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SPHERICAL FUZZY BI-IDEALS OF GAMMA NEAR-RINGS

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ABSTRACT. The purpose of the article is to study about spherical fuzzy bi-ideal (SFBI) of Gamma-near-ring ($\mathcal{R}$) and their relevant results. We establish the relationship between FBI and SFBI of $\mathcal{R}$. Some characteristics of normal SFBI of $\mathcal{R}$ in terms of SFBIs are attained. Some properties of homomorphism, epimorphism and direct product between Gamma near-rings are also discussed.

1. INTRODUCTION

The fuzzy set(FS) was introduced by Zadeh[18] in 1965. It is identified as better tool for the scientific study of uncertainty, and came as a boost to the researchers working in the field of uncertainty. Many extensions and generalizations of FS was conceived by a number of researchers and a large number of real-life applications were developed in a variety of areas. In addition to this, parallel analysis of the classical results of many branches of Mathematics was also undisturbed in the fuzzy settings. One such abstract area in the branch of fuzzy algebra. It was initiated by Rosenfeld, who coined the idea of fuzzy subgroup(FSG) of a group in 1971 and studied basic properties of this structure. Liu[8] defined fuzzy invariant subgroups(FISGs) and fuzzy ideals(FIs) and discussed some properties. Pliz[11] introduced the algebraic structure near-rings as a generalization of rings and gave many examples and classification of near-rings. Kim and Kim[7] defined FIs of near-rings. Further properties of FIs in

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near-rings was studied by Hong et al.[3]. The monograph by Chinnadurai[1] gives a detailed discussion on FIs in algebraic structures. FIs in $\mathcal{R}$ was discussed by Jun et al.[5] . Yong et al. have discussed Weak bi-ideals of near-rings of Near-rings. Manikantan[9] studied some properties of FBIs of near-rings. Meenakumari and Tamizh chelvam have defined FBI in $\mathcal{R}$ and discussed some results of this structure. Srinivas and Nagaiah have proved some results on T-FIs of Γ-near-rings. Thillaigovindan, Chinnadurai and Kadalarasi have discussed Interval valued $(\epsilon, \epsilon \ \vee q)$-Fuzzy subnear-rings of Near-rings and Interval valued Fuzzy ideals near-rings. Chinnadurai and Kadalarasi[2] have defined the direct product of FIs in near-rings. Kahraman and Gundogdu[6] introduced spherical fuzzy sets as an extension of picture fuzzy sets. In our research work, we initiate SFBI of $\mathcal{R}$ and establish its properties and study the relationship between bi-ideal(BI) and SFBI of $\mathcal{R}$.

2. SPHERICAL FUZZY BI-IDEAL OF GAMMA NEAR-RING

Here we define SFBI of $\mathcal{R}$ and study their basic results. We have obtained the condition under which an arbitrary fuzzy subset of $\mathcal{R}$ becomes to SFBI.

Definition 2.1. *A FS $A_s = (\mu, \nu, \xi)$, where $\mu : \mathcal{R} \longrightarrow [0,1]$, $\nu : \mathcal{R} \longrightarrow [0,1]$ and $\xi : \mathcal{R} \longrightarrow [0,1]$ of $\mathcal{R}$ to be SFBI of $\mathcal{R}$ and Γ be non-empty set if the given conditions are satisfied*

(i) $\mu(u - v) \geq \min\{\mu(u), \mu(v)\}$,
(ii) $\nu(u - v) \geq \min\{\nu(u), \nu(v)\}$,
(iii) $\xi(u - v) \leq \max\{\xi(u), \xi(v)\}$,
(iv) $\mu(u\alpha v\beta w) \geq \min\{\mu(u), \mu(w)\}$,
(v) $\nu(u\alpha v\beta w) \geq \min\{\nu(u), \nu(w)\}$,
(vi) $\xi(u\alpha v\beta w) \leq \max\{\xi(u), \xi(w)\}$,

for all $u, v, w \in \mathcal{R}$ and $\alpha, \beta \in \Gamma$.

