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APPROXIMATION ALGORITHMS FOR THE *H*-GRAPHS

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ABSTRACT. The pseudoachromatic number of a graph is the largest number of colors in a (not necessarily proper) vertex coloring of the graph such that every pair of distinct colors appears on the endpoints of some edge. The achromatic number is largest number of colors which can be used if the coloring must also be proper. In this paper, we present the approximation algorithms for the achromatic number and pseudoachromatic number of H-graph and we have compared their results.

1. INTRODUCTION

The achromatic number of the graph G is the largest number m such that G has a complete coloring with m colors. Equivalently there is a partition of V into disjoint independent sets $(V_1, \ldots, V_m)$ such that for each pair of distinct sets V_i, V_j, $V_i \cup V_j$ is not an independent set in G. A pseudocomplete coloring of a simple graph G is a (not necessarily proper) vertex coloring such that each pair of colors appears together on at least one edge. More precisely, a function c from a color set S to the set $V(G)$ of vertices of G such that for any pair of distinct colors; say, there is an edge (x, y) such that $c(x) = \alpha$, and $c(y) = \beta$.

A complete coloring is a pseudocomplete coloring which is also a proper vertex coloring, i.e. for any edge (x, y), $c(x) \neq c(y)$. The achromatic number

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$\psi(G)$ is the greatest number of colors in a complete coloring of G, and the pseudoachromatic number $\psi_s(G)$ is the greatest number of colors in a pseudo-complete coloring of G. If m is a positive integer, and $q(m)$ the greatest integer k such that $\binom{k}{2} \leq m$, then it is easy to see that for any graph G with m edges, $\psi_s(G) \leq q(m)$. Also, it follows immediately from the definitions that for any graph G; $\psi(G) \leq \psi_s(G)$, and consequently we have, for any G with m edges, $\psi(G) \leq \psi_s(G) \leq q(m)$: It is easy to find graphs for which ψ and ψ_s differ.

Graph coloring problem is expected to have wide variety of applications such as scheduling, frequency assignment in cellular networks, timetabling, crew assignment etc. Small Communication Time task systems show that the achromatic number of the co-comparability graph is an upper bound on the minimum number of processors [1].

2. Overview of the Paper

The achromatic number was introduced by Harary, Hedetniemi and Prins [11]. The survey articles by Hughes and MacGillivray and Edwards [5] contain huge collection of references of research papers related to achromatic problem. Computing achromatic number of a general graph was proved to be $NP-complete$ by Yannakakis and Gavril [16]. Farber et al. [8] show that the problem is NP- hard on bipartite graphs. It was further proved that the achromatic number problem remains NP- complete even for connected graphs which are both interval graphs and cographs simultaneously [3]. Cairnie and Edwards [4], Edwards and McDiarmid [6] show that the problem is NP- hard even on trees. Further, it is polynomially solvable for paths, cycles [5], union of paths [14] and [12] gives the approximation algorithm for the achromatic number problem on bipartite graphs. The parameter α in the α-approximation algorithm is called the $approximation\ ratio$. It is stated in [4] that "for achromatic numbers, there appear to be only a few results on special graphs apart from those for paths and cycles". Geller and Kronk [9] proved that there is almost optimal coloring for families of paths and cycles. This result was extended to bounded degree trees [6]. Roichman gives a lower bound on the achromatic number of Hypercubes [15].

The pseudo-achromatic number was first introduced by Gupta [10]. Later it was studied by Bhave [2]. Yegnanarayanan [17] determined pseudo-achromatic number for graphs such as cycles, paths, wheels, certain complete multipartite graphs, and for other classes of graphs. Hedetniemi [11] conjectured that the two parameters achromatic number and pseudo-achromatic number are equal for all trees which was disproved later [7]. The pseudo-achromatic problem is known to be NP-Complete even on restricted classes of graph Bodlaender [3], K. Edwards, C.McDiamid [6], Kortsarz et al. [13]. Kortsarz et al. [13] studied the approximability of the pseudo-achromatic number problem. It was proved in [13] that the problems have a randomized polynomial-time approximation algorithm which can be de-randomized in polynomial time. The pseudo-achromatic number problem was also considered from the extremal graph theoretic point of view on special classes of graphs [2, 17].

3. MAIN RESULTS

We begin with the following definition of H-graph.

