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LAPLACIAN MINIMUM DOMINATING EXTENDED ENERGY

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ABSTRACT. In this paper we introduce the concept of Laplacian minimum dominating extended energy of a graph $E^D_{ex}(G)$ and computed Laplacian minimum dominating extended energies of star graph, complete graph, bipartite graph, crown graph, cocktail party graph and friendship graphs. Upper and lower bounds for $E^D_{ex}(G)$ are also established.

1. INTRODUCTION

The concept of energy of a graph was introduced by I. Gutman [5] in the year 1978. Let G be a graph with n vertices and m edges and let $A = (a_{ij})$ be the adjacency matrix of the graph. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A, assumed in non increasing order, are the eigenvalues of the graph G. As A is real symmetric, the eigenvalues of G are real with sum equal to zero. The energy $E(G)$ of G is defined to be the sum of the absolute values of the eigenvalues of G. i.e.,$E(G) = \sum_{i=1}^{n} |\lambda_i|$. I. Gutman and B. Zhou [4] defined the Laplacian energy of a graph G in the year 2006. Rajesh Kanna et al. [8] defined minimum dominating energy of a graph. Recently K.C. Das et al. [3] defined extended energy of a graph. Motivated by these papers the present authors defined Laplacian minimum dominating extended energy of a graph.

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1.1. **Laplacian minimum dominating extended energy.** Let G be a simple graph of order n with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set E. A subset D of V is called a dominating set of G if every vertex of V-D is adjacent to some vertex in D. Any dominating set with minimum cardinality is called a minimum dominating set. Let D be a minimum dominating set of a graph G. The Laplacian minimum dominating extended matrix of G is the $n \times n$ matrix defined by $L^D_{ex}(G) := (x_{ij})$, where

$$x_{ij} = \begin{cases} -\frac{1}{2}(\frac{d_i}{d_j} + \frac{d_j}{d_i}) & \text{if } \ v_iv_j \in E(G) \\ 1 & \text{if } \ i = j \text{ and } v_i \in D. \\ 0 & \text{otherwise} \end{cases}$$

Here, d_i is the degree of the vertex v_i. Let $\rho_1, \rho_2, \ldots, \rho_n$ be the Laplacian minimum dominating extended eigenvalues of G. The Laplacian minimum dominating extended energy of G is defined as $E^D_{ex}(G) : = \sum_{i=1}^{n} \left|\rho_i - \frac{2m}{n}\right|$, where n is the number of vertices and m is the number of edges of a graph G.

Example 1. *The possible minimum dominating sets for the following graph G (Fig. 1) are:*

(*i*) $D_1 = \{v_2, v_4\}$;
(*ii*) $D_2 = \{v_3, v_4\}$.

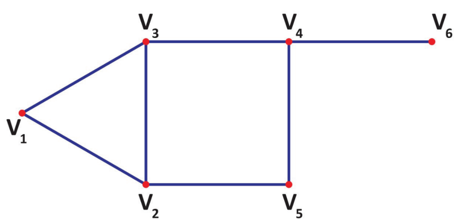

FIGURE 1

$$\text{(i) } A^{D_1}_{ex}(G) = \begin{pmatrix} 0 & 13/2 & 13/2 & 0 & 0 & 0 \\ 13/2 & 1 & 1 & 0 & 13/2 & 0 \\ 13/2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 13/12 & 5/3 \\ 0 & 13/2 & 0 & 13/2 & 0 & 0 \\ 0 & 0 & 0 & 5/3 & 0 & 0 \end{pmatrix}$$

$$D_1(G) = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \; L_{ex}^{D_1}(G) = D_1(G) - A_{ex}^{D_1}(G).$$

Laplacian minimum dominating extended energy is $E_{ex}^{D_1}(G) \approx 9.853989486478543$.

(ii) $$A_{ex}^{D_2}(G) = \begin{pmatrix} 0 & 13/2 & 13/2 & 0 & 0 & 0 \\ 13/2 & 0 & 1 & 0 & 13/2 & 0 \\ 13/2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 13/12 & 5/3 \\ 0 & 13/2 & 0 & 13/2 & 0 & 0 \\ 0 & 0 & 0 & 5/3 & 0 & 0 \end{pmatrix}$$

$$D_2(G) = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \; L_{ex}^{D_2}(G) = D_2(G) - A_{ex}^{D_2}(G).$$

Laplacian minimum dominating extended energy is $E_{ex}^{D_2}(G) \approx 9.82473787603896$. *Therefore, Laplacian minimum dominating extended energy depends on the dominating set.*

2. Properties of Laplacian minimum dominating extended eigenvalues

Theorem 2.1. *Let G be a simple graph with vertex set* $V = \{v_1, v_2, ..., v_n\}$*, edge set* E *and* $D = \{u_1, u_2, ..., u_k\}$ *be a minimum dominating set. If* $\rho_1, \rho_2, ..., \rho_n$ *are the eigenvalues of Laplacian minimum dominating extended matrix* $L_{ex}^D(G)$ *then*

(i) $\sum_{i=1}^n \rho_i = 2|E| - |D|$.

(ii) $\sum_{i=1}^n \rho_i^2 = \sum_{i=1}^n (d_i - c_i)^2 + \frac{1}{2}\sum_{i<j} \left(\frac{d_i}{d_j} + \frac{d_j}{d_i}\right)^2$,

where $c_i = \begin{cases} 1 & \text{if } v_i \in D \\ 0 & \text{otherwise.} \end{cases}$

Proof.