Example 1. *Let $\mathcal{R} = \{0, 1, 2, 3\}$ with binary operation " $+$ " on $\mathcal{R}$, $\Gamma = \{0, 1\}$ and $\mathcal{R} \times \Gamma \times \mathcal{R} \longrightarrow \mathcal{R}$ be a mapping. From the cayley table,*

TABLE 1. Cayley table

+	0 1 2 3
0	0 1 2 3
1	1 0 3 2
2	2 3 1 0
3	3 2 0 1

+	0 1 2 3
0	0 0 0 0
1	0 1 1 1
2	0 2 2 2
3	0 3 3 3

+	0 1 2 3
0	0 0 0 0
1	0 0 0 0
2	0 0 0 0
3	0 0 0 0

Define SFS $\mu : \mathcal{R} \longrightarrow [0,1]$ *by* $\mu(u) = 0.2, \mu(v) = 0.3, \mu(w) = 0.7$*;* $\nu : \mathcal{R} \longrightarrow [0,1]$ *by* $\nu(u) = 0.7, \nu(v) = 0.4, \nu(w) = 0.8$*;* $\xi : \mathcal{R} \longrightarrow [0,1]$ *by* $\xi(u) = 0.3, \xi(v) = 0.1, \mu(w) = 0.9$*. Then* A_s *is SFBI of* $\mathcal{R}$*.*

Lemma 2.1. *Let* A *be BI of a* $\mathcal{R}$*. For any* $0 < m < 1$*, there exists a SFBI* A_s *of* $\mathcal{R}$ *such that* $A_{s_m} = A$*.*

Proof. Let A be BI of $\mathcal{R}$. Define $A_s : \mathcal{R} \longrightarrow [0,1]$ by

$$A_s(u) = \begin{cases} m, & \text{if } u \in A \\ 0, & \text{if } u \notin A. \end{cases}$$

where m be a constant in $(0,1)$. Clearly $A_{s_m} = A$. Let $u, v \in \mathcal{R}$. If $u, v \in A$, then $\mu(u - v) = m \geq \min\{\mu(u), \mu(v)\}$,$\nu(u - v) = m \geq \min\{\nu(u), \nu(v)\}$ and $\xi(u - v) = m \leq \max\{\xi(u), \xi(v)\}$.

If at least one of u and v is not in A, then $u - v \notin A$ and so $\mu(u - v) = 0 = \min\{\mu(u), \mu(v)\}$, $\nu(u - v) = 0 = \min\{\nu(u), \nu(v)\}$ and $\xi(u - v) = 0 = \min\{\xi(u), \xi(v)\}$.

Let $u, v, w \in \mathcal{R}$ and $\alpha, \beta \in \Gamma$. If $u, w \in A$, then $\mu(u)$, $\nu(u)$, $\xi(u) = m$; $\mu(w)$, $\nu(w)$, $\xi(w) = m$. Also $\mu(u\alpha v\beta w) = m \geq \min\{\mu(u), \mu(w)\}$, $\nu(u\alpha v\beta w) = m \geq \min\{\nu(u), \nu(w)\}$ and $\xi(u\alpha v\beta w) = m \leq \max\{\xi(u), \xi(w)\}$.

If at least one of u and w is not in A, then $\mu(u\alpha v\beta w) \geq 0 = \min\{\mu(u), \mu(w)\}$, $\nu(u\alpha v\beta w) \geq 0 = \min\{\nu(u), \nu(w)\}$ and$\xi(u\alpha v\beta w) \leq 0 = \min\{\xi(u), \xi(w)\}$.

Thus A_s is SFBI of $\mathcal{R}$. □

Theorem 2.1. *If* A_s *and* σ_s *are SFBIs of* $\mathcal{R}$*, then* $A_s \wedge \sigma_s$ *is SFBI of* $\mathcal{R}$*.*

Proof. Let A_s and σ_s are SFBIs of $\mathcal{R}$. Let $u, v, w \in \mathcal{R}$ and $\alpha, \beta \in \Gamma$. Then,
$(\mu \wedge \sigma_s)(u - v) = \min\{\mu(u - v), \sigma_s(u - v)\}$, since by $(\mu \wedge \sigma)(u) = \min\{\mu(u), \sigma(u)\}$

$$\geq \min\{\min\{\mu(u), \mu(v)\}, \min\{\sigma_s(u), \sigma_s(v)\}\}$$

$= \min\{\min\{\min\{\mu(u), \mu(v), \sigma_s(u)\}, \sigma_s(v)\}\}$
$= \min\{\min\{\min\{\mu(u), \sigma_s(u)\}, \mu(v)\}, \sigma_s(v)\}$
$= \min\{\min\{\mu(u), \sigma_s(u)\}, \min\{\mu(v), \sigma_s(v)\}$
$= \min\{\mu \wedge \sigma(u)), (\mu \wedge \sigma(v))\}.$

Also $(\nu \wedge \sigma_s)(u - v) \geq \min\{\nu \wedge \sigma(u)), (\nu \wedge \sigma(v))\}$ and $(\xi \wedge \sigma_s)(u - v) \leq \max\{\xi \wedge \sigma(u)), (\xi \wedge \sigma(v))\}$.

