Advances in Mathematics: Scientific Journal **9** (2020), no.9, 7525–7534
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.9.99

RAYLEIGH WAVE PROPAGATION IN TWO-TEMPERATURE DUAL PHASE LAG MODEL WITH IMPEDANCE BOUNDARY CONDITIONS

SONU SHARMA, S. KUMARI[1], AND M. SINGH

ABSTRACT. In the research article, we discuss the propagation of Rayleigh wave in two temperature with dual phase lag thermo-elasticity with impedance boundary conditions. The governing equations are computed for the wave solutions and satisfy the relevant boundary condition. The frequency equation is calculated for thermally insulated and isothermal case and approximated for calculating the numerical results for the dimensionless speed of Rayleigh wave. The dimensionless velocity plotted against frequency and initial stress graphically. The impact of various parameters in presence of Impedance Boundary condition are shown graphically.

1. INTRODUCTION

Biot [1] introduced the classical dynamical coupled theory of thermo-elasticity and extended to generalized theory of thermo-elasticity by Green and Lindsay [2] and Lord and Shulman [3]. Later on, Ignaczak and Ostoja- Starzewski [4] and Hetnarski and Ignaczak [5] was reviewed these theories in detailed and conclude that wave Speeds is to finite. Tzou [6-8] introduced the modern theory known as dual phase lag thermo-elastic model, where the phonon-electron interaction on microscopic level consider as retarding source of delayed response on the macroscopic. The second law of thermo-dynamics for continuous bodies

[1]*corresponding author*

2010 *Mathematics Subject Classification.* 74S05, 74J15.
Key words and phrases. Rayleigh wave, thermo-elasticity, two-temperature, dual phase lag, impedance boundary condition, frequency.

was proposed Gurtin and Williams [9]. Chen and Gurtin [10] introduced the thermo-elasticity theory with two different temperatures i.e. as the conductive temperature and T as thermodynamic temperature, where material parameter .Warren and Chen [11] observed that the two temperatures and T are different when the problem of wave propagation is involved. Puri and Jorden [12] discussed the Harmonic plane wave propagation with the two-temperature theory. Youssef [13] studied the generalized two-temperature thermo-elasticity. Chandrasekhraiah and Srikantaiah [14] discussed the Rayleigh wave in temperature rate dependent thermoelasticity. Ahmed and Abouelregal [15] discussed the Rayleigh wave propagation under the dual phase lag thermo-elasticity in an isotropic half space. The dual phase lag delay with two-temperature theory delay was studied by Quintanilla et al. [16]. Singh et al. [17] investigated the Rayleigh wave propagation in a two-temperature generalized thermoelastic solid half–plane. The impedance boundary conditions with Rayleigh waves are expressing in many fields of science and technology. We use Impedance boundary conditions in the field of electro-magnetism and acoustics. The wave propagation in an isotropic elastic solid covered with thin film of different material which encountered impedance boundary conditions was studied by Tiersten [18]. The Rayleigh waves in an orthotropic and monoclinic half space by boundary conditions was investigated by Vinh and Hue [19].The extended Eringen's theory of nonlocal elasticity to generalized thermoelasticity with dual phase lag and voids introduced by Mondal et. al. [20]. In the present paper, Rayleigh wave studied in context of dual phase lag thermo-elasticity under the initial stress, two temperature and Impedance boundary conditions. The frequency equation obtained from the governing equation by using the Surface wave solution and satisfy the relevant boundary conditions. Rayleigh wave frequency equation evaluated numerically to explore the influence of initial stress, Impedance Boundary and two temperature parameters.

