ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **9** (2020), no.10, 8807–8814
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.10.104

AVERAGE BETWEENNESS CENTRALITY OF CARTESIAN PRODUCT OF GRAPHS

SHINY JOSEPH[1] AND V. AJITHA

ABSTRACT. The betweenness centrality of a graph measures the tendency of a single vertex to be more central than all other vertices in the graph. In many real world situation it has quite a significant role. In this paper we study the average betweenness centrality of the cartesian product of two graphs and calculate the betweenness centrality of some cartesian product graphs .

1. INTRODUCTION

The common way to express the importance of network objects is to quantify it by evaluating a specific centrality index on the vertices of the graph representing a given network; where the vertices with the higher values of centrality are perceived as being more important [1]. The most frequently used centrality measures are vertex degree, closeness centrality, betweenness centrality and eigen vector centrality. Betweenness centrality plays an important role in analysis of social networks, computer networks and many other types of network data models [1, 6, 7]. It measures the extent to which a vertex lies on the shortest paths between pairs of other vertices [8].

Let $G(V(G), E(G))$ be a simple connected undirected graph with vertex set $V(G)$ and edge set $E(G)$, n and m denote the number of its vertices and edges

[1]*corresponding author*

2020 *Mathematics Subject Classification.* 05C76, 05C12.

Key words and phrases. betweennss centrality, betweenness-uniform graphs, cartesian product.

respectively. For any two vertices $u, v \in V(G)$, the distance $d(u,v)$ between u and v is the length of a shortest path between u and v in G. The eccentricity of a vertex u is the number $e(u) = \max\{d(u,v) : v \in V\}$. The maximum eccentricity of the vertices of G is called the diameter of G and is denoted by D.

The cartesian product of two graphs G and H, denoted $G\Box H$, is a graph with vertex set $V(G) \times V(H)$. The edge set of $G\Box H$ consists of all pairs $(u_1, v_1)(u_2, v_2)$ of vertices with $u_1u_2 \in E(G)$ and $v_1 = v_2$, or $u_1 = u_2$ and $v_1v_2 \in E(H)$ [6]. An automorphism of a graph G is a permutation φ of $V(G)$ with the property that uv is an edge if and only if $\varphi(u)\varphi(v)$ is an edge. The automorphism group of a graph G is transitive if there exists an automorphism φ to any pair u, v of vertices in G such that $\varphi(u) = v$. In this case G is called vertex transitive [5].

Definition 1.1. *[2]. Let $G(V,E)$ be a simple connected undirected graph with vertex set V. The betweenness centrality of a vertex $v \in V$ is given by*

$$B(v) = \sum_{s \neq v \neq t} \frac{\sigma_{st}(v)}{\sigma_{st}},$$

where σ_{st} is the number of shortest paths with vertices s and t as their end vertices and $\sigma_{st}(v)$ is the number of those shortest paths that include vertex v.

Definition 1.2. *[7] The average vertex betweenness centrality of a graph $G(V,E)$ of order n is defined as*

$$\bar{B}(G) = \frac{1}{n} \sum_{v \in V} B(v).$$

Since the process of counting the shortest path is very difficult as the network-size increases, the calculation of the betweenness centrality of a vertex is not simple . In betweenness uniform graphs the average betweenness of the graph gives the betweenness centrality of a vertex in the graph. The graph product is an algebraic method for constructing networks and are useful to designing networks in terms of small subgraphs. In cartesian product graphs the calculation of betweenness cenrality using factor graphs is useful. In this paper we calculate the betweenness centrality of the vertices of cycle ladder graph and hypercube graphs simply by using the cartesian product of graphs.

2. Average Betweenness centrality of the Cartesian product

In this section we present the average betweenness centrality of the cartesian product of two graphs.

Lemma 2.1. *[6] Let G and H be two connected graphs and let $(u, v), (u^{'}, v^{'})$ be vertices of $G\Box H$. Then*

$$d((u, v), (u^{'}, v^{'}) \mid G\Box H) = d((u, u^{'}) \mid G) + d((v, v^{'}) \mid H).$$

Theorem 2.1. *Let G be a graph of order n and H be a graph of order m . Then the average betweenness centrality of the cartesian product of G and H is given by*

$$\bar{B}(G\Box H) = m\bar{B}(G) + n\bar{B}(H) + \frac{(m-1)(n-1)}{2}.$$

Proof. Let $u_1, u_2, ..., u_n$ be the vertices of G and $v_1, v_2, ..., v_m$ be the vertices of H. The average betweenness centrality of G and H are

$$\bar{B}(G) = \frac{1}{n}\sum_{u_j \in V(G)} B(u_j) = \frac{1}{n}\sum_{u_j \in V(G)} \sum_{\substack{u_r \neq u_k \neq u_j \\ u_r, u_k \in V(G)}} \frac{\sigma_{u_r u_k}(u_j)}{\sigma_{u_r u_k}}$$

and

$$\bar{B}(H) = \frac{1}{m}\sum_{v_j \in V(H)} B(v_j) = \frac{1}{m}\sum_{v_j \in V(H)} \sum_{\substack{v_l \neq v_s \neq v_j \\ v_l, v_s \in V(H)}} \frac{\sigma_{v_l v_s}(v_j)}{\sigma_{v_l v_s}}.$$