Since $(\mu(u\alpha v\beta w) \geq \min\{\mu(u), \mu(v)),$

$(\mu \wedge \sigma_s)(u\alpha v\beta w) = \min\{\mu(u\alpha v\beta w), \sigma_s(u\alpha v\beta w))$
$\geq \min\{\min\{\mu(u), A_s(w)\}, \min\{\sigma_s(u), \sigma_s(w)\}\}$
$= \min\{\min\{\mu(u), \sigma_s(u)\}, \min\{\mu(w)\}\sigma_s(w)\}\}$
$= \min\{(\mu \wedge \sigma_s)(u), (\mu \wedge \sigma_s)(w)\}.$

Also $(\nu \wedge \sigma_s)(u\alpha v\beta w) \geq \min\{(\nu \wedge \sigma_s)(u), (\nu \wedge \sigma_s)(w))$ and $(\xi \wedge \sigma_s)(u\alpha v\beta w) \leq \max\{(\xi \wedge \sigma_s)(u), (\xi \wedge \sigma_s)(w))$.

Hence $(A_s \wedge \sigma_s)$ is a SFBI of $\mathcal{R}$. □

Lemma 2.2. *Let U is fuzzy subset of $\mathcal{R}$. Then U is BI of $\mathcal{R}$ if and only if μ_U is SFBI of $\mathcal{R}$.*

Proof. Let U be BI of $\mathcal{R}$. For $u, v \in U$, $u - v \in U$.

(i) Let $u, v \in \mathcal{R}$.

case(a): If $u, v \in U$, then $\mu_U(u) = 1$ and $\mu_U(v) = 1$.
Thus $\mu_U(u - v) = 1 \geq \min\{\mu(u), \mu(v)\}$.

case(b): If $u \in U$ and $v \notin U$, then $\mu_U(u) = 1$ and $\mu_U(v) = 0$.
Thus $\mu_U(u - v) = 0 \geq \min\{\mu(u), \mu(v))$.

case(c): If $u \notin U$ and $v \in U$, then $\mu_U(u) = 0$ and $\mu_U(v) = 1$.
Thus $\mu_U(u - v) = 0 \geq \min\{\mu(u), \mu(v)\}$.

case(d): If $u \notin U$ and $v \notin U$, then $\mu_U(u) = 0$ and $\mu_U(v) = 0$.
Thus $\mu_U(u - v) = 0 \geq \min\{\mu(u), \mu(v)\}$.

In the above four cases $\nu_U(u - v) \geq \min\{\nu(u), \nu(v)\}$ and $\xi_U(u - v) \leq \max\{\xi(u), \xi(v)\}$.

(ii) Let $u, v, w \in \mathcal{R}$.

case(a): If $u \in U$ and $w \in U$, then $\mu_U(u) = 1$ and $\mu_U(w) = 1$.
Thus $\mu_U(u\alpha v\beta w) = 1 \geq \min\{\mu(u), \mu(w)\}$.

case(b): If $u \in U$ and $w \notin U$, then $\mu_U(u) = 1$ and $\mu_U(w) = 0$.
Thus $\mu_U(u\alpha v\beta w) = 0 \geq \min\{\mu(u), \mu(w)\}$.

case(c): If $u \notin U$ and $w \in U$, then $\mu_U(u) = 0$ and $\mu_U(w) = 1$.

Thus $\mu_U(u\alpha v\beta w) = 0 \geq \min\{\mu(u), \mu(w)\}$.

case(d): If $u \notin U$ and $w \notin U$, then $\mu_U(u) = 0$ and $\mu_U(w) = 0$.

Thus $\mu_U(u\alpha v\beta w) = 0 \geq \min\{\mu(u), \mu(w)\}$.

Also $\nu_U(u\alpha v\beta w) \geq \min\{\nu(u), \nu(w)\}$ and $\xi_U(u\alpha v\beta w) \leq \max\{\xi(u), \xi(w)\}$ Thus A_{s_U} is a SFBI of $\mathcal{R}$.

