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SOME PROPERTIES OF GENERALIZED HYPERGEOMETRIC POLYNOMIALS IN TWO VARIABLES

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ABSTRACT. An attempt has been made to derive certain classical properties of two variable generalized hypergeometric polynomials (2VGHP) $I_n(\alpha;\beta : x, y)$, namely, recurrence relations of ascending, descending type , ordinary differential equation and linear generating relation, which are needed in order to obtain many other properties of $I_n(\alpha;\beta : x, y)$. Furthermore, two variable Laguerre polynomials are deduced from $I_n(\alpha;\beta : x, y)$ as a special case, which are of great importance in the basic quantum analysis of hydrogen atoms.

1. INTRODUCTION

Hypergeometric polynomials in one or more variables arise frequently in a wide variety of problems in theoretical physics, applied mathematics, engineering sciences,statistics and operational research. It is then obvious that a datailed study of the analytical behaviour of such polynomials will be of great importance. There are many directions to study such polynomials but the theory of generating functions has been developed into various directions and found wide applications ([1-20]).

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2. Definition

The polynomial set $\{I_n(\alpha;\beta:x,y)\}_{n=0,1,2...}$ is defined as

$$\begin{aligned} I_n(\alpha;\beta:x,y) =&(xy)^n\ {}_2F_1\left[-n,\alpha;\beta;\frac{x}{y}\right]\\ =&\sum_{k=0}^{n}\frac{(-n)_k(\alpha)_k x^{n+k}y^{n-k}}{k!(\beta)_k}\\ =&\sum_{k=0}^{n}\frac{(-1)^k n!(\alpha)_k x^{n+k}y^{n-k}}{(n-k)!k!(\beta)_k}, \end{aligned}$$

which is valid under the following conditions:

(i) α is a real number,

(ii) β is neither zero nor a negative integer,

(iii) n is a non-negative integer,

(iv) α and β are independent of n, because for the polynomial so many properties which are valid for α, β independent of n fail to be valid for α , β dependent upon n,

(v) x is any finite complex variable such that $x \neq 1$.

Deduction

The following special case of $I_n(\alpha;\beta:x,y)$ has been obtained:

$$\lim_{\alpha\to\infty}\left\{\alpha^{-n}I_n\left(\alpha;1+\gamma;x;\frac{\alpha}{y}\right)\right\{=\frac{n!}{(1+\gamma)_n}x^n L_n^{(\gamma)}(x,y),$$

where $L_n^{(\gamma)}(x,y)$ is the Laguerre polynomial.

3. Simple Generating Relations

Theorem 3.1. *The following relation holds:*

$$\sum_{n=0}^{\infty}\frac{I_n(\alpha;\beta;x,y)t^n}{n!}=e^{xyt}\ {}_1F_1\left[\alpha;\beta;-x^2t\right].$$

Proof. Consider the series

$$\sum_{n=0}^{\infty}\frac{I_n(\alpha;\beta;x,y)t^n}{n!}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-n)_k(\alpha)_k x^{n+k}y^{n-k}t^n}{n!k!(\beta)_k}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k (\alpha)_k x^{n+k} y^{n-k} t^n}{(n-k)! k! (\beta)_k}$$
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k (\alpha)_k x^{n+2k} y^n t^{n+k}}{n! k! (\beta)_k}$$
$$= \sum_{n=0}^{\infty} \frac{x^n y^n t^n}{n!} \sum_{k=0}^{\infty} \frac{(-1)^k (\alpha)_k (x^2 t)^k}{k! (\beta)_k}$$
$$= \exp(xyt)_1 F_1 \left[\alpha; \beta; -x^2 t\right].$$

Hence the proof.

□

Application

One can derived the linear generating function for the two variable Laguerre polynomial from the above generating relation:

$$\sum_{n=0}^{\infty} \frac{x^n L_n^{(\gamma)}(x,y) z^n}{(1+\gamma)_n} = \exp(xyz)\ {}_0F_1 \left[-; 1+\gamma; \frac{x}{y}\right],$$

which is equivalent to

$$\sum_{n=0}^{\infty} \frac{L_n^{(\gamma)}(x,y) t^n}{(1+\gamma)_n} = \exp(yt)\ {}_0F_1 \left[-; 1+\gamma; \frac{t}{zy}\right].$$

Theorem 3.2. *The following relation holds:*

$$\sum_{n=0}^{\infty} \frac{(\gamma)_n I_n(\alpha;\beta;x,y) t^n}{n!} = (1-xyt)_2^{-\gamma} F_1 \left[\gamma, \alpha; \beta; \frac{-tx^2}{1-xyt}\right].$$