2. Basic Equations

Following Singh et al. [17] the basic equations of transversely isotropic dual–phase–lag thermo-elastic model in the absence of body forces and heat sources are:

(i) The equation of motion:

$$\sigma_{ji,j} + \rho F_i = \rho \ddot{u}_i, \tag{2.1}$$

(ii) The relationship between strain-stress-temperature:

$$\sigma_{ij} = (c_{ijkl} + \delta_{kl} p_{ij}) e_{kl} \Theta, \tag{2.2}$$

(iii) The energy equation:

$$-q_{i,j} = \rho T_0 \dot{S}, \tag{2.3}$$

(iv)The relation of strain and displacement:

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{2.4}$$

(v) The modified Fourier's law:

$$-K_{ij}(\Theta_{,j} + \tau_\theta \dot{\Theta}_{,j}) = q_i + \tau \dot{q}_i, \tag{2.5}$$

(vi)Relationship between Entropy-strain-temperature:

$$\rho S = \frac{\rho c_E}{T_0}\Theta - a_{ij} e_{ij}, \tag{2.6}$$

Using equation (5) and (6) in equation (3) we obtain

$$(1 + \tau_\theta \frac{\partial 1}{\partial t}) K_{ij}(\Phi_{11} + \Phi_{33}) = (1 + \tau_\theta \frac{\partial 1}{\partial t})(\rho c_E \frac{\partial \Theta}{\partial t} + T_0 \beta_{ij} \frac{\partial e_{ij}}{\partial t}) \tag{2.7}$$

The two-temperature relation:

$$\Phi - \Theta = a^\star \Phi_{ii} \tag{2.8}$$

where ρ is the mass density, u_i is the components of the displacement vector, K_{ij} are the components of the thermal conductivity component, σ_{ij} is the stress component, c_{ijkl} is the components of elastic constants, q_i is component of heat conduction, S is the entropy per unit mass, c_E is the component of specific heat, a_{ij} is the constitutive coefficients, p_{ij} is the initial stress component, $\Theta = T - T_0$ is small increment, a^* is two temperature parameter, T is absolute temperature, T_0 is uniform temperature chosen such that $|\frac{\Theta}{T_0} \leq 1|$, Φ is the conductive temperature, τ_q is the phase-lag heat flux and τ_θ is the phase-lag of the gradient of the temperature where $0 \leq \tau_\theta \leq \tau_q$. If we replace $\tau_\theta = 0, \tau_q = 0$ then the dual phase lag model is reduced to classical theory of thermo-elasticity. If we replace $\tau_\theta = 0$ and τ_q by τ_θ then the dual phase lag thermo-elasticity is reduced to the Lord Shulman theory of thermo-elasticity.

3. Formulation of problem and solution

Consider a dual phase lag with initial stress and two-temperature thermo-elastic solid half-space in transversely isotropic homogeneous medium and taking the x-z plane with $(u_1, 0, u_3)$. Using equations (2.1) to (2.7) we get,

$$(c_{11}+p_{11})u_{1,11}+(c_{13}+c_{44}+p_{11})u_{3,13}+(c_{44}+p_{11})u_{1,33}-\beta_1\Theta_{,}1=\rho\ddot{u}_1 \tag{3.1}$$

$$(c_{44}+p_{11})u_{3,11}+(c_{13}+c_{44}+p_{11})u_{1,13}+(c_{33}+p_{33})u_{3,33}-\beta_3\Theta_{,3}=\rho\ddot{u}_3 \tag{3.2}$$

$$(1+\tau_\theta\frac{\partial}{\partial t})[K_{11}\Phi_{11}+K_{33}\Phi_{33}]=(1+\tau_q\frac{\partial}{\partial t})[\rho c_E\dot{\Theta}+\beta_1T_0\dot{u}_{1,1}+\beta_3T_0\dot{u}_{3,3}]\,. \tag{3.3}$$

In this case of thermo-elastic Rayleigh wave half space is propagating in x-direction, the function (u_1, u_3, Φ) are considered as follows:

$$(u_1,\ u_3,\ \Phi)=(\phi_1(z),\phi_3(z),\psi(z))\exp\iota k(x-ct) \tag{3.4}$$

By using equation (2.8) into equations (3.1)-(3.3) and then using equation (3.4), we obtain the three homogeneous equations in ϕ_1,ϕ_2 and ψ are as follows:

(3.5)

$$[k^2(D^2+\frac{\rho c^2}{c_{44}+p_{11}}-\frac{c_{11}+p_{11}}{c_{44}+p_{11}})]\phi_1+\iota k(\frac{c_{13}}{c_{44}+p_{11}}+1)D\phi_3-\frac{\beta_1}{c_{44}+p_{11}}\iota k[1-a^*(-k^2+D^2)]\psi=0$$

