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THE SKEW DISTANCE CHARACTERISTIC POLYNOMIAL OF AN ORIENTED TREE WITH CANONICAL ORIENTATION AND AN ORIENTED EVEN CIRCUITS WITH REACHABLE AND CANONICAL ORIENTATION

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ABSTRACT. A directed graph G^ϕ is a finite simple undirected graph G with an orientation ϕ, which assigns to each edge a direction so that G^ϕ becomes a directed graph. G is called the underlying graph of G^ϕ and we denote by $SD(G^\phi)$, the Skew-Distance matrix of G^ϕ. The eigen values $\lambda_1, \lambda_2 \cdots, \lambda_n$ of the $SD(G^\phi)$ are said to be the skew distance eigen values or the SD-Eigen values of G^ϕ. The Skew Distance Energy, $E_{SD}(G^\phi) = \sum_{i=1}^{n} |\lambda_i - \overline{\lambda}|$, where $\overline{\lambda} = \left[\frac{\lambda_1+\lambda_2+\cdots+\lambda_n}{n}\right]$. In this paper, we find the coefficients of the skew distance characteristic polynomial of a oriented tree T with canonical orientation and an oriented even circuit C_{2n} with reachable and canonical orientation.

1. INTRODUCTION

Let G be a finite simple connected graph with n vertices and m edges. Let G^ϕ be a graph with an orientation ϕ, which assigns to each edge of G a direction so that G^ϕ becomes a directed graph. The skew adjacency matrix of the directed graph G^ϕ is the $n \times n$ matrix, $S(G^\phi) = (S_{ij})$, where $S_{ij} = 1 = -S_{ji}$ if $v_i \to v_j$ is an arc of G^ϕ, otherwise $S_{ij} = S_{ji} = 0$.

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Let d_{ij}^{ϕ} be the distance between the vertices v_i and v_j in G^{ϕ}. The Skew Distance Matrix (SD-matrix) of G^{ϕ}, $SD(G^{\phi}) = (Sd_{ij})$ is real skew symmetric matrix, where

$$Sd_{ij} = \begin{cases} d_{ij}^{\phi} & \text{if } d_{ij}^{\phi} \leq d_{ji}^{\phi} \\ -d_{ji}^{\phi} & \text{if } d_{ij}^{\phi} > d_{ji}^{\phi} \\ 0 & \text{if no path between } v_i \text{ and } v_j \end{cases}$$

and $Sd_{ij} = -Sd_{ji}$.

Suppose there is only one path from v_i to v_j or v_j to v_i, then $Sd_{ij} = d_{ij}^{\phi}$ or $Sd_{ij} = -d_{ji}^{\phi}$.

Let $G(V, E)$ be a bipartite graph with bi-partition (X, Y). An orientation is said to be **canonical** if it orients all the edges from one partition set to the other. It is immaterial if it is from X to Y or from Y to X.From this point onwards, σ stands for the canonical orientation with respect to a bipartite graph G with a fixed bi-partition (X,Y).

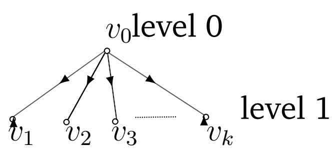
Let G be a finite simple connected graph with n vertices and m edges. An orientation is said to be a **reachable orientation** in P_n if all the edges in P_n are in one direction. An orientation is said to be a reachable orientation in G if any two vertices in G has at least one path with reachable orientation. Reachable orientation is denoted by R.

In this paper we find the skew distance characteristic polynomial of an oriented tree T having vertices upto height 3 with canonical orientation and an oriented even circuit C_{2n} with reachable orientation R and canonical orientation.

2. Tree

Theorem 2.1. *Let T_{k+1}^{σ} be a canonical oriented tree with $(k+1)$ vertices in which k vertices are of level 1 and one vertex is in level zero. Then $P[SD(T_{k+1}^{\sigma}); x] = (-x)^{k+1} + k(-x)^{k-1} \forall\, k \geq 4$.*

Proof. T_{k+1}^{σ} is



$$SD(T^{\sigma}_{k+1}) = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{(k+1)\times(k+1)} . \tag{2.1}$$

By [2], Theorem 3.10,

$$P[SD(T^{\sigma}_{k+1}); x] = (-x)^{k+1} + k(-x)^{k-1} \forall\ k \geq 4. \tag{2.2}$$