Proof. Consider the series

$$\sum_{n=0}^{\infty} \frac{(\gamma)_n I_n(\alpha;\beta;x,y) t^n}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-n)_k (\alpha)_k (\gamma)_n x^{n+k} y^{n-k} t^n}{n! k! (\beta)_k}$$
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k (\alpha)_k (\gamma)_{n+k} x^{n+2k} y^n t^{n+k}}{n! k! (\beta)_k}$$
$$= \sum_{k=0}^{\infty} \left[\sum_{n=0}^{\infty} \frac{(\gamma+k)_n (xyt)^n}{n!}\right] \frac{(\gamma)_k (\alpha)_k (-x^2 t)^k}{k! (\beta)_k}$$

$$= (1-xyt)^{-\gamma}\ {}_2F_1\left[\gamma,\alpha;\beta;\frac{-x^2t}{1-xyt}\right].$$

Hence the proof. □

Corollary 3.1. *If $\gamma=\beta$, then*

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(\beta)_n I_n(\alpha;\beta;x,y)t^n}{n!} =&(1-xyt)_1^{-\beta}F_0\left[\alpha;-;\frac{-x^2t}{1-xyt}\right]\\ =&(1-xyt)^{\alpha-\beta}(1-xyt+x^2t)^{-\alpha}.\end{aligned}$$

Therefore

$$\sum_{n=0}^{\infty}\frac{(\beta)_n I_n(\alpha;\beta;x,y)t^n}{n!} = (1-xyt)^{\alpha-\beta}(1-xyt+x^2t)^{-\alpha}. \tag{3.1}$$

4. Recurrence relations

(i) Let

$$G=\sum_{n=0}^{\infty}\frac{(\beta)_n I_n(\alpha;\beta;x,y)t^n}{n!}.$$

Then from (3.1)

$$G=(1-xyt)^{\alpha-\beta}(1-xyt+x^2t)^{-\alpha}.$$

Differentiating partially w.r.t. y and t respectively, we have

$$\frac{\partial G}{\partial y} = -(\alpha-\beta)xt(1-xyt)^{-1}G+\alpha xt(1-yt+x^2t)^{-1}G \tag{4.1}$$

and

$$\frac{\partial G}{\partial t} = -xy(\alpha-\beta)(1-xyt)^{-1}G-\alpha x(x-y)(1-xyt+x^2t)^{-1}G. \tag{4.2}$$

This implies

$$\begin{aligned}&y\frac{\partial G}{\partial y}-t\frac{\partial G}{\partial t}=\alpha tx^2(1-xyt+x^2t)^{-1}G\\ =&y\sum_{n=0}^{\infty}\frac{(\beta)_n I_n(\alpha;\beta;x,y)t^n}{n!}+xy(x-y)\sum_{n=0}^{\infty}\frac{(\beta)_n I_n'(\alpha;\beta;x,y)t^{n+1}}{n!}\\ &-\sum_{n=1}^{\infty}\frac{(\beta)_n I_n(\alpha;\beta;x,y)t^n}{(n-1)!}-x(x-y)\sum_{n=1}^{\infty}\frac{(\beta)_n I_n'(\alpha;\beta;x,y)t^{n+1}}{(n-1)!}\\ =&\alpha x^2\sum_{n=0}^{\infty}\frac{(\beta)_n I_n(\alpha;\beta;x,y)t^{n+1}}{n!},\end{aligned}$$

on comparing the comparing coefficient of t^n,we finally have

$$(4.3)\quad \begin{aligned}&(\beta+n-1)yI_n'(\alpha;\beta;x,y)+nxy(x-y)I_{n-1}'(\alpha;\beta;x,y)\\ &-(\beta+n-1)nI_n(\alpha;\beta;x,y)-nx[(n-1)(x-y)-\alpha x]I_{n-1}=0.\end{aligned}$$

(ii) Further, from (4.1) and (4.2), we have

$$(x-y)\frac{\partial G}{\partial y}+t\frac{\partial G}{\partial t}=-x^2t(\alpha-\beta)1-xyt)^{-1}G$$

or,

$$\begin{aligned}&(x-y)\sum_{n=0}^{\infty}\frac{(\beta)_nI_n(\alpha;\beta;x,y)t^n}{n!}-xy(x-y)\sum_{n=0}^{\infty}\frac{(\beta)_nI_n'(\alpha;\beta;x,y)t^{n+1}}{n!}\\ &+\sum_{n=1}^{\infty}\frac{(\beta)_nI_n(\alpha;\beta;x,y)t^n}{(n-1)!}-xy\sum_{n=1}^{\infty}\frac{(\beta)_nI_n(\alpha;\beta;x,y)t^{n+1}}{(n-1)!}\\ &=-x^2(\alpha-\beta)\sum_{n=0}^{\infty}\frac{(\beta)_nI_n(\alpha;\beta;x,y)t^{n+1}}{n!}.\end{aligned}$$

