

EQUIVALENCE RELATIONS ON IMPLICATION-BASED FUZZY AUTOMATON OVER A FINITE GROUP

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ABSTRACT. The study explores *fuzzy semiautomaton* based on implications more extensively over finite group. Different forms of relations are elucidated on *fuzzy semiautomaton* based on implications. They are analyzed to be congruence relations. The classes of equivalence induced by these relations form semigroups. Another approach of external product of *implication-based fuzzy subgroups* over finite groups using the apprehension of *implication-based fuzzy subgroup* of a finite group is conceptualized. Therefore, the product thus defined is also an *implication-based fuzzy subgroup* of the finite group.

1. INTRODUCTION

Zadeh [1] initially proposed the *fuzzy set* concept in 1965. Earlier, in 1969, it was Wee [2] who interpreted the notion of *fuzzy automata*. Rosenfeld [3] had advanced the study in 1971 with the application of Zadeh's definition of fuzzy sets to groups. It contributed to the incorporation of several research works that were steered on their algebraic structures. [4–6] had also carried out some more analyses on the *fuzzy normal subgroups*. Asok Kumar [7] has made an intensive analysis on the products of fuzzy subgroups. In 1997, Malik [8] prompted a research into the algebraic techniques of the fuzzy finite state machine in fuzzy automata theory. Hofer [9] and Fong [10] had carried a comprehensive investigation into group semiautomaton. P. Das [11] fuzzified the notion of

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group semiautomaton which he considered over a finite group with the idea of *fuzzy semiautomaton*. Yuan [12] has conceived an apprehension regarding the *implication-based fuzzy subgroup*. Selvarathi [13–15] had asserted an approach over a finite group to the *implication-based fuzzy normal subgroup*, *implication-based fuzzy semiautomaton* and *implication-based intuitionistic fuzzy subgroup*. The theoretical study in this paper includes further analysis in the field above and introduction of a finite group's definition of equivalence relations, congruence relationships and semigroups of implication-based fuzzy semiautomaton. The paper exemplifies the homomorphic and isomorphic relationship between different semigroups.

2. PRELIMINARIES

Definition 2.1. [3] Let $(\Omega, \cdot)$ be a group. Let a fuzzy set in Ω be a function Υ from Ω to $[0, 1]$. Υ will be called a fuzzy subgroup of Ω , if for all ξ_1, ξ_2 in Ω , $\Upsilon(\xi_1 \xi_2) \geq \min(\Upsilon(\xi_1), \Upsilon(\xi_2))$ and $\Upsilon(\xi_1^{-1}) \geq \Upsilon(\xi_1)$.

Let χ be an universe of discourse and $(\Omega, \cdot)$ be a group. In fuzzy logic, $[\alpha]$ is used to denote the truth value of fuzzy proposition α . The fuzzy logical and the corresponding set theoretical notations used in this paper are

$$\begin{aligned} (\xi \in \Upsilon) &= \Upsilon(\xi); \\ (\alpha \wedge \beta) &= \min\{[\alpha], [\beta]\}; \\ (\alpha \rightarrow \beta) &= \min\{1, 1 - [\alpha] + [\beta]\}; \\ (\forall \xi \alpha(\xi)) &= \inf_{\xi \in X} [\alpha(\xi)]; \\ (\exists \xi \alpha(\xi)) &= \sup_{\xi \in X} [\alpha(\xi)]; \text{ and} \\ \models \alpha &\text{ if and only if } [\alpha] = 1 \text{ for all valuations.} \end{aligned}$$

The truth valuation rules used here are that of the Łukasiewicz system of continuous-valued logic.

The concept of λ - tautology is $\models_{\lambda} \alpha$ if and only if $[\alpha] \geq \lambda$ for all valuation by Ying [16].

Definition 2.2. [12] Let Υ be a fuzzy subset of a finite group Ω and $\lambda \in (0, 1]$ is a fixed number. If for any $\xi_1, \xi_2 \in \Omega$, $\models_{\lambda} (\xi_1 \in \Upsilon) \wedge (\xi_2 \in \Upsilon) \rightarrow (\xi_1 \xi_2 \in \Upsilon)$ and $\models_{\lambda} (\xi_1 \in \Upsilon) \rightarrow (\xi_1^{-1} \in \Upsilon)$. Then Υ is called an *implication-based fuzzy subgroup* of Ω .

Definition 2.3. [13] Let Υ be an implication-based fuzzy subgroup of Ω , $\lambda \in (0, 1]$ is a fixed number and $f : \Omega \rightarrow \Omega$ be a function defined on Ω . Then the implication-based fuzzy subgroup Φ of $f(\Omega)$ is defined by $\models_{\lambda} (\exists \xi \{(\xi \in \Upsilon)\}; \xi \in f^{-1}(\psi)) \rightarrow (\psi \in \Phi)$, for all $\psi \in f(\Omega)$.

Similarly if Φ is an implication-based fuzzy subgroup of $f(\Omega)$ then the implication-based fuzzy subgroup $\Upsilon = f \circ \Phi$ in Ω is defined as $\models_{\lambda} (f(\xi) \in \Phi) \rightarrow (\xi \in \Upsilon)$ for all $\xi \in \Omega$ and is called the pre-image of Φ under f .

Definition 2.4. [13] An implication-based fuzzy subgroup Υ of Ω is called an implication-based fuzzy normal subgroup if $\models_{\lambda} (\xi\psi \in \Upsilon) \rightarrow (\psi\xi \in \Upsilon) \quad \forall \xi, \psi \in \Omega$ where $\lambda \in (0, 1]$ is a fixed number.