(3.6)

$$\iota k(\frac{c_{13}}{c_{44}+p_{11}}+1)D\phi_1+[k^2(\frac{\rho c^2}{c_{44}+p_{11}}-1)+(\frac{c_{33}+p_{33}}{c_{44}+p_{11}})D^2]\phi_3-\frac{\beta_3}{c_{44}+p_{11}}[1-a^*(-k^2+D^2)]\psi=0$$

(3.7)

$$\iota k^3\epsilon\frac{\rho c^2}{c_{44}+p_{11}}\phi_1+\bar{\beta}\epsilon k^2\frac{\rho c^2}{c_{44}+p_{11}}D\phi_3+[k^2\frac{\rho c^2}{c_{44}+p_{11}}(1-a^*(-k^2+D^2))-K_1{}^*+K_3{}^*D^2]\psi=0$$

$$\epsilon=\frac{\beta_1{}^2}{\rho^2c_Ec_1{}^2},\ \tau^*=\frac{\tau_q+\frac{\iota}{w}}{1-\iota w\tau_\theta},\ K_1{}^*=\frac{K_11}{c_E(c_{44}+p_{11})\tau^*},\ K_2{}^*=\frac{K_33}{c_E(c_{44}+p_{11})\tau^*},\ \bar{\beta}=\frac{\beta_3}{\beta_1}\,.$$

The following auxiliary equation is the result of equations (3.5) to (3.7), as given below:

$$(D^6-PD^4+QD^2-R)(\phi_1,\phi_3,\psi)=0\,. \tag{3.8}$$

The general solutions of equation (3.8) are as follows:

$$u_1(z) = [\sum_{i=1}^{3} P_i \exp^{-m_i z} + P_i \exp^{m_i z}] \exp^{ik(x-ct)} \tag{3.9}$$

$$u_3(z) = [\sum_{i=1}^{3} Q_i \exp^{-m_i z} + Q_i \exp^{m_i z}] \exp^{ik(x-ct)} \tag{3.10}$$

$$\psi(z) = [\sum_{i=1}^{3} R_i \exp^{-m_i z} + R_i \exp^{m_i z}] \exp^{ik(x-ct)}, \tag{3.11}$$

where P_i, Q_i, R_i are the arbitrary constants, the roots of the equation are m_i :

$$m^6 - Pm^4 + Qm^2 - R = 0. \tag{3.12}$$

The Equation (3.12) is cubic in m^2 and its roots ${m_1}^2, {m_2}^2, {m_3}^2$ are related as:

$$\begin{aligned} &{m_1}^2 + {m_2}^2 + {m_3}^2 = P \\ &{m_1}^2{m_2}^2 + {m_2}^2{m_3}^2 + {m_3}^2{m_1}^2 = Q \\ &{m_1}^2{m_2}^2{m_3}^2 = R\,. \end{aligned}$$

The roots are complex in general and here we have considered the surface waves, W.L.O.G., we can assume that Re$(m_i) > 0$ We choose only that form of m_i, which satisfies the radiation condition:

$$u_1(z), u_3(z), \phi(z) \to 0 \; as \; z \to \infty\,. \tag{3.13}$$

With the help of condition (3.13), the solutions (3.9) to (3.11) reduces to specific solutions in half plane $z \geq 0$ as:

$$u_1(z) = [\sum_{i=1}^{3} P_i \exp^{-m_i z}] \exp^{ik(x-ct)} \tag{3.14}$$

$$u_3(z) = [\sum_{i=1}^{3} Q_i \exp^{-m_i z}] \exp^{ik(x-ct)} \tag{3.15}$$

$$\psi(z) = [\sum_{i=1}^{3} R_i \exp^{-m_i z}] \exp^{ik(x-ct)}, \tag{3.16}$$

where $Q_i = F_i P_i$ and $R_i = {F_i}^* P_i$

$$F_i = \frac{-\iota m_i}{k}[\frac{\bar{\beta}\rho c^2 - (c_{11}+p_{11}) + \frac{m_i^2}{k}(c_{44}+p_{11}) + (c_{13}+c_{44}+p_{11})}{\bar{\beta}\frac{m_i^2}{k^2}(c_{13}+c_{44}+p_{11}) - \rho c^2 - (c_{11}+p_{11}) + \frac{m_i^2}{k^2}(c_{33}+p_{11})}]$$