The average betweenness centrality of the cartesian product $G\Box H$ is

$$\bar{B}(G\Box H) = \frac{1}{mn}\sum_{(u_i, v_j) \in V(G\Box H)} B(u_i, v_j).$$

By definition we have,

$$B((u_i, v_j)) = \sum_{\substack{(u_r, v_s) \neq (u_k, v_l) \neq (u_i, v_j) \\ (u_k, v_l), (u_r, v_s) \in V(G\Box H)}} \frac{\sigma_{(u_k, v_l)(u_r, v_s)}((u_i, v_j))}{\sigma_{(u_k, v_l)(u_r, v_s)}}.$$

Hence,

$$\bar{B}(G\Box H) = \frac{1}{mn}\sum_{(u_i, v_j) \in V(G\Box H)} \sum_{\substack{(u_r, v_s) \neq (u_k, v_l) \neq (u_i, v_j) \\ (u_k, v_l), (u_r, v_s) \in V(G\Box H)}} \frac{\sigma_{(u_k, v_l)(u_r, v_s)}((u_i, v_j))}{\sigma_{(u_k, v_l)(u_r, v_s)}}.$$

Now, for each pair $(u_k, v_l), (u_r, v_s)$ of vertices in $V(G\Box H)$, the summation taken over all elements in $V(G\Box H)$ except (u_k, v_l) and (u_r, v_s) of $\frac{\sigma_{(u_k,v_l)(u_r,v_s)}((u_i,v_j))}{\sigma_{(u_k,v_l)(u_r,v_s)}}$ gives the distance between (u_k, v_l) and (u_r, v_s) less 1. Hence,

$$\begin{aligned}
\bar{B}(G\Box H) &= \frac{1}{mn} \sum_{\substack{(u_r,v_s)\neq(u_k,v_l)\\(u_k,v_l),(u_r,v_s)\in V(G\Box H)}} \sum_{(u_i,v_j)\in V(G\Box H)\setminus\{(u_k,v_l),(u_r,v_s)\}} \frac{\sigma_{(u_k,v_l)(u_r,v_s)}((u_i,v_j))}{\sigma_{(u_k,v_l)(u_r,v_s)}}\\
&= \frac{1}{mn} \sum_{\substack{(u_r,v_s)\neq(u_k,v_l)\\(u_k,v_l),(u_r,v_s)\in V(G\Box H)}} (d((u_r,v_s),(u_k,v_l) \mid G\Box H) - 1)\\
&= \frac{1}{mn} \sum_{\substack{(u_r,v_s)\neq(u_k,v_l)\\(u_k,v_l),(u_r,v_s)\in V(G\Box H)}} (d((u_r,v_s),(u_k,v_l) \mid G\Box H) - \frac{1}{mn}\tbinom{mn}{2})
\end{aligned}$$

By Lemma 2.1, we get

$$\begin{aligned}
\bar{B}(G\Box H) &= \frac{1}{mn} \sum_{\substack{(u_r,v_s)\neq(u_k,v_l)\\(u_k,v_l),(u_r,v_s)\in V(G\Box H)}} (d((u_r,u_k) \mid G) + d((v_s,v_l) \mid H)) - \frac{1}{mn}\tbinom{mn}{2}\\
&= \frac{1}{mn} \sum_{\substack{(u_r,v_s)\neq(u_k,v_l)\\(u_k,v_l),(u_r,v_s)\in V(G\Box H)}} d((u_r,u_k) \mid G)\\
&\quad + \frac{1}{mn} \sum_{\substack{(u_r,v_s)\neq(u_k,v_l)\\(u_k,v_l),(u_r,v_s)\in V(G\Box H)}} d((v_s,v_l) \mid H) - \frac{mn-1}{2}\\
&= \frac{m^2}{mn}(\sum_{u_j\in V(G)} B(u_j) + \tbinom{n}{2}) + \frac{n^2}{mn}(\sum_{v_j\in V(H)} B(v_j) + \tbinom{m}{2}) - \frac{mn-1}{2}\\
&= m(\bar{B}(G) + \frac{n-1}{2}) + n(\bar{B}(H) + \frac{m-1}{2}) - \frac{mn-1}{2}\\
&= m\bar{B}(G) + n\bar{B}(H) + \frac{(m-1)(n-1)}{2}.
\end{aligned}$$