□

Theorem 2.2. *Let* T^{σ}_n *be a canonical oriented tree with 1 vertex is of level 0,* k *vertices are of level one and* $n_i (i = 1, 2, \dots, k)$ *vertices are of level two s.t* $n = 1 + k + n_1 + n_2 + \cdots + n_k$. *Then*

$$\begin{aligned} P[SD(T^{\sigma}_n); x] &= (-x)^{n_1+n_2+\cdots+n_k+(k+1)} \\ &+[k + \sum_{i=1}^{k} n_i](-x)^{n_1+n_2+\cdots+n_k+(k-1)} \\ &+[(k-1)\sum_{i=1}^{k} n_i + \sum_{i\neq j=1}^{k} n_i n_j](-x)^{n_1+n_2+\cdots+n_k+(k-3)} \\ &+[(k-2)\sum_{i\neq j=1}^{k} n_i n_j + \sum_{i\neq j\neq t=1}^{k} n_i n_j n_t](-x)^{n_1+n_2+\cdots+n_k+(k-5)} \\ &+\cdots+[2\sum n_1 n_2 \cdots n_{k-2} + \sum n_1 n_2 \cdots n_{k-1}](-x)^{n_1+n_2+\cdots+n_k-(k-3)} \\ &+[\sum n_1 n_2 \cdots n_{k-1} + n_1 n_2 \cdots n_k](-x)^{n_1+n_2+\cdots+n_k-(k-1)}. \end{aligned}$$

Proof. Let T^{σ}_n be a canonical oriented tree with 1 vertex is of level 0, k vertices are of level one and $n_i (i = 1, 2, \dots, k)$ vertices are of level two s.t $n = 1 + k + n_1 + n_2 + \cdots + n_k$. Then

(2.3)
$$|SD(T_n^{\sigma}) - xI| = \begin{vmatrix} S_{k+1} & V_1 & V_2 & \cdots & V_k \\ -V_1^T & D_{n_1} & 0 & \cdots & 0 \\ -V_2^T & 0 & D_{n_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -V_k^T & 0 & 0 & \cdots & D_{n_k} \end{vmatrix}$$

Here,

$$|S_{k+1}| = \begin{vmatrix} -x & 1 & 1 & \cdots & 1 \\ -1 & -x & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & 0 & 0 & \cdots & -x \end{vmatrix}_{(k+1)\times(k+1)} \qquad |D_{n_i}| = \begin{vmatrix} -x & 0 & 0 & \cdots & 0 \\ 0 & -x & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & -x \end{vmatrix}_{n_i\times n_i};$$

$$V_i = \begin{matrix} \\ \\ \\ i \\ (i+1) \\ (i+2) \\ \\ \\ \end{matrix} \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ -1 & -1 & -1 & \cdots & -1 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{(k+1)\times n_i} ; 1 \le i \le k.$$

Let $S_{k+1}^{(i)}, i = 1, 2, \dots, k$ be the matrix obtained by deleting any i number of rows from the rows 2nd, 3rd, . . . (k+1)th row of S_{k+1} and replacing i number of rows of the corresponding matrices $-V_1^T, -V_2^T, \dots, -V_k^T$ respectively. [Delete the j^{th} row $2 \le j \le k+1$ of S_{k+1} and replacing any one row from $-V_{j-1}^T \forall j$].

For $1 \le i \le k, D'_{n_i}$ be the matrix obtained from D_{n_i} by replacing any one row by $(-1, -1, \dots, -1)_{1\times n_i}$.

By Laplace's expansion,

$$|SD(T_n^{\sigma}) - xI| = |S_{k+1}||D_{n_1}| \cdots |D_{n_k}| + |S_{k+1}^{(1)}| \sum_{i=1}^{k} n_i |D_{n_1}| \cdots |D'_{n_i}| \cdots |D_{n_k}|$$

$$+ |S_{k+1}^{(2)}| \sum_{i \ne j = 1}^{k} n_i n_j |D_{n_1}| \cdots |D'_{n_i}| \cdots |D'_{n_j}| \cdots |D_{n_k}| + \cdots + |S_{k+1}^{(k-1)}| \sum n_1 n_2$$