$$F_i{}^* = \frac{-\iota k\epsilon}{\frac{\beta_1}{c_{44}+p_{11}}}[\frac{\bar{\beta}\rho c^2 - (c_{11}+p_{11}) + \frac{m_i^2}{k}(c_{44}+p_{11}) + (c_{13}+c_{44}+p_{11})}{(1-a^*(-k^2+m_i^2))(\epsilon\bar{\beta}(c_{44}+p_{11})+c_{13}) + (c_{13}+c_{44}+p_{11})(-\frac{K_1^2}{\rho c^2}+\frac{m_i^2K_3^*}{k^2\rho c^2})}],$$

where $i = (1, 2, 3)$.

4. Impedance boundary condition

The thermal and mechanical conditions at free surface $z = 0$ by using impedance boundary conditions are:

(4.1)
$$\sigma_{zz} + wz_2u_3 = o$$

(4.2)
$$\sigma_{zx} + wz_1u_1 = 0$$

(4.3)
$$\frac{\partial\Theta}{\partial z} + h\Theta = 0\,,$$

where $h \to 0$ correlate with the thermally insulated surface and $h \to \infty$ correlate with the isothermal surface,

$$\sigma_{33} = (c_{33} + p_{33}u_{3,3} + c_{13}u_{1,1} - \beta_1[\Phi - a^*(\Phi_{11} + \Phi_{33})]$$

$$\sigma_{31} = c_{44}(u_{1,3} + u_{3,1})\,.$$

Making use of solutions (3.14) to (3.16) in impedance boundary conditions (4.1) to (4.3), we find the system of three homogeneous equations in P_1, P_2 and P_3. The determinant of the co-efficient must vanish for the solution of the homogeneous equation.

(4.4)
$$\begin{gathered}1 - a^*(k^2 - m_1^2)m_1F_1^*(X_2Y_3 - X_3Y_2) - 1 - a^*(k^2 - m_2^2)m_2F_2^*(X_1Y_3 - X_3Y_1)\\ + 1 - a^*(k^2 - m_3^2)m_3F_3^*(X_1Y_2 - X_2Y_1)\\ = hm_1F_1^*P_1(X_2Y_3 - X_3Y_2) - m_2F_2^*P_2(X_1Y_3 - X_3Y_1) + m_3F_3^*P_3(X_1Y_2 - X_2Y_1)\,,\end{gathered}$$

where

$$\begin{aligned}X_i &= (c_{33} + p_{33})m_iF_i - \iota kc_{13} + \beta_3 1 - a^*(-k^2 + m_i^2)F_i^* + wz_2u_3\\ Y_i &= (m_i - \iota kF_i + wz_1u_1)\,,\end{aligned}$$

where $(i = 1, 2, 3)$. The equation (4.4) is the required frequency equation of the Rayleigh wave in two-temperature, dual phase lag with impedance boundary condition of transversely isotropic thermo-elastic model.

Particular cases

(a) The thermally insulated case $h \to 0$ the frequency equation (4.4) reduces to

$$1 - a^*(-k^2 - {m_1}^2)m_1{F_i}^*(X_2Y_3 - X_3Y_2) - 1 - a^*(k^2 - {m_2}^2)m_2{F_2}^*(X_1Y_3 - X_3Y_1) + 1 - a^*(k^2 - {m_3}^2)m_3{F_3}^*(X_1Y_2 - X_2Y_1) = 0\,.$$

(b) For isothermal case $h \to 0$ the frequency equation (4.4) reduces to

$$m_1F_1^*(X_2Y_3 - X_3Y_2) - m_2F_2^*(X_1Y_3 - X_3Y_1) + m_3F_3^*(X_1Y_2 - X_2Y_1) = 0\,.$$

Special Cases

Case 1. In absence of two-temperature, impedance boundary parameter and initial stress the frequency equation for the isotropic dual phase lag and initial stress thermo-elastic half space (4.4) takes the form,

$$m_1{F_1}^*(X_2Y_3 - X_3Y_2) - m_2{F_2}^*(X_1Y_3 - X_3Y_1) + m_3{F_3}^*(X_1Y_2 - X_2Y_1) = 0\,,$$

where $X_i, Y_i, F_i, {F_i}^*$ and m_i calculated accordingly, the obtained equation is same as the frequency equation of the Singh et. al. [17].