(i) We know that the sum of the eigenvalues of $L_{ex}^D(G)$ is the trace of $L_{ex}^D(G)$. Therefore $\sum_{i=1}^n \rho_i = \sum_{i=1}^n a_{ii} = \sum_{i=1}^n d_i - |D| = 2|E| - |D| = 2m - k$.

(ii) Similarly the sum of squares of the eigenvalues of $L_{ex}^D(G)$ is the trace of $[L_{ex}^D(G)]^2$. Therefore,

$$\begin{aligned}\sum_{i=1}^n \rho_i^2 &= \sum_{i=1}^n \sum_{j=1}^n a_{ij}a_{ji} = \sum_{i=1}^n (a_{ii})^2 + \sum_{i\neq j} a_{ij}a_{ji} = \sum_{i=1}^n (a_{ii})^2 + 2\sum_{i<j}(a_{ij})^2 \\ &= \sum_{i=1}^n (d_i - c_i)^2 + 2\sum_{i<j}\Big(\frac{1}{2}\Big(\frac{d_i}{d_j} + \frac{d_j}{d_i}\Big)\Big)^2, \\ &= \sum_{i=1}^n (d_i - c_i)^2 + \frac{1}{2}\sum_{i<j}\Big(\frac{d_i}{d_j} + \frac{d_j}{d_i}\Big)^2\end{aligned}$$

where $c_i = \begin{cases} 1 & \text{if } v_i \in D \\ 0 & \text{otherwise} \end{cases}$. □

The question of when does the graph energy becomes a rational number was answered by Bapat and S.Pati in their article [2]. Similar result for Laplacian minimum dominating extended energy is obtained in the following theorem.

Theorem 2.2. *If the sum of the absolute eigenvalues of Laplacian minimum dominating extended matrix $L_{ex}^D(G)$ is a rational number, then $\sum_{i=1}^n |\rho_i| \equiv |D| (mod\ 2)$.*

Proof. Let $\rho_1, \rho_2, ..., \rho_n$ be eigenvalues of Laplacian minimum dominating extended matrix $L_{ex}^D(G)$ of a graph G, of which $\rho_1, \rho_2, ..., \rho_r$ are positive and the rest are non-positive, then

$$\begin{aligned}\sum_{i=1}^n |\rho_i| &= (\rho_1 + \rho_2 + \cdots + \rho_r) - (\rho_{r+1} + \cdots + \rho_n) \\ &= 2(\rho_1 + \rho_2 + \cdots + \rho_r) - (\rho_1 + \rho_2 + \cdots + \rho_n) \\ &= 2(\rho_1 + \rho_2 + \cdots + \rho_r) - \sum_{i=1}^n \rho_i = 2(\rho_1 + \rho_2 + \cdots + \rho_r) - (2|E| - |D|) \\ &= 2(\rho_1 + \rho_2 + \cdots + \rho_r - |E|) - |D|.\end{aligned} \tag{2.1}$$

Therefore $\sum_{i=1}^n |\rho_i| \equiv |D| (mod 2)$. □

Theorem 2.3. *Let G be a graph with n vertices and m edges. If the sum of the absolute Laplacian minimum dominating extended eigenvalues is a rational* *number, then* $E^D_{ex}(G) \in (2t - 2m, 2t + 2m)$*, where* t *is an integer such that* $\sum_{i=1}^{n} \mid \rho_i \mid \equiv |D|(mod 2)$.

Proof. We know that $\sum_{i=1}^{n} \left|\rho_i - \frac{2m}{n}\right| \leq \sum_{i=1}^{n} \mid \rho_i \mid + 2m$, i.e.,

$$E^D_{ex}(G) \leq \sum_{i=1}^{n} \mid \rho_i \mid + 2m = 2t + 2m \geq \sum_{i=1}^{n} \mid \rho_i \mid - 2m = 2t - 2m$$

(from equation (2.1)), and $E^D_{ex}(G) \in (2t - 2m, 2t + 2m)$. □

3. Bounds for Laplacian minimum dominating extended energy

McLelland's [6] gave upper and lower bounds for ordinary energy of a graph. Similar bounds for $E^D_{ex}(G)$ are given in the following theorem.

Theorem 3.1. *(Upper bound) Let G be a simple graph with n vertices and m edges then*

$$E^D_{ex}(G) \leq \sqrt{n\Big(\sum_{i=1}^{n}(d_i - c_i)^2 + \frac{1}{2}\sum_{i<j}\Big(\frac{d_i}{d_j} + \frac{d_j}{d_i}\Big)^2\Big)} + 2m.$$

Proof. Cauchy Schwarz inequality is

$$\Big(\sum_{i=1}^{n} a_i b_i\Big)^2 \leq \Big(\sum_{i=1}^{n} a_i^2\Big)\Big(\sum_{i=1}^{n} b_i^2\Big).$$