Now, comparing the coefficient of t^n, then

$$(4.4)\quad \begin{aligned}&(x-y)(\beta+n-1)I_n'(\alpha;\beta;x,y)-nxy(x-y)I_{n-1}'(\alpha;\beta;x,y)\\ &+n(\beta+n-1)I_n(\alpha;\beta;x,y)-nx[y(n-1)-x(\alpha-\beta)]I_{n-1}(\alpha;\beta;x,y)=0.\end{aligned}$$

Eliminating $I_n'(\alpha;\beta;x,y)$ and $I_{n-1}'(\alpha;\beta;x,y)$ respectively from (4.3) and (4.4),we have the following recurrence relations.

$$(4.5)\quad \begin{aligned}DI_n(\alpha;\beta;x,y)=&\frac{1}{xy(x-y)}\left[(\beta+n)I_{n+1}(\alpha;\beta;x,y)\right.\\ &\left.+(n+\alpha)x^2-(\beta+2n)xy]\right]I_n(\alpha;\beta;x,y),\end{aligned}$$

$$(4.6)\quad DI_n(\alpha;\beta;x,y)=nxI_{n-1}(\alpha;\beta;x,y).$$

Therefore (4.5) and (4.6) are two independent differential recurrence relations. From these equations we can determine the following linear ordinary differential equation

$$\begin{aligned}&\left\{y(x-y)D^2-[(n+\alpha-1)x-(\beta+2n-2)y]D\right.\\ &\left.-n(\beta+n-1)\right\}I_n(\alpha;\beta;x,y)=0\end{aligned}$$

where $D\equiv\dfrac{d}{dy}$.

Mixed Recurrence Relation

(a) We know that

$$I_n(\alpha;\beta;x,y) = (xy)^n\ {}_2F_1\left[-n,\alpha;\beta;\frac{x}{y}\right].$$

Differentiating the above equation with respect to y, we have

$$\begin{aligned}\frac{d}{dy}I_n(\alpha;\beta;x,y) =& nx^n y_2^{n-1}F_1\left[-n,\alpha;\beta;\frac{x}{y}\right]\\ &+\frac{n\alpha x^2}{\beta y}(xy)^{n-1}\ {}_2F_1\left[-(n-1),\alpha+1;\beta+1;\frac{x}{y}\right]\end{aligned}$$

or

$$\frac{d}{dy}I_n(\alpha;\beta;x,y) = \frac{n}{y}I_n(\alpha;\beta;x,y) + \frac{n\alpha x^2}{\beta y}I_{n-1}(\alpha+1;\beta+1;x,y),$$

which is a mixed recurrence relation for $I_n(\alpha;\beta;x,y)$.

(b) Differentiating the recurrence relation with respect to y , we get

$$\begin{aligned}I_n''(\alpha;\beta;x,y) =& \frac{n}{y}I_n'(\alpha;\beta;x,y) - \frac{n}{y^2}I_n(\alpha;\beta;x,y)\\ &+\frac{n\alpha x^2}{\beta y}I_{n-1}'(\alpha+1;\beta+1;x,y)\\ &-\frac{n\alpha x^2}{\beta y^2}I_{n-1}(\alpha+1;\beta+1;x,y).\end{aligned} \tag{4.7}$$

The differential equation for $I_n(\alpha;\beta;x,y)$ is

$$\begin{aligned}&y(x-y)\frac{d^2}{dy^2}I_n(\alpha;\beta;x,y) - [(n+\alpha-1)x-(\beta+2n-2)y]\frac{d}{dy}I_n(\alpha;\beta;x,y)\\ &\quad - n(\beta+n-1)I_n(\alpha;\beta;x,y) = 0.\end{aligned} \tag{4.8}$$

Eliminating $I_n''(\alpha;\beta;x,y)$ from the equations (4.7) and (4.8) and replacing n by $n+1$, α by $\alpha-1$ and β by $\beta-1$, we have

$$\begin{aligned}\frac{d}{dy}I_n(\alpha;\beta;x,y) =& \frac{(\alpha-1)x^2-(\beta+n-1)xy}{xy(x-y)}I_n(\alpha;\beta;x,y)\\ &+\frac{(\beta-1)}{xy(x-y)}I_{n+1}(\alpha-1;\beta-1;x,y).\end{aligned}$$

5. Conclusion

In this paper, certain classical properties for the two variable generalized hypergeometric polynomials have derived successfully by series manipulation method.

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