Conversely, suppose A_{s_U} is a SFBI of $\mathcal{R}$. Then by lemma A_{s_U} has two value. Hence U is BI of $\mathcal{R}$. □

Theorem 2.2. *Let A_s be SFBI of $\mathcal{R}$ and A_s^* be FS in $\mathcal{R}$ defined by $A_s^*(u) = \frac{A_s(u)}{A_s(1)}$ for all $u \in \mathcal{R}$. Then A_s^* is normal SFBI of $\mathcal{R}$ containing A_s.*

Proof. Let A_s be SFBI of $\mathcal{R}$. For any $u, v, w \in \mathcal{R}$ and $\alpha \in \Gamma$, we have

$$\mu^*(u - v) = \frac{\mu(u-v)}{\mu(1)} \geq \frac{\min\{\mu(u),\mu(v)\}}{\mu(1)}$$
$$=\min\{\frac{\mu(u)}{\mu(1)}, \mu(v), \mu(1)\} = \min\{\mu^*(u), \mu^*(v)\},$$

and also

$$\nu^*(u - v) \geq \min\{\nu^*(u), \nu^*(v)\},\ \xi^*(u - v) \leq \max\{\xi^*(u), \xi^*(v)\}.$$
$$\mu^*(u\alpha v\beta w) = \frac{\mu(u\alpha v\beta w)}{\mu(1)} \geq \frac{\min\{\mu(u),\mu(w)\}}{\mu(1)}$$
$$= \min\{\frac{\mu(u)}{\mu(1)}, \mu(w), \mu(1)\} = \min\{\mu^*(u), \mu^*(w)\},$$

and also

$$\nu^*(u\alpha v\beta w) \geq \min\{\nu^*(u), \nu^*(w)\},\ \xi^*(u\alpha v\beta w) \geq \min\{\xi^*(u), \xi^*(w)\}.$$

Hence A_s^* is SFBI of $\mathcal{R}$. Clearly $A_s^*(1) = \frac{A_s(1)}{A_s(1)} = 1$ and $A_s \subset A_s^*$. Therefore A_s^* is normal SFBI of $\mathcal{R}$ contains A_s. □

Theorem 2.3. *Let A_s be SFBI of $\mathcal{R}$ and let A_s^+ be FS in $\mathcal{R}$ defined by $A_s^+(u) = A_s(u) + 1 - A_s(1)$ for all $u \in \mathcal{R}$. Then A_s^+ is normal SFBI of $\mathcal{R}$ containing A_s.*

Proof. The proof is straightforward. □

Theorem 2.4. *Let $\mathcal{R}$ and $\mathcal{S}$ are Gamma near-rings and let $\zeta : \mathcal{R} \longrightarrow \mathcal{S}$. An onto homomorphic image of SFBI with* sup *property $A_s(m_0) = \sup_{m\in\mathcal{N}} A_s(m)$ is SFBI.*

Proof. Let $\mathcal{R}$ and $\mathcal{S}$ be Gamma near-rings. Let $\zeta : \mathcal{R} \longrightarrow \mathcal{S}$, epimorphism and A_s be SFBI of $\mathcal{R}$ with sup property.

Let $u', v' \in \mathcal{S}$, $u_0 \in \zeta^{-1}(u')$, $v_0 \in \zeta^{-1}(v')$ and $w_0 \in \zeta^{-1}(w')$ be such that

$$A_s(u_0) = \sup_{n\in\zeta^{-1}(u')} A_s(n), \quad A_s(v_0) = \sup_{n\in\zeta^{-1}(v')} A_s(n), \quad A_s(w_0) = \sup_{n\in\zeta^{-1}(w')} A_s(n),$$

respectively. Then for any $\alpha \in \Gamma$, we have

$$\mu^\zeta(u' - v') = \sup_{w\in\zeta^{-1}(u',v')} \mu(w)$$

$$= \mu(u_0 - v_0)$$
$$\geq \min\{\mu(u_0), \mu(v_0)\}$$
$$= \min\{ \sup_{n\in\zeta^{-1}(u')} \mu(n), \sup_{n\in\zeta^{-1}(v')} \mu(n) \}$$
$$= \min\{\mu^{\zeta}(u'), \mu^{\zeta}(v')\},$$