If $a_i = 1$ and $b_i = \mid \rho_i \mid$ then from Theorem 2.1:

$$\Big(\sum_{i=1}^{n} \mid \rho_i \mid\Big)^2 \leq \Big(\sum_{i=1}^{n} 1\Big)\Big(\sum_{i=1}^{n} \rho_i^2\Big)$$

$$\Big(\sum_{i=1}^{n} \mid \rho_i \mid\Big)^2 \leq n\Big(\sum_{i=1}^{n}(d_i - c_i)^2 + \frac{1}{2}\sum_{i<j}\Big(\frac{d_i}{d_j} + \frac{d_j}{d_i}\Big)^2\Big)$$

$$\Longrightarrow \sum_{i=1}^{n} \mid \rho_i \mid \leq \sqrt{n\Big(\sum_{i=1}^{n}(d_i - c_i)^2 + \frac{1}{2}\sum_{i<j}\Big(\frac{d_i}{d_j} + \frac{d_j}{d_i}\Big)^2\Big)}.$$

By triangle inequality,

$$\left|\rho_i - \frac{2m}{n}\right| \leq \mid \rho_i \mid + \left|\frac{2m}{n}\right| \leq \mid \rho_i \mid + \frac{2m}{n} \leq \sum_{i=1}^{n} \mid \rho_i \mid + \sum_{i=1}^{n} \frac{2m}{n},$$

for all $i = 1, 2, \ldots, n$, i.e.,

$$\sum_{i=1}^{n}\Big|\rho_i - \frac{2m}{n}\Big| \leq \sqrt{n\Big(\sum_{i=1}^{n}(d_i - c_i)^2 + \frac{1}{2}\sum_{i<j}\Big(\frac{d_i}{d_j} + \frac{d_j}{d_i}\Big)^2\Big)} + 2m.$$

The proof of the theorem follows. □

Theorem 3.2. *(Upper bound) Let G be a simple graph with n vertices and m edges then*

$$E_{ex}^{D}(G) \leq \sqrt{n\left(\sum_{i=1}^{n}(d_i - c_i)^2 + \frac{1}{2}\sum_{i<j}\Big(\frac{d_i}{d_j} + \frac{d_j}{d_i}\Big)^2 - \frac{4m^2}{n} + \frac{4m|D|}{n}\right)}.$$

Proof. Cauchy- Schwarz inequality is

$$\Big(\sum_{i=1}^{n} a_i b_i\Big)^2 \leq \Big(\sum_{i=1}^{n} a_i^2\Big)\Big(\sum_{i=1}^{n} b_i^2\Big).$$

Put $a_i = 1$ and $b_i = \left|\rho_i - \frac{2m}{n}\right|$ then

$$\left(\sum_{i=1}^{n}\Big|\rho_i - \frac{2m}{n}\Big|\right)^2 \leq \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n}\Big|\rho_i - \frac{2m}{n}\Big|^2\right),$$

i.e.,

$$\begin{aligned}
[E_{ex}^{D}(G)]^2 &\leq n\left(\sum_{i=1}^{n}\rho_i^2 + \sum_{i=1}^{n}\frac{4m^2}{n^2} - \frac{4m}{n}\sum_{i=1}^{n}\rho_i\right) \\
&= n\left(\sum_{i=1}^{n}(d_i - c_i)^2 + \frac{1}{2}\sum_{i<j}\Big(\frac{d_i}{d_j} + \frac{d_j}{d_i}\Big)^2 + \frac{4m^2}{n^2}.n - \frac{4m}{n}(2m - |D|)\right) \\
&= n\left(\sum_{i=1}^{n}(d_i - c_i)^2 + \frac{1}{2}\sum_{i<j}\Big(\frac{d_i}{d_j} + \frac{d_j}{d_i}\Big)^2 + \frac{4m^2}{n} - \frac{8m^2}{n} + \frac{4m|D|}{n}\right) \\
&= n\left(\sum_{i=1}^{n}(d_i - c_i)^2 + \frac{1}{2}\sum_{i<j}\Big(\frac{d_i}{d_j} + \frac{d_j}{d_i}\Big)^2 - \frac{4m^2}{n} + \frac{4m|D|}{n}\right).
\end{aligned}$$

The proof of the theorem follows. □

Theorem 3.3. *(Lower bound) Let G be a simple graph with n vertices and m edges and $P = |detL^D_{ex}(G)|$, then*

$$E^D_{ex}(G) \geq \sqrt{\sum_{i=1}^{n}(d_i - c_i)^2 + \frac{1}{2}\sum_{i<j}\left(\frac{d_i}{d_j} + \frac{d_j}{d_i}\right)^2 + n(n-1)P^{\frac{2}{n}} - 2m}.$$

Proof. Consider

$$\left(\sum_{i=1}^{n} \mid \rho_i \mid\right)^2 = \left(\sum_{i=1}^{n} \mid \rho_i \mid\right).\left(\sum_{j=1}^{n} \mid \rho_j \mid\right) = \sum_{i=1}^{n} \mid \rho_i \mid^2 + \sum_{i \neq j} \mid \rho_i \mid\mid \rho_j \mid$$

Therefore

$$\sum_{i \neq j} \mid \rho_i \mid\mid \rho_j \mid = \left(\sum_{i=1}^{n} \mid \rho_i \mid\right)^2 - \sum_{i=1}^{n} \mid \rho_i \mid^2.$$