$$\cdots n_{k-1}|D'_{n_1}|\cdots|D'_{n_{(k-1)}}||D_{n_k}| + |S^{(k)}_{k+1}|n_1n_2\cdots n_k|D'_{n_1}|\cdots|D'_{n_k}|$$
$$= (-x)^{n_1+n_2+\cdots+n_k+(k+1)} + [k+\sum_{i=1}^{k} n_i](-x)^{n_1+n_2+\cdots+n_k-(k-1)}$$
$$+[(k-1)\sum_{i=1}^{k} n_i + \sum_{i\neq j=1}^{k} n_in_j](-x)^{n_1+n_2+\cdots+n_k+(k-3)}[(k-2)\sum_{i\neq j=1}^{k} n_in_j +$$
$$\sum_{i\neq j\neq t=1}^{k} n_in_jn_t](-x)^{n_1+n_2+\cdots+n_k+(k-5)} + \cdots + [2\sum n_1n_2\cdots n_{k-2}$$
$$+\sum n_1n_2\cdots n_{k-1}](-x)^{n_1+n_2+\cdots+n_k-(k-3)}$$
$$+[\sum n_1n_2\cdots n_{k-1} + n_1n_2\cdots n_k](-x)^{n_1+n_2+\cdots+n_k-(k-1)} \qquad (*)$$

Hence the proof. □

Problem 2.1. *Find the skew distance characteristic polynomial of a canonical oriented tree* T_n^σ *with 1 vertex is of level* **0**, *k vertices are of level* **1**, $n_i (1 \leq i \leq k)$ *vertices are of level* **2** *and* $11r_1, 12r_2, \ldots, 1n_1r_{n_1}, 21s_1, 22s_2, \ldots, 2n_2s_{n_2}, \ldots, k1t_1, k2t_2, \ldots, kn_kt_{n_k}$ *number of vertices are of level* **3** *such that* $n = (1) + (k) + (n_1 + n_2 + \cdots + n_k) + (11r_1 + 12r_2 + \cdots + n_1r_{n_1} + 21s_1 + 22s_2 + \cdots + 2n_2s_{n_2} + \cdots + k1t_1 + k2t_2 + \cdots + kn_kt_{n_k})$.

Let T_1 be the characteristic matrix of the skew distance matrix of the canonical oriented tree with upto level **one** vertices.

Let T_2 be the characteristic matrix of the skew distance matrix of the canonical oriented tree with upto level **two** vertices.

Let $T_2^{(i)}$ be the matrix oriented from T_2 by deleting any i number of rows from r_{k+2} to $r_{k+1+n_1+n_2+\cdots+n_k}$ of T_2 and replacing i number of rows from the corresponding matrices $-V^T_{11r_1}, -V^T_{12r_2}, \ldots, -V^T_{1n_1r_{n_1}}, -V^T_{21s_1}, -V^T_{22s_2}, \ldots, -V^T_{2n_2s_{n_2}}, \ldots -V^T_{k1t_1}, -V^T_{k2t_2}, \ldots, -V^T_{kn_kt_{n_k}}$ respectively.

For $1 \leq i \leq k$; $1 \leq j \leq n_i$ D'_{ijl} be the diagonal matrix obtained from the diagonal matrix D_{ijl} by replacing any one row by $(1, 1, \ldots, 1)_{1\times l}$ where l takes the values

$r_1, r_2, \ldots, r_{n_1}, s_1, s_2, \ldots, s_{n_2}, \ldots, t_1, t_2, \ldots, t_{n_k}$.

Let

$$T_1 = \begin{vmatrix} -x & 1 & 1 & \cdots & 1 \\ -1 & -x & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & 0 & 0 & \cdots & -x \end{vmatrix}_{(k+1)\times(k+1)}$$

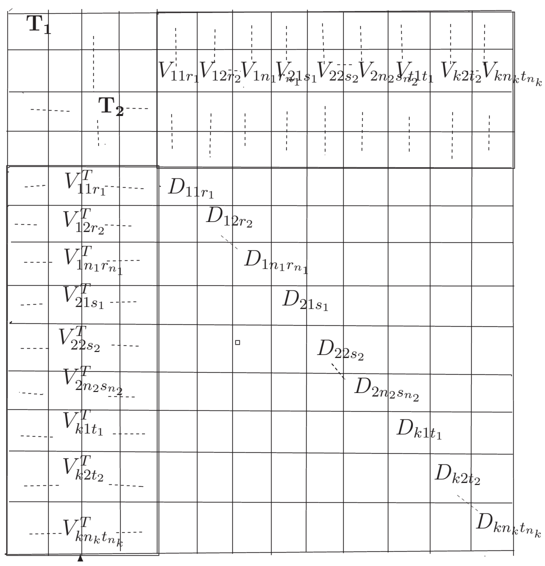
and T_2 = L. H.S of (2.3) in matrix form.