and also

$$\nu^{\zeta}(u' - v') \geq \min\{\nu^{\zeta}(u'), \nu^{\zeta}(v')\}, \ \xi^{\zeta}(u' - v') \leq \max\{\xi^{\zeta}(u'), \xi^{\zeta}(v')\}.$$
$$\mu^{\zeta}(u'\alpha v'\beta w') = \sup_{w\in\zeta^{-1}(u'\alpha v'\beta w')} \mu(w)$$
$$= \mu(u_0\alpha v_0\beta w_0)$$
$$\geq \min\{\mu(u_0), \mu(v_0)\}$$
$$= \min\{ \sup_{n\in\zeta^{-1}(u')} \mu(n), \sup_{n\in\zeta^{-1}(w')} \mu(n) \}$$
$$= \min\{\mu^{\zeta}(u'), \mu^{\zeta}(w')\}, \text{and also}$$
$$\nu^{\zeta}(u'\alpha v'\beta w') \geq \min\{\nu^{\zeta}(u'), \nu^{\zeta}(w')\}, \ \xi^{\zeta}(u'\alpha v'\beta w') \leq \max\{\xi^{\zeta}(u'), \xi^{\zeta}(w')\}.$$

□

Theorem 2.5. *An epimorphic pre-image of a SFBI of $\mathcal{R}$ is SFBI.*

Proof. Let $\mathcal{R}$ and $\mathcal{S}$ be Gamma near rings. Let $\zeta : \mathcal{R} \longrightarrow \mathcal{S}$ be an epimorphism. Let γ be SFBI of $\mathcal{S}$ and μ be the pre-image of γ under ζ. Then for any $u, v, w \in \mathcal{R}$ and $\alpha \in \Gamma$, we have

$$\mu(u - v) = (\gamma \circ \zeta)(u - v)$$
$$= \gamma(\zeta(u - v))$$
$$= \gamma(\zeta(u) - \zeta(v))$$
$$\geq \min\{\gamma(\zeta(u)), \gamma(\zeta(v))\}$$
$$= \min\{(\gamma \circ \zeta)(u), (\gamma \circ \zeta)(v)\}$$
$$= \min\{\mu(u), \mu(v)\},$$

and also

$$\nu(u - v) \geq \min\{\nu(u), \nu(v)\}, \ \xi(u - v) \leq \max\{\xi(u), \xi(v)\}$$
$$\mu(u\alpha v\beta w) = (\gamma \circ \zeta)(u\alpha v\beta w)$$
$$= \gamma(\zeta(u\alpha v\beta w))$$
$$= \gamma(\zeta(u)\alpha\zeta(v)\beta\zeta(w))$$
$$\geq \min\{\gamma(\zeta(u)), \gamma(\zeta(w))\}$$
$$= \min\{(\gamma \circ \zeta)(u), (\gamma \circ \zeta)(w)\}$$
$$= \min\{\mu(u), \mu(w)\}, \text{ and also}$$
$$\nu(u\alpha v\beta w) \geq \min\{\nu(u), \nu(w)\}, \ \xi(u\alpha v\beta w) \leq \min\{\xi(u), \xi(w)\}.$$

□

Theorem 2.6. *If A_s be SFBI of $\mathcal{R}$, then the complement A'_s is also SFBI of $\mathcal{R}$.*

Proof. For $u, v, w \in \mathcal{R}$ and $\alpha, \beta \in \Gamma$, we have

$$\mu'(u-v) = 1-\mu(u-v) \geq 1-\min\{\mu(u), \mu(v)\} = \min\{1-\mu(u), 1-\mu(v)\}$$
$$= \min\{\mu'(u), \mu'(v)\},$$

and also

$$\nu'(u-v) \geq \min\{\nu'(u), \nu'(v)\},\ \xi'(u-v) \leq \max\{\xi'(u), \xi'(v)\},$$
$$\mu'(u\alpha v\beta w) = 1-\mu(u\alpha v\beta w) \geq 1-\min\{\mu(u), \mu(w)\} = \min\{1-\mu(u), 1-\mu(w)\}$$
$$= \min\{\mu'(u), \mu'(w)),$$

and also

$$\nu'(u\alpha v\beta w) \geq \min\{\nu'(u), \nu'(w)\},\ \xi'(u\alpha v\beta w) \leq \max\{\xi'(u), \xi'(w)\}.$$