Applying inequality between the arithmetic and geometric means for $n(n-1)$ terms, we have:

$$\frac{\sum_{i \neq j} \mid \rho_i \mid\mid \rho_j \mid}{n(n-1)} \geq \left[\prod_{i \neq j} \mid \rho_i \mid\mid \rho_j \mid\right]^{\frac{1}{n(n-1)}} \geq n(n-1)\left[\prod_{i \neq j} \mid \rho_i \mid\mid \rho_j \mid\right]^{\frac{1}{n(n-1)}}.$$

Using (2.1) we get:

$$\left(\sum_{i=1}^{n} \mid \rho_i \mid\right)^2 - \sum_{i=1}^{n} \mid \rho_i \mid^2 \geq n(n-1)\left[\prod_{i=1}^{n} \mid \rho_i \mid^{2(n-1)}\right]^{\frac{1}{n(n-1)}}$$

$$\left(\sum_{i=1}^{n} \mid \rho_i \mid\right)^2 - \sum_{i=1}^{n}(d_i - c_i)^2 - \frac{1}{2}\sum_{i<j}\left(\frac{d_i}{d_j} + \frac{d_j}{d_i}\right)^2 \geq n(n-1)\left[\prod_{i=1}^{n} \mid \rho_i \mid\right]^{\frac{2}{n}}$$

$$\left(\sum_{i=1}^{n} \mid \rho_i \mid\right)^2 \geq \sum_{i=1}^{n}(d_i - c_i)^2 + \frac{1}{2}\sum_{i<j}\left(\frac{d_i}{d_j} + \frac{d_j}{d_i}\right)^2 + n(n-1)\left[\prod_{i=1}^{n} \rho_i\right]^{\frac{2}{n}}$$

$$\therefore \sum_{i=1}^{n} \mid \rho_i \mid \geq \sqrt{\sum_{i=1}^{n}(d_i - c_i)^2 + \frac{1}{2}\sum_{i<j}\left(\frac{d_i}{d_j} + \frac{d_j}{d_i}\right)^2 + n(n-1)P^{\frac{2}{n}}}.$$

We know that

$$\mid \rho_i \mid - \left|\frac{2m}{n}\right| \leq \left|\rho_i - \frac{2m}{n}\right| \quad \forall i,$$

$$\mid \rho_i \mid - \frac{2m}{n} \leq \left|\rho_i - \frac{2m}{n}\right| \quad \forall i,$$

$$\sum_{i=1}^{n} \mid \rho_i \mid - \sum_{i=1}^{n} \frac{2m}{n} \leq \sum_{i=1}^{n} \Big| \rho_i - \frac{2m}{n} \Big|$$

$$\sum_{i=1}^{n} \mid \rho_i \mid -2m \leq E_{ex}^{D}(G) \geq \sum_{i=1}^{n} \mid \rho_i \mid -2m$$

Therefore,

$$E_{ex}^{D}(G) \geq \sqrt{\sum_{i=1}^{n}(d_i - c_i)^2 + \frac{1}{2}\sum_{i<j}\left(\frac{d_i}{d_j} + \frac{d_j}{d_i}\right)^2 + n(n-1)P^{\frac{2}{n}}} - 2m.$$

□

Theorem 3.4. *If $\rho_1(G)$ is the largest Laplacian minimum dominating extended eigenvalue of $L_{ex}^{D}(G)$, then*

$$\rho_1(G) \geq \frac{2|E| - \mid D \mid -2\sum_{i<j}\left(\frac{d_i}{d_j} + \frac{d_j}{d_i}\right)}{n}.$$

Proof. Let X be any nonzero vector. Then by [1], we have $\rho_1(A) = \max_{X \neq 0}\left\{\frac{X'AX}{X'X}\right\}$. Therefore,

$$\begin{aligned} \rho_1(A) \geq \frac{J'AJ}{J'J} &= \frac{\sum_{i=1}^{n} d_i - \mid D \mid -2\sum_{i<j}\left(\frac{d_i}{d_j} + \frac{d_j}{d_i}\right)}{n} \\ &= \frac{2|E| - \mid D \mid -2\sum_{i<j}\left(\frac{d_i}{d_j} + \frac{d_j}{d_i}\right)}{n}, \end{aligned}$$

where J is a unit matrix $[1, 1, 1, \ldots, 1]'$. □

Milovanović [7] bounds for Laplacian modified Schultz energy of a graph are given in the following theorem.

Theorem 3.5. *Let G be a graph with n vertices and m edges. Let $|\rho_1| \geq |\rho_2| \geq \cdots \geq |\rho_n|$ be a non-increasing order of Laplacian minimum dominating extended eigenvalues of $L_{ex}^{D}(G)$ then*

$$E_{ex}^{D}(G) \geq \sqrt{n\Big(\sum_{i=1}^{n}(d_i - c_i)^2 + \frac{1}{2}\sum_{i<j}\left(\frac{d_i}{d_j} + \frac{d_j}{d_i}\right)^2\Big) - \alpha(n)(|\rho_1| - |\rho_n|)^2} - 2m,$$

where $\alpha(n)$= $n[\frac{n}{2}](1 - \frac{1}{n}[\frac{n}{2}])$ and $[x]$ denotes the integral part of a real number.