Case 2. In absence of dual phase lag, two temperature, Impedance boundary parameter, initial stress for thermally insulated case the frequency equation (4.4) reduces to frequency equation of Rayleigh wave.

$$(2 - \frac{c^2}{{c_2}^2})^2 = \sqrt{4(1 - \frac{c^2}{{c_2}^2})(1 - \frac{c^2}{{c_2}^2})}\,.$$

5. Numerical Results and Discussion

Following Chadwick and Sheet the Zinc parameter are considered for numerical calculation as
$C_{11} = 1.628{*}10^{11}Nm^-2,\ C_{33} = 1.562{*}10^{11}Nm^-2,\ C_{13} = 0.508{*}10^{11}Nm^-2,\ C_{44} = 0.385{*}10^{11}Nm^-2,\ K_1 = 1.24{*}10^2Wm^-1deg^-1,\ K_3 = 1.34{*}10^2Wm^-1deg^-1,\ \beta_1 = 5.75{*}10^6Nm^-2deg^-1,\ \beta_1 = 5.17{*}10^6Nm^-2deg^-1,\ c_E = 3.9{*}10^2JKg^-1deg^-1,\ \rho = 7.14 * 10^3Kgm^-3\tau_q = 0.005s,\ \tau_\theta = 0.0005s,\ T_0 = 296K$

The non dimensional speed $\dfrac{\rho c^2}{c_{11} + p_{11}}$ is plotted against the frequency for the various values of the two-temperature parameter $a^* = 0, 0.5, 1$ in presence of

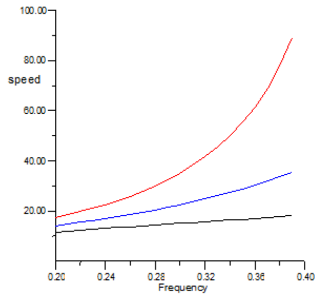

FIGURE 1

initial stress and impedance parameter. When $a^* = 0, 0.5$ the speed slowly increases as increase of the velocity. When $a^* = 1$ the speed sharply increases with increase of the frequency.

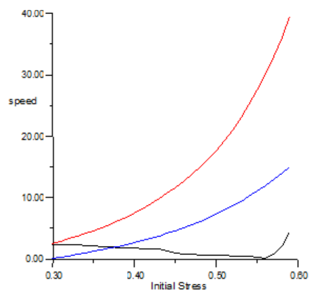

FIGURE 2

The non dimensional speed $\dfrac{\rho c^2}{c_{11} + p_{11}}$ is plotted against the initial stress parameter for the various values of two-temperature parameter $a^* = 0.02, 0.5$ and 1.0 in presence of impedance parameter and frequency. Its shows that when $a^* = 1$ the non dimensional speed increase with increase of the initial stress and $a^* = 0.5$ it increases slowly with increase of initial stress and when $a^* = 0.02$ speed decrease slowly and then slowly increase with increasing the initial stress parameter.

6. CONCLUSION

The governing equation of Rayleigh wave lag with initial stress, two temperature thermo-elasticity with impedance boundary condition are specified in context of dual phase lag model, Lord-Shulman and Green-Lindsay theory. These governing equation solved by the surface wave solution and satisfied the required boundary condition and frequency equation obtained.

(i) It has been observed that for the different values of two-temperature the non dimensional speed increases with increase of the frequency in presence of impedance boundary condition.

(ii) It has been observed that for different values of two temperature the non dimensional speed against the initial stress in presence of impedance boundary condition. For $a^* = 1, 0.5$ the non dimensional speed increases with increase of initial stress. For $a^* = 0.02$, the non dimensional speed decrease and then increases with increase of initial stress.