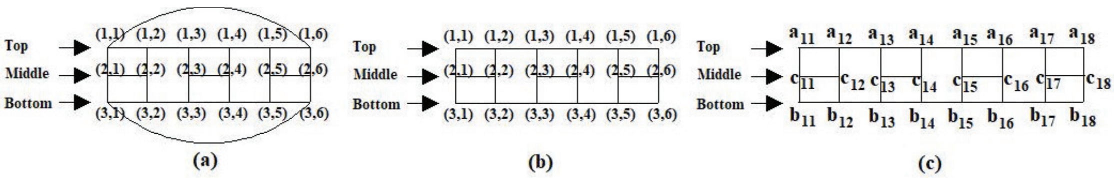

FIGURE 1. (a) $H(3)$ graph (b) $3H$-band (c) Definition of re-labeling for $4H$-band

Definition 3.1. *An H graph $H(r)$ is a 3-regular graph with vertex set $\{(i,j) : 1 \le i \le 3, 1 \le j \le n\}$ and edge set*

$$\left\{ \begin{array}{l} ((i,j),(i,j+1)), i = 1,3 \cup \{((2,j),(2,j+1)) : j \text{ odd } 1 \le j \le n-1\} \cup \\ \{((1,1),(1,n)),((3,1),(3,n))\} \cup \{((i,j),(i+1,j)), i = 1,2, 1 \le j \le n\} \end{array} \right\},$$

where $n = 2r$. See Figure 1(a).

Definition 3.2. *A tH-band is the subgraph obtained from H-graph $H(t)$ by removing edges $\{((1,1),(1,n)),((3,1),(3,n))\}, n = 2t$. See Figure 1(b).*

Notation: For convenience sake, we label the spine vertices in G_i as a_{i1}, $a_{i2}, \ldots,$ a_{in}, corresponding middle vertices as $c_{i1}, c_{i2}, \ldots, c_{in}$ and their bottom vertices as $b_{i1}, b_{i2}, \ldots, b_{in}$. See Figure 1(c).

The following algorithm gives the algorithm of any tH – band.

ALGORITHM $tH - band$:

(i) Find maximum s such that $s \leq \lceil \frac{-1+\sqrt{8t+1}}{2} \rceil$.
(ii) Partition the band into $D^k \cup D^{k-1} \cup \ldots \cup D^1$ where each D^k consists of k-layers of the band with $k = s - (i - 1), i = 1, 2, \ldots s$.

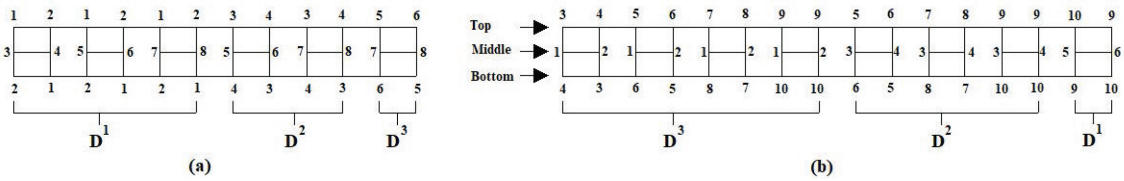

FIGURE 2. (a) Achromatic labelling of $6H$-band (b) Pseudo-Achromatic labelling of $5H$-band

Algorithm Achromatic tH – band:

For $0 \leq j \leq k - 1$, label D^{k-j} as follows:

Label the top and bottom vertices of D^{k-j} as $2j + 1$ and $2j + 2$ alternately. The remaining $2(k - j)$ middle vertices in D^{k-j} are labeled $2j + 3, 2j + 4, \ldots, 2j + [2(k - j) + 2]$ from left to right. Here $2j + [2(k - j) + 2] = 2k + 2$. See Figure 2(a).

Proof of correctness: Since vertices labeled $2j + 1$ and $2j + 2$ are adjacent in D^{k-j}, they in turn are adjacent to vertices labeled $2j + 3, \ldots, 2k + 2$. Since none of the adjacent vertices in D^{k-j} receive the same label, the labeling is proper. Therefore, the labeling of the vertices induces an achromatic labeling yielding the achromatic number for D^k to be at least $2k + 2$.
Using the fact that tH-band is an induced subgraph in H-graphs, the following results are obtained.

Theorem 3.1. *The* H*-graph* $H(r)$ *has* $\lfloor n/2 \rfloor$ *number of* H*-band where* $n = 2r$.