Proof. Let $a, a_1, a_2, \ldots, a_n, A$ and $b, b_1, b_2, \ldots, b_n, B$ be real numbers such that $a \leq a_i \leq A$ and $b \leq b_i \leq B \ \forall\ i = 1, 2, \ldots, n$. Then the following inequality is valid:

$$\Big|n\sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i\Big| \leq \alpha(n)(A-a)(B-b).$$

If $a_i = |\rho_i|$, $b_i = |\rho_i|$, $a = b = |\rho_n|$ and $A = B = |\rho_1|$,

$$\Big|n\sum_{i=1}^{n}|\rho_i|^2 - \Big(\sum_{i=1}^{n}|\rho_i|\Big)^2)\Big| \leq \alpha(n)(|\rho_1| - |\rho_n|)^2$$

implies

$$n\Big(\sum_{i=1}^{n}(d_i - c_i)^2 + \frac{1}{2}\sum_{i<j}\Big(\frac{d_i}{d_j} + \frac{d_j}{d_i}\Big)^2\Big) - \Big(\sum_{i=1}^{n}|\rho_i|\Big)^2 \leq \alpha(n)(|\rho_1| - |\rho_n|)^2,$$

and

$$\Big(\sum_{i=1}^{n}|\rho_i|\Big) \geq \sqrt{\Big(\sum_{i=1}^{n}(d_i - c_i)^2 + \frac{1}{2}\sum_{i<j}\Big(\frac{d_i}{d_j} + \frac{d_j}{d_i}\Big)^2\Big)n - \alpha(n)(|\rho_1| - |\rho_n|)^2},$$

since $E^D_{ex}(G) = \sum_{i=1}^{n}\left|\rho_i - \frac{2m}{n}\right| \geq \sum_{i=1}^{n}(\left|\rho_i\right| - \left|\frac{2m}{n}\right|)$. Hence the proof of the theorem follows. □

Theorem 3.6. *Let G be a graph with n vertices and m edges. Let $|\rho_1| \geq |\rho_2| \geq \cdots \geq |\rho_n| > 0$ be a non-increasing order of eigenvalues of $L^D_{ex}(G)$. Then*

$$E^D_{ex}(G) \geq \frac{\sum_{i=1}^{n}(d_i - c_i)^2 + \frac{1}{2}\sum_{i<j}\left(\frac{d_i}{d_j} + \frac{d_j}{d_i}\right)^2 + n|\rho_1||\rho_n|}{(|\rho_1| + |\rho_n|)} - 2m.$$

Proof. Let $a_i \neq 0$, b_i, r and R be real numbers satisfying $ra_i \leq b_i \leq Ra_i$, then the following inequality holds (Theorem 2, [7]):

$$\sum_{i=1}^{n} b_i^2 + rR\sum_{i=1}^{n} a_i \leq (r+R)\sum_{i=1}^{n} a_i b_i.$$

Put $b_i = |\rho_i|$, $a_i = 1$, $r = |\rho_n|$ and $R = |\rho_1|$ then

$$\sum_{i=1}^{n}|\rho_i|^2 + |\rho_1||\rho_n|\sum_{i=1}^{n}1 \leq (|\rho_1| + |\rho_n|)\sum_{i=1}^{n}|\rho_i|,$$

i.e.,

$$\sum_{i=1}^{n}(d_i - c_i)^2 + \frac{1}{2}\sum_{i<j}\left(\frac{d_i}{d_j} + \frac{d_j}{d_i}\right)^2 + |\rho_1||\rho_n|n \leq (|\rho_1| + |\rho_n|)\sum_{i=1}^{n}|\rho_i|.$$

Therefore,

$$\sum_{i=1}^{n}|\rho_i| \geq \frac{\sum_{i=1}^{n}(d_i - c_i)^2 + \frac{1}{2}\sum_{i<j}\left(\frac{d_i}{d_j} + \frac{d_j}{d_i}\right)^2 + n|\rho_1||\rho_n|}{(|\rho_1| + |\rho_n|)}.$$

We know that $E_{ex}^D(G) = \sum_{i=1}^{n}\left|\rho_i - \frac{2m}{n}\right|$. So, $E_{ex}^D(G) \geq \sum_{i=1}^{n}\left(\left|\rho_i\right| - \left|\frac{2m}{n}\right|\right)$. Hence, the proof of the theorem follows. □

4. Laplacian minimum dominating extended energy of some standard graphs:

Theorem 4.1. *For $n \geq 2$, the Laplacian minimum dominating extended energy of a star graph $K_{1,n-1}$ is*

$$\frac{(3n^2 - 8n + 4)}{n} + \frac{\sqrt{n^5 - 4n^4 + 4n^3 + 6n^2 - 12n + 5}}{n - 1}.$$

Proof. Consider a star graph $K_{1,n-1}$ with vertex set $V = \{v_0, v_1, v_2, \ldots, v_{n-1}\}$ and center v_0. Then minimum dominating extended set is $D = \{v_0\}$,

$$A_{ex}^D(K_{n-1}) = \begin{pmatrix} 1 & \frac{1}{2}(\frac{n-1}{1} + \frac{1}{n-1}) & \frac{1}{2}(\frac{n-1}{1} + \frac{1}{n-1}) & \ldots & \frac{1}{2}(\frac{n-1}{1} + \frac{1}{n-1}) \\ \frac{1}{2}(\frac{n-1}{1} + \frac{1}{n-1}) & 0 & \frac{1}{2}(\frac{n-1}{1} + \frac{1}{n-1}) & \ldots & \frac{1}{2}(\frac{n-1}{1} + \frac{1}{n-1}) \\ \frac{1}{2}(\frac{n-1}{1} + \frac{1}{n-1}) & \frac{1}{2}(\frac{n-1}{1} + \frac{1}{n-1}) & 0 & \ldots & \frac{1}{2}(\frac{n-1}{1} + \frac{1}{n-1}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}(\frac{n-1}{1} + \frac{1}{n-1}) & \frac{1}{2}(\frac{n-1}{1} + \frac{1}{n-1}) & \frac{1}{2}(\frac{n-1}{1} + \frac{1}{n-1}) & \ldots & 0 \end{pmatrix}$$

