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PROBABILITY ON A PARTIALLY ORDERED SET

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ABSTRACT. The paper proposes to introduces the notion of probability on a poset and discuss some elements properties.

1. INTRODUCTION

One may ask the question why we want to consider non-classical probability in our study. The answer is that the non-classical comes from the logical point of view, an essential feature of quantum mechanics or more generally the uncertainty relations we have in quantum mechanics. The model of quantum system has been investigated by several mathematician and physicist [1-8] and they have come to an agreement that it is at least an orthomodular poset. Thus our aim is to study probability on a partial ordered set.

2. PROBABILITY POSET

Definition 2.1. *A probability on a poset $(P, \leq)$ is a function p of P into the closed unit interval $[0, 1]$ which satisfies the following properties:*

(p_1) $p(x) \geq 0, \forall x \in P$;
(p_2) $p(x) \geq p(y), \forall x, y \in P$;
(p_3) *if m is a maximal element of P, then $p(m) = 1$.*

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A probability poset consists of a poset $(P, \geq)$ together with a probability p defined on P and is denoted by (P, p). It should be noted that in a supremum lattice bounded above. We have $p(m) = 1$, for m, in this case, is the universal element.

If the following additional property also holds in (P, p) whenever $x \vee y$ and $x \wedge y$ exist in P for every pair of elements $x, y \in P$:

(p_4) $p(x \wedge y) \leq p(x) + p(y) \leq p(x \vee y)$,

then we call (P, p) a probability lattice. We are led to the introduction of probability on a pseudo complemented lattice, which is a new concept, in which probability satisfies analogous properties to those on a Boolean lattice.

Theorem 2.1. *If $(P, \leq)$ is a pseudo complemented lattice, then p satisfies the following properties for all $x, y \in P$:*

- (I) $x \leq y \Rightarrow p(x^\star) \geq p(y^\star)$ *where $x^\star$ and $y^\star$ are pseudo complements of x and y respectively.*
- (II) $p(x) \leq p(x^{\star\star})$;
- (III) $p(x^\star) \leq 1 - p(x)$;
- (IV) $p(x \vee y)^\star \leq 1 - p(x \wedge y)$;
- (V) $p(x) \leq p(x^{\star\star}) \leq 1 - p(x^\star) = 1 - p(x^{\star\star\star})$;
- (VI) $p(x \vee y^\star) \geq p(x) - p(y)$;
- (VII) $p(n) = 0$, *where n is the null element of P;*
- (VIII) $p(x^{\star\star}) + p(y^{\star\star}) \geq p(x \vee y)^{\star\star}$.

Proof. The proof of (I), (II), (V) and (VIII) are obvious.
(III) We have $p_4 \Rightarrow$

$$\begin{aligned} p(x) + p(x^\star) &\leq p(x \vee x^\star) \leq 1 \\ \Rightarrow p(x^\star) &\leq 1 - p(x). \end{aligned}$$

(IV) This follows from the fact that

$$\begin{aligned} p(x \vee y\star) &\leq 1 - p(x \vee y) \\ &\leq 1 - p(x) + p(y) \text{ from } p_4 \\ &\leq 1 - p(x \wedge y). \end{aligned}$$

(VI) Using p_4 we find that

$$p(x \vee y^\star) \geq p(x) + p(y^\star)$$

and

$$0 \leq p(y \vee y^\star) \leq p(y) + p(y^\star) \Rightarrow p(y^\star) \geq -p(y).$$

Thus

$$p(x \vee y^\star) \geq p(x) - p(y^\star).$$

(VII)

$$\begin{aligned} p(n^\star) &\leq 1 - p(n) \\ \Rightarrow p(e) &\leq 1 - p(n), \end{aligned}$$

where $n^\star = e$ is the universal element in P

$$\Rightarrow 1 \leq 1 - p(n).$$

But $1 - p(n) \leq 1$ Which means that

$$1 - p(n) = 1 \Rightarrow p(n) = 0.$$

□

Remark 2.1.

(i) *Probability on a pseudocomplemented lattice is strictly positive, i.e.,* $P(x) \geq 0$ *and* $x = n \Leftrightarrow P(x) = 0$*, for* $x \in P$.