□

Corollary 2.1. *The average betweenness centrality of rook's graph is given by*

$$\bar{B}(K_m\Box K_n) = \frac{(m-1)(n-1)}{2}.$$

Proof. By Theorem 2.1, the average betweenness centrality of $\bar{B}(K_m \Box K_n)$ is given by

$$\bar{B}(K_m \Box K_n) = n\bar{B}(K_m) + m\bar{B}(K_n) + \frac{(m-1)(n-1)}{2} = \frac{(m-1)(n-1)}{2}.$$

□

Corollary 2.2. *The average betweenness centrality of ladder graph is given by* $\bar{B}(K_2 \Box P_n) = \frac{1}{2n} \sum_{k=1}^{n} [n(4k-3) - (2k-1)^2]$.

Proof. By Theorem 2.1, the average betweenness centrality of $\bar{B}(K_2 \Box P_n)$ is given by,

$$\begin{aligned}
\bar{B}(K_2 \Box P_n) &= n\bar{B}(K_2) + 2\bar{B}(P_n) + \frac{(n-1)}{2} \\
&= 2\frac{1}{n}\sum_{k=1}^{n}(k-1)(n-k) + \frac{(n-1)}{2} \\
&= \frac{2}{n}\sum_{k=1}^{n}[(k-1)(n-k) + \frac{(n-1)}{4}] \\
&= \frac{1}{2n}\sum_{k=1}^{n}[n(4k-3) - (2k-1)^2].
\end{aligned}$$

□

Corollary 2.3. *The average betweenness centrality of grid graph is given by*

$$\bar{B}(P_m \Box P_n) = \frac{n}{m}\sum_{k=1}^{m}(k-1)(m-k) + \frac{m}{n}\sum_{k=1}^{n}(k-1)(n-k) + \frac{(m-1)(n-1)}{2}.$$

3. Some Results on betweenness uniform Graphs

In this section we present the cartesian product of betweenness uniform graphs and using this result calculate the betweenness centrality of some graphs.

Definition 3.1. *[4] Graphs with vertices having the same betweenness centrality are called betweenness uniform graphs.*

Theorem 3.1. *If $G\Box H$ is the cartesian product of two betweenness uniform graphs G and H with order n and m respectively. If $G\Box H$ is a betweenness uniform graph, then the betweenness centrality of a vertex $(u,v)\in V(G\Box H)$ is given by*

$$B((u,v)) = mB(u) + nB(v) + \frac{(m-1)(n-1)}{2}.$$

Theorem 3.2. *[8] The betweenness centrality of a vertex in a cycle C_n is given by*

$$B(v) = \begin{cases} \frac{(n-2)^2}{8}, & \text{if } n \text{ is even} \\ \frac{(n-1)(n-3)}{8}, & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 3.3. *[5] A cartesian product of connected graphs is vertex transitive if and only if every factor is vertex transitive.*

Following theorem can be proved by using Theorem 3.1.

Theorem 3.4. *[8] The betweenness centrality of a vertex v in a circular ladder graph CL_n is*

$$B(v) = \begin{cases} \frac{(n-1)^2+1}{4}, & \text{if } n \text{ is even} \\ \frac{(n-1)^2}{4}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. The circular laddergraph CL_n is a 3-regular simple graph isomorphic to the cartesian product $K_2\Box C_n$. Since it is a vertex tranitive graph it is betweenness uniform [3]. Hence, by Theorem 3.1,

$$\begin{aligned} B((u,v)) &= mB(u) + nB(v) + \frac{(m-1)(n-1)}{2} \\ &= \begin{cases} 0 + \frac{2(n-2)^2}{8} + \frac{(n-1)}{2}, & \text{if } n \text{ is even} \\ 0 + \frac{2(n-1)(n-3)}{8} + \frac{(n-1)}{2}, & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Hence, $B(v) = \frac{(n-1)^2+1}{4}$, if n is even, and $B(v) = \frac{(n-1)^2}{4}$, if n is odd. □

Theorem 3.5. *The betweenness centrality of a vertex (u,v) in the graph $K_n\Box K_n$ is given by $B((u,v)) = \frac{(n-1)^2}{2}$.*

Proof. Since $K_n\Box K_n$ is a betweenness uniform graph

$$B((u,v)) = \bar{B}(K_n\Box K_n) = n\bar{B}(K_n) + n\bar{B}(K_n) + \frac{(n-1)(n-1)}{2} = \frac{(n-1)^2}{2}.$$

□

Following theorem can be proved by using Theorem 3.1.