Proof. Every subgraph induced by vertices $a_{1i}, a_{1i+1}, b_{1i}, b_{1i+1}, c_{1i}, c_{1i+1}$ for $1 \leq i \leq n - 1$, i being odd gives a tH-band. See Figure 1(c). □

The following theorem is straight forward as the number of edges in H-graph is $9n$.

Theorem 3.2. *Let* $H(r)$ *be a* H*-graph of dimension* r*. There is an O(1)- approximation algorithm to determine the achromatic number of* H*-graph.*

Proof. The expected achromatic number for H-graph is $\frac{1\pm\sqrt{1+72r}}{2}$ and the lower bound realized is $\psi(G) \geq \lceil \frac{2}{3}\frac{1\sqrt{1+72r}}{2} \rceil$. This proves the theorem. □

Algorithm Pseudo - Achromatic tH – band:

Label the middle vertices of D^{k-1} as $2j+1$ and $2j+2$ from left to right. In D^k, the first $2(k-j)$ top and bottom vertices are labeled as $2j+3$ and $2j+4$ alternatively. In D^{k-1}, the first $2(k-j)$ top and bottom vertices are labelled alternately as $2j+3$ and $2j+4$, $2j+6$ and $2j+5$, $2j+7$ and $2j+8$ respectively. In D^{k-j}, $2 \leq j \leq k-1$, the first $2(k-j)$ top and bottom vertices are labeled alternately as $2j+6$ and $2j+5$. The last H of each D^{k-j}, where $k-j \neq 1$ is labeled such that the top and bottom vertices are colored by $2j+[2(k-j)+3]$ and $2j+[2(k-j)+4]$ respectively. See Figure 2(b). Here $2j+[2(k-j)+4] = 2(k+2)$.

Proof of correctness: Since vertices labelled $2j+1$ and $2j+2$ are adjacent in D^{k-j}, they in turn are adjacent to vertices labelled $2j+3, \ldots, 2(k+2)$. Since the last H of every D^{k-j}, receives the same label, the labeling is pseudo. Therefore, the labeling of the vertices induce a pseudo-achromatic labeling yielding the pseudo-achromatic number for D^k to be at least $2(k+2)$.

Using the fact that tH-band is an induced subgraph in H-graphs, the following results are obtained.

Theorem 3.3. *Let $H(r)$ be a H-graph of dimension r. There is an O(1)- approximation algorithm to determine the pseudo-achromatic number of H-graph.*

Proof. The expected pseudo-achromatic number for H-graph is $\frac{1\pm\sqrt{1+72r}}{2}$ and the lower bound realized is $\psi(G) \geq \lfloor \frac{3}{4}(\frac{1\pm\sqrt{1+72r}}{2}) - 3 \rfloor$. This proves the theorem. □

We extend the achromatic labelling and pseudo-achromatic labeling for H-graph to union of H-graphs.

Theorem 3.4. *Let $G = \cup_{i=1}^{s} G_i$ be the disjoint union of extended H-graphs $G_1, G_2, \ldots, G_s$. Then $\lceil \frac{2}{3}(\frac{1\pm\sqrt{1+72r}}{2}) \rceil \leq \psi(G) \leq (\frac{1\pm\sqrt{1+72r}}{2})$ also $\lfloor \frac{3}{4}(\frac{1\pm\sqrt{1+72r}}{2}) - 3 \rfloor \leq \psi_s(G) \leq (\frac{1\pm\sqrt{1+72r}}{2})$*

Proof. We extend the achromatic labeling for H-graph to union of H-graphs as in the case of *algorithm Achromaticto union* of tH - band and *algorithm pseudo-Achromaticto union* of tH - band. □

Theorem 3.5. *There is an $O(1)$- approximation algorithm to determine the achromatic number and pseudo-achromatic number of union of H-graphs.*

We conclude that $\lceil \frac{2}{3}(\frac{1\pm\sqrt{1+72r}}{2})\rceil \leq \psi(G) \leq \lfloor \frac{3}{4}(\frac{1\pm\sqrt{1+72r}}{2}) - 3\rfloor \leq \psi_s(G) \leq (\frac{1\pm\sqrt{1+72r}}{2})$.

4. CONCLUSION

In this paper, we present an $O(1)$ - approximation algorithm to determine the achromatic number and the pseudo-achromatic number of H-graph. Finding efficient approximation algorithms to determine the achromatic number and pseudo-achromatic number for other interconnection networks is quite challenging.

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