(ii) *For all* x, y *in a pseudocomplemented distributive lattice, we have*

$$(a)\ p(x \vee y)^\star = p(x^\star \wedge y^\star),\ \textit{since}\ (x \vee y)^\star = x^\star \wedge y^\star,$$

$$(b)\ p(x \wedge y)^\star \leq p(x \wedge y^\star),\ \textit{since}\ x \wedge (x \wedge y)^\star = x \wedge y^\star,$$

$$(c)\ p(x \wedge y)^\star \geq p(x^\star \wedge y^\star),\ \textit{since}\ (x \wedge y)^\star \geq x^\star \wedge y^\star,$$

$$(d)\ p(x \vee y)^\star \leq p(x^\star) + p(y^\star) \leq p(x \wedge y)^\star,$$

This follows from (b) *and* (c) *and* p_4.

$$(e)\ p(x \vee y)^{\star\star} = p(x^{\star\star} \wedge y^{\star\star}),\ \textit{since}\ (x \wedge y)^{\star\star} = x^{\star\star} \wedge y^{\star\star}.$$

Theorem 2.2. *If* $(P, \rightarrow)$ *is a Browerian lattice with null element* n *and the Browerian complement of an element* $x \in P$ *is the pseudocomplement* $x^\star$*, where* $z \leq x \rightarrow y$ *if* $z \wedge x \leq y, \forall x, y, z \in P$*, then* P *satisfies the following additional properties:*

(a) $p(x \rightarrow y) \geq p(x \wedge y) - p(x)$
(b) $p((x \rightarrow y) \wedge (x \rightarrow z)) > p(x \wedge y \wedge z) - p(x \wedge y)$
(c) $p((x \rightarrow z) \wedge (y \rightarrow z)) \geq p(x \wedge z)p(y \wedge z) - p(x \vee y)$
(d) $p(x \rightarrow (y \rightarrow z)) \geq p(x \wedge y \wedge z) - p(x \wedge y)$.

Proof.

(a) $y \wedge x \leq y \wedge x$

$$\begin{aligned} &\Rightarrow \quad y \leq (x \to y) \wedge z \\ &\Rightarrow \quad p(y) \leq p((x \to y) \wedge x) \leq p(x \to y) + p(x) \\ &\Rightarrow \quad p(x \wedge y) \leq p(x \to y) + p(x) \ (p(x \wedge y) \leq p(y)) \\ &\Rightarrow \quad p(x \to y) \geq p(x \wedge y) - p(x) \end{aligned}$$

(b) $(x \to y) \wedge (x \to z) = x \to (y \wedge z)$

$$p((x \to y) \wedge (x \to z)) = p(x \to (y \wedge z))$$

and $p((x \to (y \wedge z)) \geq p(x \wedge y \wedge z) - p(x)$ from Theorem 2.2(a).

(c) $(x \to z) \wedge (y \to z) = (x \vee y) \to z$, together with p_2

$$p((x \to z) \wedge (y \to z)) = p((x \vee y) \to z)$$

But $p(x \vee y \to z) \geq p(z) - p(x \vee y) \geq p(x \wedge z) + p(y \wedge z) - p(x \vee y)$
{Since $(x \wedge z) \vee (y \wedge z) = (x \vee y) \wedge z \geq z$}

(d) For, $x \to (y \to z) = (x \wedge y) \to z$.

□

Corollary 2.1.

(a) *If* $x = y$, *then* $p(x \to y) = 1, \forall x, y \in P$.
(b) *If* $x \leq y$, *then* $p(x \to y) = 1, \forall x, y \in P$.

Proof. Proof of (a) is obvious. For (b),we have

$$\begin{aligned} x \leq y \quad &\Rightarrow \quad e \wedge x \leq y \\ &\Rightarrow \quad e \leq x \to y \\ &\Rightarrow \quad e = x \to y \\ &\Rightarrow \quad p(e) = p(x \to y) \\ &\Rightarrow \quad p(x \to y) = 1. \end{aligned}$$

□

Definition 2.2. *Let* $(P, \leq)$ *be lattice with the universal element* e. *The pseudodual complement of an element* $a \in P$ *denoted by* $a_\star$, *is the smallest element* x *in* P, *such that* $x \vee a = 1$.

The proof of the following succeeding theorem are obvious dually.

Theorem 2.3. *If $(P, \leq)$ be a pseudo dual complemented lattice. Then the following properties hold for all $x, y \in P$:*

(a) $x \leq y \Rightarrow p(x_\star) \leq p(y_\star)$;
(b) $p(x_\star) \leq 1 - p(x_\star)$;
(c) $p(x_{\star\star}) \leq p(x) \leq 1 - p(x_\star) \leq 1 - p(x_{\star\star})$;
(d) $p(x \vee y)_\star \leq 1 - p(x \vee y)$.

Theorem 2.4. *If $(P, \leq)$ be a pseudo dual complemented distributive lattice, then p satisfies some more properties than that listed in the previous theorem for all $x, y \in P$:*

(a) *If* $x \leq y$, *then* $p(x \wedge y_\star) \leq p(y) - p(x), \forall x, y \in P$;
(b) $p(x \wedge y)_\star \leq p(x \wedge y_\star)$;
(c) $p(n) = 0$, *where* n *is the null element in* P.;
(d) $p(x \wedge y_\star) \leq p(x_\star) + p(y_\star) \leq p(x \wedge y)_\star$;
(e) $p(x_{\star\star}) + p(y_{\star\star}) \leq p(x \wedge y)_{\star\star}$.

