively. A star graph is a complete bipartite graph $K_{1,n}$. A star graph with n vertices is called a n -star graph. The graph G is said to be acyclic if it has no cycles. If G is a connected acyclic graph then it is called a tree. The distance $d(x, y)$ between two distinct vertices x and y of G is the length of a shortest path joining x and y , if any. Otherwise $d(x, y) = \infty$. The diameter $diam(G)$ of G is $\sup\{d(x, y) \mid x \text{ and } y \text{ are vertices of } G\}$. Girth of the graph G denoted by $gr(G)$ is the length of a shortest cycle of G , if G contains a cycle; otherwise $gr(G) = \infty$.

Let G be a graph. If there exists a walk in G that traverses each line exactly once, goes through all the vertices, and ends at the starting vertex, then G is called Eulerian. In G a subset S of the vertex set of G is said to be a dominating set for G if every vertex not in S is adjacent to atleast one member of S . The domination number is defined as $\min\{|S| : S \text{ is a dominating set in } G\}$.

A non-empty set S is called a semiring if it is equipped with two binary operations $+$ and $\cdot$, called addition and multiplication respectively such that $(S, +)$ is a commutative monoid and $(S, \cdot)$ is a monoid with respective identity elements 0 and 1; moreover multiplication distributes over addition from either side and 0 is multiplicatively absorbing. If S commutes with respect to multiplication then it is said to be a commutative semiring.

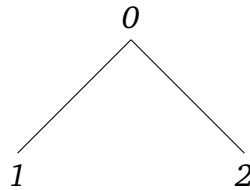
A left ideal of a semiring S is a non-empty subset I of S such that for $a, b \in I, s \in S, a + b \in I$ and $sa \in I$; also $I \neq S$. A right ideal is defined in a similar manner. A non-empty subset of a semiring S is said to be an ideal if it is both left and right ideal. The ideals of a semiring are proper, namely S is not an ideal of itself. An ideal I of a semiring S is called a subtractive ideal or a k -ideal if and only if $x \in I$ and $x + y \in I$ imply $y \in I$. An ideal P of a semiring S is called prime if and only if whenever $AB \subseteq P$, for any ideals A, B of S , then either $A \subseteq P$ or $B \subseteq P$. The semiring S is said to be a prime semiring if 0 is a prime ideal.

3. ON THE PRIME GRAPH OF A SEMIRING

Throughout this section S will represent a semiring (not necessarily commutative), unless otherwise mentioned. Here, we study some fundamental properties like connectedness, diameter, girth of the prime graph $PG(S)$ of S .

Definition 3.1. *The prime graph of a semiring S is defined as the undirected graph $PG(S)$ with all elements of S as vertices and any two distinct vertices x and y are adjacent if and only if either $xSy = 0$ or $ySx = 0$.*

Example 1. *Let us consider the semiring $B(n, i) = \{0, 1, 2, \dots, (n-1)\}$ where $2 \leq n$ is an integer and $0 \leq i < n$. The operations $\oplus$ and $\odot$ on $B(n, i)$ are defined as follows: for $x, y \in B(n, i)$, $x \oplus y = x + y$ if $x + y \leq n - 1$ and, otherwise, $x \oplus y = l$ where $l \equiv (x + y) \pmod{(n - i)}$ and $i \leq l \leq n - 1$. In a similar manner the operation $\odot$ is defined on $B(n, i)$. If we take $n = 3$ and $i = 1$ then $S = B(3, 1)$ is a semiring with the operations as defined above. The graph $PG(S)$ of S is as given in Figure 1.*

FIGURE 1. $PG(S)$

Theorem 3.1. *The graph $PG(S)$ is connected and $\text{diam}(PG(S)) \leq 2$.*

Proof. We have $0Sx = 0$ for all non-zero $x \in S$ and so in the graph $PG(S)$ there is an edge from the vertex 0 to all the other vertices of the graph. Now, if we consider any two non-zero elements x, y of S , each of x and y will be connected to the vertex 0 and hence x and y are also connected. Thus the graph $PG(S)$ is connected.