Hence A'_s is also SFBI of $\mathcal{R}$. □

Theorem 2.7. *Let $\zeta : \mathcal{R}_1 \longrightarrow \mathcal{R}_2$ is an onto homomorphism of $\mathcal{R}$ and if A_s is SFBI of $\mathcal{R}_1$, then $\zeta(A_s)$ is SFBI of $\mathcal{R}_2$.*

Proof. Let $A_{s_1} = \zeta^{-1}(v_1)$ and $A_{s_2} = \zeta^{-1}(v_2)$, where $v_1, v_2 \in \mathcal{R}_2$ are fuzzy subsets of R_2. Similarly let, $A_{s_3} = \zeta^{-1}(v_1 - v_2)$. Consider the set

$$A_{s_1} - A_{s_2} = \{l_1 - l_2; l_1 \in A_{s_1}, l_2 \in A_{s_2}\}.$$

If $u \in A_{s_1} - A_{s_2}$, then $u = u_1 - u_2$ for some $u_1 \in A_{s_1}$ and $u_2 \in A_{s_2}$ and so

$$\zeta(u) = \zeta(u_1 - u_2) = \zeta(u_1) - \zeta(u_2) = v_1 - v_2,$$

which implies $u \in \zeta^{-1}(v_1 - v_2) = A_{s_3}$. Thus $A_{s_1} - A_{s_2} \subseteq A_{s_3}$. That is $\{u : u \in \zeta^{-1}(v_1 - v_2)\} \geq \{u_1 - u_2 : u_1 \in \zeta^{-1}(v_1), v_2 \in \zeta^{-1}(v_2)\}$. Let $\alpha, \beta \in \Gamma$ and $v_3 \in \mathcal{R}_2$. Then

$$\zeta(\mu)(v_1 - v_2) = \sup\{\mu(u) : u \in \zeta^{-1}(v_1 - v_2)\}$$
$$\geq \sup\{\mu(u_1 - u_2) : u_1 \in \zeta^{-1}(v_1), u_2 \in \zeta^{-1}(v_2)\}$$
$$\geq \sup\{\min\{\mu(u_1), \mu(u_2)\} : u_1 \in \zeta^{-1}(v_1), u_2 \in \zeta^{-1}(v_2)\}$$
$$= \min\{\sup\{\mu(u_1) : u_1 \in \zeta^{-1}(v_1)\}, \sup\{\mu(u_2) : u_2 \in \zeta^{-1}(v_2)\}\}$$
$$= \min\{\zeta(\mu)(v_1), \zeta(\mu)(v_2)\}$$

and also

$$\zeta(\nu)(v_1 - v_2) \geq \min\{\zeta(\nu)(v_1), \zeta(\nu)(v_2)\},$$
$$\zeta(\xi)(v_1 - v_2) \leq \max\{\zeta(\xi)(v_1), \zeta(\xi)(v_2)\},$$
$$\zeta(\mu)(v_1\alpha v_2\beta v_3) = \sup\{\mu(u) : u \in \zeta^{-1}(v_1\alpha v_2\beta v_3)\}$$
$$\geq \sup\{\mu(u_1\alpha u_2\beta u_3) : u_1 \in \zeta^{-1}(v_1), u_2 \in \zeta^{-1}(v_2), u_3 \in \zeta^{-1}(v_3)\}$$
$$\geq \sup\{\min\{\mu(u_1), \mu(u_3)\} : u_1 \in \zeta^{-1}(v_1), u_3 \in \zeta^{-1}(v_3)\}$$

$$= \min\{\sup\{\mu(u_1) : u_1 \in \zeta^{-1}(v_1)\}, \sup\{\mu(u_3) : u_3 \in \zeta^{-1}(v_3)\}\}$$
$$= \min\{\zeta(\mu)(v_1), \zeta(\mu)(v_3)\}$$

and also

$$\zeta(\nu)(v_1\alpha v_2\beta v_3) \geq \min\{\zeta(\nu)(v_1), \zeta(\nu)(v_3)\},$$
$$\zeta(\xi)(v_1\alpha v_2\beta v_3) \leq \max\{\zeta(\xi)(v_1), \zeta(\xi)(v_3)\}.$$

Hence $\zeta(A_s)$ is SFBI of $\mathcal{R}_2$. □

Theorem 2.8. *Let A_s and σ_s be SFBIs of $\mathcal{R}$. Then $A_s + \sigma_s$ is the smallest SFBI of $\mathcal{R}$ that contains both A_s and σ_s.*