Theorem 2.5. *If $(P, \leq)$ be a dual Brouwian lattice with universal element e in which the operator $\leftarrow$ in P is defined as follow:*

$$z \geq x \leftarrow y \text{ iff } z \vee x \geq y, \forall x, y \in P$$

and the dual Brouwian complement of $x \in P$ is the pseudodual complement $x_\star$. Then p satisfies again some more properties than listed in the theorems 2.3 and 2.4.

(I) $p(x \leftarrow y) \leq p(y) - p(x)$
(II) $p((x \leftarrow y) \cup (x \leftarrow z)) \leq p(y \vee z) - p(x)$
(III) $p((x \leftarrow z) \vee (y \leftarrow z)) \leq p(z) - p(x \wedge y)$
(IV) $p((x \leftarrow (y \leftarrow z)) = p((x \wedge y) \leftarrow z)) = p(((x \wedge y) \leftarrow (x \leftarrow z))$
(V) $p(x_{\star\star}) \leq p(x) \leq p(x^{\star\star})$
(VI) $p(x \rightarrow y) \geq p(x \leftarrow y)$.

Corollary 2.2.

(a) *If* $x = y$, *then* $p(x \leftarrow y) = 0$.
(b) *If* $x \geq y$, *then* $p(x \leftarrow y) = 0$.

Remark 2.2. *Let $(P, \leq)$ be a probability poset with probability p and P_0 is a subset of P, then the restriction of p to P_0 is probability p_{p_0} on P_0.*

Definition 2.3. *Let P_0 be a non-empty subset of a poset $(P, \leq)$, then there exists a smallest subset P_s of $(P, \leq)$ containing P_0, the probability poset (P_s, p_{p_s}) is then called the probability poset generated by P_0 is in (P, p).*

Definition 2.4. *A probability poset (P_1, p_1) is said to be homometric to (P_2, p_2) iff there exists a mapping$f : (P_1, p_1) \to (P_2, p_2)$ such that f is an order homomorphism and $p_1(x) = p_2(f(x))$.*

Definition 2.5. *[1] Let (L, p) be a probability lattice and L_0 a subset of L, then we say that L_0 is $p-$dense in L, iff for every $x \in L$ and for every positive real number $\epsilon > 0$, there exists an element $a = a(x, \epsilon) \in L_0$, such that $p(x \vee a) < \epsilon$. A $p-$lattice (L, p) is called $p-$separable iff there exists a countable class C of elements of L, which is $p-$dense in L.*

Every $p-$sub lattice of $p-$separable $p-$lattice is also $p-$separable.

Theorem 2.6. *[1] The probability interval lattice (L, m) is $m-$separable.*

Proof. Let L_0 be a sub lattice of L generated by the class of all intervals I_α for every α. Then L_0 is a countable set and it is $m-$separable. □

Theorem 2.7. *Let (L, p) be pseudo complemented probability lattice. Let ρ be a real valued function defined on $L \times L$ as follows:*

$$\begin{aligned} \rho(a, b) &= p(a \vee b) \\ \textit{and } \rho(a, b) &= 0 \textit{ iff } a = b. \end{aligned}$$

Then the following conditions hold for all $a, b, c \in L$

(i) $\rho(a, b) \geq 0$ *and* $\rho(a, b) = 0$ *iff* $a = b$
(ii) $\rho(a, b) = \rho(b, a)$
(iii) $\rho(a, b) \leq \rho(a, c) + \rho(c, b)$

Proof. Here (i) and (ii) are trivially true. We shall prove (iii), We have

$$\begin{aligned} \rho(a, b) = p(a \vee b) &\leq p(a) + p(b) \\ \text{Also } p(a) + p(b) &\leq \rho(a \vee c) + \rho(c \vee b) \\ \text{i.e. } \rho(a, b) &\leq \rho(a, c) + \rho(c, b). \end{aligned}$$

Hence (iii) is true. □

Hence the lattice L can be considered as a metric topological space and the concept of metric convergence or equivalently $p-$convergence can be introduced in the usual way, namely, a sequence $a_v \in L, v = 1, 2, ...$ is said to $p-$convergent to an element a if and only if

$$\lim a_v = a.$$

A $p-$convergent sequence $a_v \in L, v = 1, 2, ...$ satisfies the $p-$cauchy condition i.e. for every $\epsilon > 0,$ there exists a natural number $N(\epsilon)$ such that

$$p(a_v \vee a_u) < \epsilon,$$

for every $u, v \geq N(\epsilon), u = v + p,$ and p is a positive integer $\geq 1.$

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