Moreover, we get that the degree of the 0 vertex is $|S| - 1$ and also $d(0, x) = 1$ and $d(x, y) \leq 2$ for any two non-zero elements x, y in S . This implies that $\text{diam}(PG(S)) \leq 2$. $\square$

Theorem 3.2. *$gr(PG(S)) = 3$, if $PG(S)$ contains a cycle.*

Proof. Suppose the graph $PG(S)$ contains a cycle. We already have any non zero vertex of the graph is adjacent to the vertex 0. Now if $PG(S)$ contains a cycle then we must have two non-zero vertices x, y which are adjacent. But the vertices x and y are adjacent to the 0 vertex. Thus we get a 3-cycle and hence $gr(PG(S)) = 3$. $\square$

Theorem 3.3. *For any two vertex x, y in S , $xSy = 0$ or $ySx = 0$ if and only if $d(x, y) = 1$ or $x = 0$ or $y = 0$.*

Proof. First we suppose that $xSy = 0$ and $x \neq 0 \neq y$. Then x and y are adjacent in $PG(S)$ and so $d(x, y) = 1$.

For the converse part, we suppose $d(x, y) = 1$ or $x = 0$ or $y = 0$. If $x = 0$ or $y = 0$, then clearly $xSy = 0$ (or $ySx = 0$). If $d(x, y) = 1$, and $x \neq 0 \neq y$, then x and y are adjacent in $PG(S)$ which implies $xSy = 0$ or $ySx = 0$. $\square$

Theorem 3.4. *Let $PG(S)$ be the prime graph of S . Then $xSy \neq 0$ if and only if $d(x, y) = 2$.*

Proof. First we suppose that for x, y in S , $xSy \neq 0$. Then x and y are not adjacent in the graph $PG(S)$ and so $d(x, y) > 1$. We know that every non-zero vertex of a prime graph is adjacent to the vertex 0 and so both x and y are adjacent to 0 and hence $d(x, y) = 2$.

Conversely, we suppose that $d(x, y) = 2$. Then clearly there is no edge between x and y in $PG(S)$ and hence $xSy \neq 0$. $\square$

Theorem 3.5. *If S is a commutative semiring with non-zero identity 1, then there exists an edge between any two vertices x and y in $PG(S)$ if and only if $xy = 0$.*

Proof. First we suppose that there exists an edge between any two vertices x and y in $PG(S)$. Then by the definition of $PG(S)$ we have $xSy = 0$. Since S is commutative, $xyS = 0$ and as $1 \in S$, $xy = 0$.

Conversely, suppose $xy = 0$. This implies $xyS = 0$ and as S is commutative, we have $xSy = 0$. Thus there exists an edge between the two vertices x and y in $PG(S)$. $\square$

Some Observations:

Let S be a semiring and $PG(S)$ be its prime graph.

- (i) The graph $PG(S)$ is a simple graph, i.e. it does not contain any self loops or multiple edges.

- (ii) If u, v are two non-zero elements in S such that $uSv = 0$, then the subgraph generated by $\{0, u, v\}$ is a triangle graph.
- (iii) For the graph $PG(S)$ the set $\{0\}$ is a dominating set. This implies that the domination number of $PG(S)$ is 1.
- (iv) Since 0 is adjacent to all the other vertices of $PG(S)$, so n -star graph is a subgraph of $PG(S)$.

4. PRIME SEMIRINGS AND PRIME GRAPHS

Theorem 4.1. *For a semiring S , following conditions are equivalent:*

- (i) S is a prime semiring.
- (ii) $PG(S)$ is a star graph.
- (iii) $PG(S)$ is a tree.

Proof. (i) $\Rightarrow$ (ii)

Let x and y be two elements of S such that they are adjacent in $PG(S)$. Then by the definition of the graph $PG(S)$, $x \neq y$ and either $xSy = 0$ or $ySx = 0$. Which implies that either $x = 0$ or $y = 0$, as S is a prime semiring. This shows that 0 is an endpoint of every edge of the graph $PG(S)$ and hence $PG(S)$ is a star graph.

(ii) $\Rightarrow$ (iii)

Follows obviously as every star graph is a tree.

(iii) $\Rightarrow$ (i)

Given that the graph $PG(S)$ is a tree. To show that S is a prime semiring. If possible we suppose that S is not a prime semiring. Then we have $xSy = 0$, for two non-zero elements x, y in S . This implies that x and y are adjacent in $PG(S)$. Also we know that any non-zero element of S is adjacent to 0 in $PG(S)$ and so x and y are adjacent to 0. Thus it follows that $\{0, x, y\}$ forms a cycle in $PG(S)$. This contradicts the fact that $PG(S)$ is a tree and hence we get that S is a prime semiring. $\square$

Corollary 4.1. *Let S be a semiring with $|S| \geq 2$. Then S is prime if and only if the diameter of the graph $PG(S)$ is 2.*

Corollary 4.2. *Let S be a semiring. Then the following conditions are equivalent:*

- (i) S is not prime.
- (ii) $PG(S)$ is not a tree.