$n \times n$,

$$D(K_{1,n-1}) = \begin{pmatrix} n-1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}_{n\times n}$$

and $L_{ex}^D(K_{1,n-1}) = D(K_{1,n-1}) - A_{ex}^D(K_{1,n-1})$. Laplacian minimum dominating extended spectrum is

$$Spec(L^D_{ex}(K_{1,n-1})) = \begin{pmatrix} -1 & \frac{-(n^2-2n+1)+\sqrt{n^5-4n^4+4n^3+6n^2-12n+5}}{2n-2} & \frac{-(n^2-2n+1)-\sqrt{n^5-4n^4+4n^3+6n^2-12n+5}}{2n-2} \\ (n-2) & 1 & 1 \end{pmatrix}.$$

Hence, Laplacian minimum dominating extended energy,

$$E^D_{ex}(K_{1,n-1}) = \frac{(3n^2-8n+4)}{n} + \frac{\sqrt{n^5-4n^4+4n^3+6n^2-12n+5}}{n-1}.$$

□

Theorem 4.2. *For $n \geq 2$, the Laplacian minimum dominating extended energy of complete graph is $(2n^2-5n+2)+\sqrt{n^2-2n+5}$.*

Proof. Consider the complete graph K_n with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The minimum dominating set is $D = \{v_1\}$,

$$A^D_{ex}(K_n) = \begin{pmatrix} 1 & \frac{1}{2}(\frac{n-1}{n-1}+\frac{n-1}{n-1}) & \frac{1}{2}(\frac{n-1}{n-1}+\frac{n-1}{n-1}) & \ldots & \frac{1}{2}(\frac{n-1}{n-1}+\frac{n-1}{n-1}) \\ \frac{1}{2}(\frac{n-1}{n-1}+\frac{n-1}{n-1}) & 0 & \frac{1}{2}(\frac{n-1}{n-1}+\frac{n-1}{n-1}) & \ldots & \frac{1}{2}(\frac{n-1}{n-1}+\frac{n-1}{n-1}) \\ \frac{1}{2}(\frac{n-1}{n-1}+\frac{n-1}{n-1}) & \frac{1}{2}(\frac{n-1}{n-1}+\frac{n-1}{n-1}) & 0 & \ldots & \frac{1}{2}(\frac{n-1}{n-1}+\frac{n-1}{n-1}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}(\frac{n-1}{n-1}+\frac{n-1}{n-1}) & \frac{1}{2}(\frac{n-1}{n-1}+\frac{n-1}{n-1}) & \frac{1}{2}(\frac{n-1}{n-1}+\frac{n-1}{n-1}) & \ldots & 0 \end{pmatrix},$$

$$D(K_n) = \begin{pmatrix} n-1 & 0 & 0 & \ldots & 0 \\ 0 & n-1 & 0 & \ldots & 0 \\ 0 & 0 & n-1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & n-1 \end{pmatrix}_{n\times n},$$

and $L^D_{ex}(K_n) = D(K_n) - A^D_{ex}(K_n)$. Minimum dominating extended spectrum is

$$Spec(L^D_{ex}(K_n)) = \begin{pmatrix} -n & \frac{-(n-1)+\sqrt{n^2-2n+5}}{2} & \frac{-(n-1)-\sqrt{n^2-2n+5}}{2} \\ (n-2) & 1 & 1 \end{pmatrix}.$$

Laplacian minimum dominating extended energy is $E^D_{ex}(K_n) = (2n^2-5n+2)+\sqrt{n^2-2n+5}$. □

Theorem 4.3. *For $n \geq 2$, the Laplacian minimum dominating extended energy of crown graph is $(4n^2-12n+8)+\sqrt{n^2-2n+5}-\sqrt{n^2+2n-3}$.*

Proof. Consider the crown graph S_n^0 with vertex set $V = \{u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$. The minimum dominating extended set is