For $1 \leq i \leq k$; $1 \leq j \leq n_i$ and $r_1 \leq l \leq r_{n_1}$, $s_1 \leq l \leq s_{n_2}, \ldots, t_1 \leq l \leq t_{n_k}$.

$$V_{ijl}(i=1; j=1; l=r_1) = \begin{matrix} 0 \\ \vdots \\ (k+1) \\ 11 \\ \vdots \\ 1n_1 \\ \vdots \\ k1 \\ \vdots \\ kn_k \end{matrix} \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ \hline 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ \hline \vdots & \vdots & \vdots & & \vdots \\ \hline 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Then

$$|SD(T_n^{\sigma}) - xI| =$$



By Laplace's Expansion, we get the proof.

3. ORIENTED EVEN CIRCUIT

Theorem 3.1. *Let C_{2n}^R be an oriented even circuit with $2n$ vertices and with an orientation $v_1 \to v_2 \to v_3 \to \cdots \to v_{2n-2} \to v_{2n-1} \to v_{2n} \to v_1$. Let $P[SD(C_{2n}^R); x] = C_0x^{2n}+C_2x^{2n-2}+\cdots+C_{2n-2}x^2+C_{2n}$ be the skew distance characteristic polynomial of C_{2n}^R. Then*

(i) $C_0 = 1$;
(ii) $C_2 = \sum\limits_{i<j} sd_{ij}^2 = \frac{n^2(2n^2+1)}{3}$;
(iii) $C_4 = \sum$ *All minors of order* $4(sd_{12}sd_{34} + sd_{13}sd_{42} + sd_{14}sd_{23})^2$;
(iv) $C_{2r} =$ *sum of determinant of the principal minors of order* $(2n - 2r)$;
(v) $C_{2n} = (2n)^{2n-2}$.

Proof. Denote the given reachable orientation $v_1 \to v_2 \to \cdots v_{2n} \to v_1$ by R. Now

$SD(C_{2n}^R) =$

	v_1	v_2	v_3	$\cdots$	v_{n-1}	v_n	v_{n+1}	$\cdots$	v_{2n-2}	v_{2n-1}	v_{2n}
v_1	0	1	2	$\cdots$	$(n-2)$	$(n-1)$	n	$\cdots$	-3	-2	-1
v_2	-1	0	1	$\cdots$	$n-3$	$n-2$	$n-1$	$\cdots$	-4	-3	-2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_n	$-(n-1)$	$-(n-2)$	$-(n-3)$	$\cdots$	-1	0	1	$\cdots$	$n-2$	$n-1$	n
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_{2n-1}	2	3	4	$\cdots$	$-n$	$-(n-1)$	$-(n-2)$	$\cdots$	-1	0	1
v_{2n}	1	2	3	$\cdots$	$n-1$	$-n$	$-(n-1)$	$\cdots$	-2	-1	0

$P[SD(C_{2n}^R); x] = |SD(C_{2n}^R) - xI| =$

	v_1	v_2	v_3	$\cdots$	v_{n-1}	v_n	v_{n+1}	$\cdots$	v_{2n-2}	v_{2n-1}	v_{2n}
v_1	-x	1	2	$\cdots$	$n-2$	$n-1$	n	$\cdots$	-3	-2	-1
v_2	-1	-x	1	$\cdots$	$n-3$	$n-2$	$n-1$	$\cdots$	-4	-3	-2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_n	$-(n-1)$	$-(n-2)$	$-(n-3)$	$\cdots$	-1	-x	1	$\cdots$	$n-2$	$n-1$	n
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_{2n-1}	2	3	4	$\cdots$	$-n$	$-(n-1)$	$-(n-2)$	$\cdots$	-1	-x	1
v_{2n}	1	2	3	$\cdots$	$n-1$	$-n$	$-(n-1)$	$\cdots$	-2	-1	-x