REFERENCES

[1] BIOT: *Thermoelasticity and irreversible thermodynamics*, J. of Appl. Phys., **2** (1956), 240–253.

[2] A. E. GREEN, K. A. LINDSAY: *Thermoelasticity*, J. of Elasticity, **2** (1972), 1–7.

[3] H. LORD, Y. SHULMAN: *A generalized dynamical theory of thermoelasticity*, J. of Mech. and phys. of sci., **15** (1967), 299–309.

[4] J. IGNACZAK, M. OSTOJA-STARZEWSKI: *Thermoelasticitywith finite wave speeds* , Oxford University Press, Oxford, 2009.

[5] R. B. HETNARSKI, J. IGNACZAK: *Generalized thermoelasticity*, J. of Thermal Stress , **22** (1999), 451–476.

[6] D. Y. TZOU: *A unified approach for heat conduction from macro- to micro- scales*, J. of Heat Transfer, **177**(1995), 8–16.

[7] D. Y. TZOU: *Experimental support for the lagging behavior in heat propagation*, J. of Thermophysics and Heat Transfer, **9** (1995), 686–693.

[8] D. Y. TZOU: *Macro- to microscale heat transfer* , The lagging behavior Washington DC, Taylor and Francis, 1996.

[9] M. E. GURTIN, W. O. WILLIAMS: *On the Clausius–Duhem Inequality*, Zeitschrift fr Angewandte Mathematik und Physik, **17** (1966), 626–633.

[10] P. J. CHEN, M. E. GURTIN : *On a theory of heat conduction involving two-temperature*, Zeitschrift fr Angewandte Mathematik und Physik , **19** (1968), 614–627.

[11] W. E. WARREN, P. J. CHEN : *Wave propagation in the two-temperature theory of thermoelasticity*, Acta Mechanica , **16** (1973), 21–33.

[12] P. PURI, P. M. JORDAN: *On the propagation of harmonic plane waves underthe two-temperature theory*, Int. J. of Eng. Sci., **44** (2006), 1113–1126.

[13] H. M. YOUSSEF: *Theory of two-temperature-generalized thermoelasticity*, J. of Appl. Math., **71**(3) (2006), 383–390.

[14] D. S. CHANDRASEKHARAIAH, K. R. SRIKANTAIAH: *On temperature rate dependent thermoelastic Rayleigh waves in half–space*, Gerlands Beitrage Zur Geophysik , **93** (1984), 133–141.

[15] E. AHMED, ABOUELREGAL: *Rayleigh waves in a thermoelastic solid half space using dual-phase-lag model*, Int. J. of Eng.Sci., **49** (2011), 781–791.

[16] R. QUINTANILLA, P. M. JORDAN: *., A note on the two-temperature theory with dual-phase-lag delay: Some exact solutions*, Mech. Research Communications , **36** (2009), 796–803.

[17] B. SINGH, S. KUMARI, J. SINGH: *Propagation of Rayleigh Wave in Two–Temperature Dual Phase Lag Thermoelasticity*, Mech. and Mech. Eng., **21**(1) (2017), 105–116.

[18] H. F. TIERSTEN: *Elastic surface waves guided by thin films*, J. of Appl. Phys., **40** (1969), 770–789.

[19] P.C. VINH, T. T. T. HUE: *Rayleigh waves with impedance boundary conditions in anisotropic solids*, Wave Motion, **51** (2014), 1082–1092.

[20] S. MONDAL, N. SARKAR, N. SARKAR: *Waves in dual-phase-lag thermoelastic materials with voids based on Eringen's nonlocal elasticity*, (2019), 1–16.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CHANDIGARH
GHARUAN, PUNJAB, INDIA
Email address: uis.18msm1018@gmail.com

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CHANDIGARH
GHARUAN, PUNJAB, INDIA
Email address: Sangwan.sangeeta@gmail.com

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CHANDIGARH
GHARUAN, PUNJAB, INDIA
Email address: mandeepvegal@gmail.com