- (iii) $PG(S)$ is not a star graph.
- (iv) triangle is a subgraph of $PG(S)$.
- (v) there exists a chain of length greater than 2 in $PG(S)$.

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) follows from Theorem 4.1.

(iii) $\Rightarrow$ (iv)

Suppose that $PG(S)$ is not a star graph. In the graph $PG(S)$, the vertex 0 is adjacent to all the other vertices and so $PG(S)$ has a n -star graph as subgraph, where $n = |S|$. As $PG(S)$ is not a star graph there is an edge in the graph $PG(S)$ which is not in the n -star subgraph of $PG(S)$. Let x and y be the two non-zero vertices which forms this edge, i.e. they are adjacent in $PG(S)$ and so $\{0, x, y\}$ forms a triangle. Hence triangle is a subgraph of $PG(S)$.

(iv) $\Rightarrow$ (v)

Suppose triangle is a subgraph of $PG(S)$. Let $\{0, x, y\}$ forms a triangle in the graph $PG(S)$. Then clearly the edges connecting the vertices 0 and x , x and y , y and 0 forms a chain of length 3 in $PG(S)$.

(v) $\Rightarrow$ (ii)

Suppose there exists a chain of length greater than 2 in $PG(S)$. Let $p - q - r - s$ be a chain of length 3 in $PG(S)$. Then clearly $p \neq q$, $q \neq r$, $r \neq s$. Also $p \neq r$ and $q \neq s$, otherwise the chain will not be of length 3.

To show that $PG(S)$ is not tree, we will show that $PG(S)$ contains a triangle.

Case I: If all of p, q, r, s are non-zero, then clearly $\{p, q, 0\}$ forms a triangle.

Case II: If $p = 0$. Then $q \neq 0$ as $r \neq 0$ and so $\{p, q, r\}$ forms a triangle.

Case III: If $q = 0$. Then $r \neq 0$ as $s \neq 0$ and so $\{q, r, s\}$ forms a triangle.

Case IV: If $r = 0$. Then $p \neq 0$ as $q \neq 0$ and so $\{r, p, q\}$ forms a triangle.

Case V: If $s = 0$. Then $q \neq 0$ as $r \neq 0$ and so $\{s, q, r\}$ forms a triangle.

Thus $PG(S)$ is not a tree. □

Theorem 4.2. For a prime semiring S , $PG(S)$ is not an Eulerian graph.

Proof. Let S be a prime semiring. Then $PG(S)$ is a star graph centred at 0, by Theorem 4.1. So the degree of any non-zero vertex x of $PG(S)$ is 1, which an odd number. But we know that the degree of each vertex of an Eulerian graph is even and so $PG(S)$ can not be an Eulerian graph. □

Notation: $B(S) = \{(x, y) \mid x \neq y, x \neq 0 \neq y, xSy = 0 \text{ or } ySx = 0\} \subseteq S \times S$, where S is a semiring.

- Corollary 4.3.** (i) S is a prime semiring if and only if $B(S) = \phi$.
 (ii) The number of elements in $B(S)$ is less than or equal to the number of triangles in $PG(S)$.
 (iii) If $B(S) \neq \phi$, then the length of the longest walk is ≥ 3 .
 (iv) $B(S) \neq S \times S$.

Proof. (i) Let S be a prime semiring. Then $PG(S)$ is a tree. So for any two non-zero $x, y \in S$, $xsy \neq 0$. Thus $B(S) = \phi$.

Conversely, if $B(S) = \phi$, then clearly $PG(S)$ is a tree and so it follows from theorem 4.1 that S is prime.

- (ii) Let $(x, y) \in B(S)$. Then $\{0, x, y\}$ form a triangle in $PG(S)$. If $(x, y), (u, v) \in B(S)$ such that $(x, y) \neq (u, v)$, then $x \neq u$ or $y \neq v$ and so the triangle $\{0, x, y\}$ and $\{0, u, v\}$ (in $PG(S)$) are distinct. This shows that the number of elements in $B(S)$ is less than the number of triangles in $PG(S)$.
 (iii) Let $(x, y) \in B(S)$. Then $0x, xy, y0$ is a triangle. This walk is of length 3. Hence the length of the longest walk is greater than or equal to 3.
 (iv) Clearly $B(S) \subseteq S \times S$. Also $(x, x) \in S \times S$ but it does not belong to $B(S)$ and so $B(S) \neq S \times S$.

□

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