Let $D_1^{(2n)} = |SD(C_{2n}^R)|$ and $D_2^{(2n)} = \begin{vmatrix} -x & 0 & \cdots & 0 \\ 0 & -x & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{vmatrix}_{2n\times 2n}$. Then By 22 Chapter IV in [1],

$$\begin{aligned} &|SD(C_{2n}^R) - xI| = (D_1 + D_2)^{2n} \\ &= D_1^{(2n)} + \sum D_1^{(2n-1)} D_2^{(1)} + \sum D_1^{(2n-2)} D_2^{(2)} + \cdots + D_2^{(2n)} \\ &= D_1^{(2n)} + \sum D_1^{(2n-2)} D_2^{(2)} + \cdots + D_2^{(2n)}, \end{aligned} \tag{3.1}$$

as $SD(C_{2n}^R)$ is skew symmetric. Further,

$$D_2^{(1)} = -x, D_2^{(2)} = (-x)(-x), \ldots, D_2^{(2n)} = (-x)(-x)\cdots(-x) = (-1)^{2n}x^{2n} = x^{2n}.$$

The corresponding minors $D_1^{(2n-1)}, D_1^{(2n-2)}, \ldots$ are got by erasing in $D_1^{(2n)}$ the i^{th} row and column, the i^{th} and k^{th} rows and columns, and etc.

Thus

$$\begin{aligned} |SD(C_{2n}^R) - xI| = x^{2n} + x^{2n-2}\sum D_1^{(2n-(2n-2))} + \cdots + x^{2n-2r}\sum D_1^{(2n-(2n-2r))} \\ + \cdots + D_1^{(2n)}. \end{aligned}$$

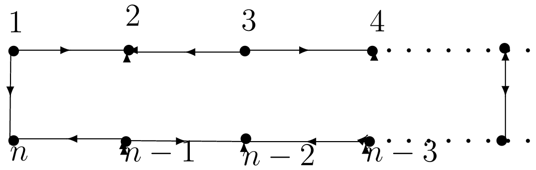
By Theorem 3.1 in [2], $C_0 = 1$ and By (3.1) , and by Theorem 3.1 in [2],

$$\begin{aligned} C_2 &= \sum D_1^{[2n-(2n-2)]} = \sum_{i<j} Sd_{ij}^2 \\ &= (2n-1)1^2 + (2n-2)2^2 + (2n-3)3^2 + \cdots + [2n-(n-1)](n-1)^2 \\ &\quad + (2n-n)n^2 + [2n-(n+1)][-(n-1)^2] + \cdots + [2n-(2n-1)](-1)^2 \\ &= 2n1^2 + \cdots + 2n(n-1)^2 + n^3 = \frac{n^2(2n^2+1)}{3}. \end{aligned}$$

By applying a sequence of elementary congredient row operations, column operation and rearranging rows and columns, we get the proof □

Theorem 3.2. *Let C_n^σ be the canonical oriented cycle with n vertices and n is even. Then $E_{SD}(C_n^\sigma) = 2\sqrt{n}$.*

Proof. C_n^σ is



$$SD(C_n^\sigma) = \begin{array}{c|cccccccc} & 1 & 2 & 3 & 4 & \cdots & n-2 & n-1 & n \\ \hline 1 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 2 & -1 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ n & -1 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 \end{array}$$

As rank $(SD(C_n^\sigma)) = 2$, we have two independent solutions. As n is even C_n^σ is bipartite. Therefore, the skew distance chatracteristic polynomial of C_n^σ is of the form $C_0x^n + C_2x^{n-2}$. As $C_0 = 1$ and $C_2 = \sum\limits_{i<j} Sd_{ij}^2 = n$.

Hence, the skew distance characteristic polynomial of C_n^σ is $x^n + nx^{n-2}$.

Its spectrum is

$$\begin{pmatrix} \sqrt{n}i & 0 & \sqrt{n}i \\ 1 & n-2 & 1 \end{pmatrix}$$

Hence, $E_{SD}(C_n^\sigma) = |-\sqrt{n}i| + |\sqrt{n}i| = 2\sqrt{n}$. □

REFERENCES

[1] R. F. SCOTT, G. B. MATHEWS, *The Theory of Determinants, Cambridge University Press*, 1904.

[2] A. WILSON BASKAR, M. DHANASEKARAN, *On the skew distance energy of a digraph*, Proceedings of the National Conference on Present Scenarion in Graph Theory, 2016, 90-107.

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