Proof. Let A_s and σ_s be SFBIs of $\mathcal{R}$. Let $u, v, w \in \mathcal{R}$ and $\alpha, \beta \in \Gamma$. Let $u = l + m$, $v = n + o$; $l, m, n, o \in \mathcal{R}$. Then we have

$$\begin{aligned}
u - v &= (l + m) - (n + o) = l + m - n - o \\
&= (m + l - m) - n + (n + m - n) - o = p + q,
\end{aligned}$$

$$\begin{aligned}
(\mu + \sigma_s)(u - v) &= \sup_{u-v=p+q} \{\min\{\mu(p), \sigma_s(q)\}\} \\
&= \sup_{u=l+m, v=n+o} \{\min\{\mu((m + l - m) - n), \sigma_s((n + m - n) - o)\} \\
&\geq \sup_{u=l+m, v=n+o} \{\min\{\min\{\mu(m + l - m), A_s(n)\}, \\
&\quad \min\{\sigma_s(n + m - n), \sigma_s(o)\}\}\} \\
&\geq \sup_{u=l+m, v=n+o} \{\min\{\min\{\mu(l), \mu(n)\}, \min\{\sigma_s(m), \sigma_s(o)\}\}\} \\
&= \sup_{u=l+m, v=n+o} \{\min\{\min\{\mu(l), \sigma_s(m)\}, \min\{\mu(n), \sigma_s(o)\}\}\} \\
&= \min\{ \sup_{u=l+m} \{(\min\{\mu(l), \sigma_s(m)\}\}, \sup_{v=n+o} \{\min\{\mu(n), \sigma_s(o)\}\}\} \\
&= \min\{(\mu + \sigma_s)(u), (\mu + \sigma_s)(v)\},
\end{aligned}$$

since by

$$(\mu + \sigma)(u) = \begin{cases} \sup(\min(\mu(l), \sigma(m)), & \text{if } u = l + m \\ 0, & \text{otherwise.} \end{cases},$$

and also

$$(\nu + \sigma_s)(u - v) \geq \min\{(\nu + \sigma_s)(u), (\nu + \sigma_s)(v)\},$$

$$(\xi + \sigma_s)(u - v) \geq \min\{(\xi + \sigma_s)(u), (\xi + \sigma_s)(v)\}$$

$$\begin{aligned}
(\mu + \sigma_s)(u\alpha v\beta w) &= \sup\{\min\{\mu(u\alpha v\beta w), \sigma_s(u\alpha v\beta w)\}\} \\
&\geq \sup\{\min\{\mu(u), \mu(w)\}, \min\{\sigma_s(u), \sigma_s(w)\}\} \\
&= \min\{\sup\{\min\{\mu(u), \sigma_s(u)\}, \min\{\mu(w), \sigma_s(w)\}\}\} \\
&= \min\{(\mu + \sigma_s)(u), (\mu + \sigma_s)(w)\}
\end{aligned}$$

and also

$$(\nu + \sigma_s)(u\alpha v\beta w) \geq \min\{(\nu + \sigma_s)(u), (\nu + \sigma_s)(w)\},$$
$$(\xi + \sigma_s)(u\alpha v\beta w) \geq \min\{(\xi + \sigma_s)(u), (\xi + \sigma_s)(w)\}.$$

Thus $A_s + \sigma_s$ is FBI of $\mathcal{R}$.

Further, as $u = u + 0$ and $u = 0 + u$, so

$$(A_s + \sigma_s)(u) \geq A_s(u) \quad \text{and} \quad (A_s + \sigma_s)(u) \geq \sigma_s(u).$$

If η is FBI of $\mathcal{R}$ such that $\eta(u) \geq A_s(u)$ and $\eta(u) \geq \sigma_s(u)$ for all $u \in \mathcal{R}$,

$$\begin{aligned}(A_s + \sigma_s)(u) &= \sup_{u=l+m} \{\min\{A_s(l), \sigma_s(m)\}\} \\ &\leq \sup_{u=l+m} \{\min\{\eta(l), \eta(m)\}\} \\ &= \sup_{u=l+m} \{\min\{(\eta(l), \eta(-m)\} \\ &\leq \sup_{u=l+m} \{\eta(l+m)\} \\ &= \eta(u).\end{aligned}$$

Thus $A_s + \sigma_s$. □

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