$D = \{u_1, v_1\}$. $A_{ex}^D(S_{2n}^0) =$

$$\left(\begin{array}{ccccc|ccccc}
u_1 & u_2 & u_3 & \dots & u_n & v_1 & v_2 & v_3 & \dots & v_n \\
\hline
1 & 0 & 0 & \dots & 0 & 0 & \frac{1}{2}(\frac{n-1}{n-1}+\frac{n-1}{n-1}) & 1 & \dots & 1 \\
0 & 0 & 0 & \dots & 0 & \frac{1}{2}(\frac{n-1}{n-1}+\frac{n-1}{n-1}) & 0 & 1 & \dots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \dots & 0 & 1 & 1 & 1 & \dots & 0 \\
\hline
0 & \frac{1}{2}(\frac{n-1}{n-1}+\frac{n-1}{n-1}) & 1 & \dots & 1 & 1 & 0 & 0 & \dots & 0 \\
\frac{1}{2}(\frac{n-1}{n-1}+\frac{n-1}{n-1}) & 0 & 1 & \dots & 1 & 0 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & & 0 & 0 & 0 & 0 & \dots & 0
\end{array}\right)_{(2n\times 2n)}$$

$$D(S_{2n}^0) = \left(\begin{array}{c|ccccc|ccccc}
 & u_1 & u_2 & u_3 & \dots & u_n & v_1 & v_2 & v_3 & \dots & v_n \\
\hline
u_1 & n-1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\
u_2 & 0 & n-1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
u_n & 0 & 0 & 0 & \dots & n-1 & 0 & 0 & 0 & \dots & 0 \\
\hline
v_1 & 0 & 0 & 0 & \dots & 0 & n-1 & 0 & 0 & \dots & 0 \\
v_2 & 0 & 0 & 0 & \dots & 0 & 0 & n-1 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
v_n & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & n-1
\end{array}\right)$$

and $L_{ex}^D(S_{2n}^0) = D(S_n^0) - A_{ex}^D(S_{2n}^0)$. Laplacian Minimum dominating extended spectrum

$Spec(L_{ex}^D(S_{2n}^0)) =$

$$\begin{pmatrix} -(n-2) & -n & \frac{-(n-1)+\sqrt{n^2-2n+5}}{2} & \frac{-(n-1)-\sqrt{n^2-2n+5}}{2} & \frac{(3n-5)+\sqrt{n^2+2n-3}}{2} & \frac{(3n-5)-\sqrt{n^2+2n-3}}{2} \\ n-2 & n-2 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Laplacian minimum dominating extended energy is

$$E_{ex}^D(S_{2n}^0) = (4n^2 - 12n + 8) + \sqrt{n^2 - 2n + 5} - \sqrt{n^2 + 2n - 3}.$$

□

Theorem 4.4. *The Laplacian minimum dominating extended energy of cocktail party graph* $K_{n\times 2}$ *is* $(8n^2 - 14n + 3) + \sqrt{4n^2 - 4n + 9}$.

Proof. Consider cocktail party graph $K_{n\times 2}$ with vertex set V = $\bigcup_{i=1}^{n-1}\{u_i, v_i\}$. The minimum dominating extended set is $D = \{u_1, v_1\}$,

$A_{ex}^D(K_{n\times 2}) =$

$$\left(\begin{array}{c|cccc|cccc} & u_1 & u_2 & \dots & u_n & v_1 & v_2 & \dots & v_n \\ \hline u_1 & 1 & \frac{1}{2}(\frac{2n-2}{2n-2}+\frac{2n-2}{2n-2}) & \dots & 1 & 0 & \frac{1}{2}(\frac{2n-2}{2n-2}+\frac{2n-2}{2n-2}) & \dots & 1 \\ u_2 & \frac{1}{2}(\frac{2n-2}{2n-2}+\frac{2n-2}{2n-2}) & 0 & \dots & 1 & \frac{1}{2}(\frac{2n-2}{2n-2}+\frac{2n-2}{2n-2}) & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ u_n & 1 & 1 & \dots & 0 & 1 & 1 & \dots & 0 \\ \hline v_1 & 0 & \frac{1}{2}(\frac{2n-2}{2n-2}+\frac{2n-2}{2n-2}) & \dots & 1 & 1 & \frac{1}{2}(\frac{2n-2}{2n-2}+\frac{2n-2}{2n-2}) & \dots & 1 \\ v_2 & \frac{1}{2}(\frac{2n-2}{2n-2}+\frac{2n-2}{2n-2}) & 0 & \dots & 1 & \frac{1}{2}(\frac{2n-2}{2n-2}+\frac{2n-2}{2n-2}) & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_n & 1 & 1 & \dots & 0 & 1 & 1 & \dots & 0 \end{array}\right)$$

$$D(K_{n\times 2}) = \left(\begin{array}{c|cccc|cccc} & u_1 & u_2 & \dots & u_n & v_1 & v_2 & \dots & v_n \\ \hline u_1 & 2n-2 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ u_2 & 0 & 2n-2 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ u_n & 0 & 0 & \dots & 2n-2 & 0 & 0 & \dots & 0 \\ \hline v_1 & 0 & 0 & \dots & 0 & 2n-2 & 0 & \dots & 0 \\ v_2 & 0 & 0 & \dots & 0 & 0 & 2n-2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_n & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 2n-2 \end{array}\right)$$

$L_{ex}^D(K_{n\times 2}) = D(K_{n\times 2}) - A_{ex}^D(K_{n\times 2}) =$
$Spec_{ex}^D(K_{n\times 2}) =$

$$\left(\begin{array}{ccccc} -(2n-3) & -(2n-2) & -2n & \frac{(-2n+1)+\sqrt{4n^2-4n+9}}{2} & \frac{(-2n+1)-\sqrt{4n^2-4n+9}}{2} \\ 1 & (n-1) & (n-2) & 1 & 1 \end{array}\right)$$