Theorem 3.6. *[8] The betweenness centrality of a vertex* (u, v) *in a hypercube graph* Q_n *is given by* $B((u, v)) = (n-2)2^{(n-2)} + \frac{1}{2}$.

Proof. Hypercube graphs are distance regular graphs and hence they are betweenness uniform [3]. Also Q_n is isomorphic to $K_2 \Box Q_{n-1}$. Hence

$$\begin{aligned}
&B((u, v)) = \bar{B}(Q_n) = \bar{B}(K_2 \Box Q_{n-1}) \\
=&2\bar{B}(Q_{n-1}) + 2^{(n-1)}\bar{B}(K_2) + \frac{(2-1)(2^{(n-1)}-1)}{2} \\
=&2\bar{B}(Q_{n-1}) + \frac{2^{(n-1)}-1}{2}
\end{aligned}$$

and

$$\begin{aligned}
\bar{B}(Q_{n-1}) &= \bar{B}(K_2 \Box Q_{n-2}) = 2\bar{B}(Q_{n-2}) + 2^{(n-2)}\bar{B}(K_2) \\
&\quad + \frac{(2-1)(2^{(n-2)}-1)}{2} = 2\bar{B}(Q_{n-2}) + \frac{2^{(n-2)}-1}{2} \\
\bar{B}(Q_1) &= \bar{B}(K_2) = 0
\end{aligned}$$

and $\bar{B}(Q_2) = \bar{B}(K_2 \Box K_2) = \frac{1}{2}$. Thus,

$$\begin{aligned}
\bar{B}(Q_n) &= \frac{2^{n-1}-1}{2} + 2 \times \frac{2^{n-2}-1}{2} + 2^2 \times \frac{2^{n-3}-1}{2} + \cdots \\
&\quad + 2^{n-3} \times \frac{2^2-1}{2} + 2^{n-2} \times \frac{2^1-1}{2} \\
&= (n-2)2^{n-2} + \frac{1}{2}
\end{aligned}$$

□

Theorem 3.7. *[8] The betweenness centrality of a vertex in a complete bipartite graph* $K_{n,n}$ *is given by ,* $B(u) = \frac{(n-1)}{2}$.

Theorem 3.8. *The betweenness centrality of a vertex* (u, v) *in a graph* $K_{n,n} \Box K_{n,n}$ *is given by,* $B((u, v)) = 4n(n-1) + \frac{1}{2}$.

Proof. Since $K_{n,n}\square K_{n,n}$ is a vertex transitive graph, it is betweenness uniform. By Theorem 3.1,

$$\begin{aligned} B(u,v) &= \bar{B}(K_{n,n}\square K_{n,n}) \\ &= 2n\bar{B}(K_{n,n}) + 2n\bar{B}(K_{n,n}) + \frac{(2n-1)(2n-1)}{2} \\ &= \frac{2n(n-1)}{2} + \frac{2n(n-1)}{2} + \frac{(2n-1)^2}{2} = 4n(n-1) + \frac{1}{2}. \end{aligned}$$

□

References

[1] M. Dehmer, V. Kraus, F. Emmert-Streib, S. Pickl: *Quantitative graph theory*, CRC press, 2014.

[2] L. C. Freeman: *A set of measures of centrality based on betweenness*, Sociometry, **40**(1) (1977), 35–41.

[3] S. Gago, J. Coroničová Hurajová, T. Madaras: *On betweenness-uniform graphs*, Czechoslovak mathematical journal, **63**(3) (2013), 629–642.

[4] J. Coroničová Hurajová, T. Madaras: *More on betweenness-uniform graphs*, Czechoslovak Mathematical Journal, **68**(2) (2018),293–306.

[5] R. Hammack, W. Imrich, Wilfried S. Klavžar: *Handbook of product graphs*, CRC Press, 2011.

[6] W. Imrich, S. Klavzar, D. F. Rall: *Topics in graph theory: Graphs and their Cartesian product*, AK Peters/CRC Press, 2008.

[7] R. Škrekovski, I. Gutman: *Vertex version of the Wiener theorem*, MATCH Commun. Math. Comput. Chem, **72**(1) (2014), 295–300.

[8] R. Unnithan, S. Kumar, B. Kannan, M. Jathavedan: *Betweenness centrality in some classes of graphs*, International Journal of Combinatorics, **2014** (2014), Article ID 241723.

Department of Mathematics
Mathematic Research centre
Mary Matha Arts and Science College, Mananthavady
Email address: shinyjoseph314@gmail.com

Department of Mathematics
University of Kannur
Mahatma Gandhi College Iritty, Kannur, Kerala, India
Email address: avmgc10166@yahoo.co