Laplacian minimum dominating extended energy is

$$E_{ex}^D(K_{n\times 2}) = (8n^2 - 14n + 3) + \sqrt{4n^2 - 4n + 9}.$$

□

Theorem 4.5. *The Laplacian minimum dominating extended energy of the complete bipartite graph is*

$$\frac{(4m-1)n^2+(4m^2-7m+1)n-2m^2+m}{m+n}$$
$$+\frac{\sqrt{mn^5+m^2n^4-2m^2n^3+(m^4-6m^3+m^2)n^2+m^5n}}{mn}.$$

Proof. For the complete bipartite graph $K_{m,n}$ $(m \leq n)$ with vertex set $V = \{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n\}$. The Minimum dominating extended matrix of complete bipartite graph is
$A^D_{ex}(K_{m,n}) =$

$$\left(\begin{array}{c|cccc|cccc}
 & v_1 & v_2 & \ldots & v_m & u_1 & u_2 & \ldots & u_n \\ \hline
v_1 & 1 & 0 & \ldots & 0 & \frac{1}{2}(\frac{m}{n}+\frac{n}{m}) & \frac{1}{2}(\frac{m}{n}+\frac{n}{m}) & \ldots & \frac{1}{2}(\frac{m}{n}+\frac{n}{m}) \\
v_2 & 0 & 1 & \ldots & 0 & \frac{1}{2}(\frac{m}{n}+\frac{n}{m}) & \frac{1}{2}(\frac{m}{n}+\frac{n}{m}) & \ldots & \frac{1}{2}(\frac{m}{n}+\frac{n}{m}) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
v_m & 0 & 0 & \ldots & 1 & \frac{1}{2}(\frac{m}{n}+\frac{n}{m}) & \frac{1}{2}(\frac{m}{n}+\frac{n}{m}) & \ldots & \frac{1}{2}(\frac{m}{n}+\frac{n}{m}) \\ \hline
u_1 & \frac{1}{2}(\frac{m}{n}+\frac{n}{m}) & \frac{1}{2}(\frac{m}{n}+\frac{n}{m}) & \ldots & \frac{1}{2}(\frac{m}{n}+\frac{n}{m}) & 0 & 0 & \ldots & 0 \\
u_2 & \frac{1}{2}(\frac{m}{n}+\frac{n}{m}) & \frac{1}{2}(\frac{m}{n}+\frac{n}{m}) & \ldots & \frac{1}{2}(\frac{m}{n}+\frac{n}{m}) & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
u_n & \frac{1}{2}(\frac{m}{n}+\frac{n}{m}) & \frac{1}{2}(\frac{m}{n}+\frac{n}{m}) & \ldots & \frac{1}{2}(\frac{m}{n}+\frac{n}{m}) & 0 & 0 & \ldots & 0
\end{array}\right)$$

$$D(K_{m,n}) = \left(\begin{array}{c|ccccc|ccc}
 & u_1 & u_2 & \ldots & u_m & v_1 & v_2 & \ldots & v_n \\ \hline
u_1 & n & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
u_2 & 0 & n & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
u_m & 0 & 0 & \ldots & n & 0 & 0 & \ldots & 0 \\ \hline
v_1 & 0 & 0 & \ldots & 0 & m & 0 & \ldots & 0 \\
v_2 & 0 & 0 & \ldots & 0 & 0 & m & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
v_n & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & m
\end{array}\right)$$

$$Spec(L^D_{ex}(K_{m,n})) = \begin{pmatrix} -(n-1) & -m & \frac{X+\sqrt{Y}}{2mn} & \frac{X-\sqrt{Y}}{2mn} \\ m-1 & n-1 & 1 & 1 \end{pmatrix}.$$

Here, $X = -mn^2 + (m - m^2)n$ and $Y = mn^5 + m^2n^4 - 2m^2n^3 + (m^4 - 6m^3 + m^2)n^2 + m^5n$. Laplacian Minimum dominating extended energy is,

$$E_{ex}^D(K_{m,n}) = \frac{(4m-1)n^2 + (4m^2 - 7m + 1)n - 2m^2 + m}{m+n} + \frac{\sqrt{mn^5 + m^2n^4 - 2m^2n^3 + (m^4 - 6m^3 + m^2)n^2 + m^5n}}{mn}.$$

□

Theorem 4.6. *The Laplacian Minimum dominating extended energy of Friendship graph is*

$$\frac{12n^2 - 4n + 1}{2n+1} + \frac{\sqrt{6n^3 + 2n}}{n}.$$

Proof. For a Friendship graph F_3^n with vertex set $V = \{v_o, v_1, v_2, \ldots, v_n, v_{n+1}, v_{n+2}, \ldots, v_{2n}\}$. The minimum dominating extended matrix of Friendship graph is

$$A_{ex}^D(F_3^n) = L_{ex}^D(F_3^n) = D(F_3^n) - A_{ex}^D(F_3^n)$$
$$= Spec(L_{ex}^D(F_3^n) = \begin{pmatrix} 1 & -1 & \frac{2n-\sqrt{6n^3+2n}}{2n} & \frac{2n+\sqrt{6n^3+2n}}{2n} \\ n-1 & n & 1 & 1 \end{pmatrix}.$$

Therefore Laplacian Minimum dominating extended energy is,

$$E_{ex}^D(F_3^n) = \frac{12n^2 - 4n + 1}{2n+1} + \frac{\sqrt{6n^3 + 2n}}{n}.$$

□

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