Definition 2.5. [14] Let Υ and Φ be two implication-based fuzzy subgroups of the finite group $(\Omega, \cdot)$, then the internal product $\Upsilon \cdot \Phi$ is defined by

$$\models_{\lambda} (\exists \psi, \tau \{(\psi \in \Upsilon) \wedge (\tau \in \Phi)\}; \psi\tau = \xi; \psi, \tau \in \Omega) \rightarrow (\xi \in \Upsilon \cdot \Phi), \quad \xi \in \Omega.$$

Proposition 2.1. [14] If Φ is an implication-based fuzzy normal subgroup of Ω and Ψ is an implication-based fuzzy subgroup of Ω then (i) $\Phi \cdot \Psi = \Psi \cdot \Phi$ (ii) $\Phi \cdot \Psi$ is an implication-based fuzzy subgroup of Ω .

Definition 2.6. [14] Let Υ be an implication-based fuzzy subgroup over a finite group Ω . An implication-based fuzzy semiautomaton over the finite group $(\Omega, \cdot)$ is a triple $\Theta = (\Omega, \Delta, \Upsilon)$ where Δ denotes the set of all logic variables. (i.e.,) $\Upsilon : \Omega \times \Delta \times \Omega \rightarrow [0, 1]$.

Let Δ^* denote the set of all combination of these logic variables along with the 0 function.

Definition 2.7. [14] Define $\Upsilon^* : \Omega \times \Delta^* \times \Omega \rightarrow [0, 1]$ by

$$\begin{aligned} \models_{\lambda} ((\alpha, \mathbf{0}, \beta) \in \Upsilon^*) &\rightarrow 1 && (\text{If } \alpha = \beta) \\ \models_{\lambda} ((\alpha, \mathbf{0}, \beta) \in \Upsilon^*) &\rightarrow 0 && (\text{Here } \lambda = 0 \text{ and } \alpha \neq \beta) \\ \models_{\lambda} (\exists \gamma \{((\beta, \xi, \gamma) \in \Upsilon^*) \wedge ((\gamma, \omega, \alpha) \in \Upsilon^*)\}; \gamma \in \Omega) &\rightarrow ((\beta, \xi \odot \omega, \alpha) \in \Upsilon^*), \end{aligned}$$

for all $\alpha, \beta \in \Omega; \xi \in \Delta^*; \omega \in \Delta$. Hereafter $\Theta = (\Omega, \Delta, \Upsilon)$ be an implication-based fuzzy semiautomaton over the finite Ω where Ω is a finite group with the identity element ' ε ' and $\lambda \in (0, 1]$ is a fixed number.

3. SEMIGROUPS OF IMPLICATION-BASED FUZZY SEMIAUTOMATON OF FINITE GROUPS

Definition 3.1. Define a relation $\supseteq$ on Δ^* by for all $\xi, \psi \in \Delta^*$ as follows $\xi \supseteq \psi$ if and only if

$$\begin{aligned} \models_{\lambda} ((\zeta, \xi, \eta) \in \Upsilon^*) &\rightarrow ((\zeta, \psi, \eta) \in \Upsilon^*), \\ \models_{\lambda} ((\zeta, \psi, \eta) \in \Upsilon^*) &\rightarrow ((\zeta, \xi, \eta) \in \Upsilon^*), \quad \forall \zeta, \eta \in \Omega. \end{aligned}$$

Theorem 3.1. Let $\Theta = (\Omega, \Delta, \Upsilon)$ be an implication-based fuzzy semiautomaton over the finite group Ω . Define a relation $\supseteq$ on Δ^* by for all $\xi, \psi \in \Delta^*$ as follows $\xi \supseteq \psi$ if and only if

$$\begin{aligned} (i) \models_{\lambda} ((\zeta, \xi, \eta) \in \Upsilon^*) &\rightarrow ((\zeta, \psi, \eta) \in \Upsilon^*), \\ (ii) \models_{\lambda} ((\zeta, \psi, \eta) \in \Upsilon^*) &\rightarrow ((\zeta, \xi, \eta) \in \Upsilon^*), \quad \forall \zeta, \eta \in \Omega \end{aligned}$$

Then $\supseteq$ is a congruence relation on Δ^* .

Proof. Let $\xi, \psi, \tau \in \Delta^*$ and $\zeta, \eta \in \Omega$. Clearly (i) and (ii) are satisfied if $\xi = \psi$, implying $\Rightarrow \xi \supseteq \xi$. Therefore $\supseteq$ is reflexive.

Let $\xi \supseteq \psi$.

$$\begin{aligned} \Rightarrow \models_{\lambda} ((\zeta, \xi, \eta) \in \Upsilon^*) &\rightarrow ((\zeta, \psi, \eta) \in \Upsilon^*) \\ \models_{\lambda} ((\zeta, \psi, \eta) \in \Upsilon^*) &\rightarrow ((\zeta, \xi, \eta) \in \Upsilon^*) \quad \forall \zeta, \eta \in \Omega \end{aligned}$$

$$\begin{aligned} \Rightarrow \models_{\lambda} ((\zeta, \psi, \eta) \in \Upsilon^*) &\rightarrow ((\zeta, \xi, \eta) \in \Upsilon^*) \\ \models_{\lambda} ((\zeta, \xi, \eta) \in \Upsilon^*) &\rightarrow ((\zeta, \psi, \eta) \in \Upsilon^*) \quad \forall \zeta, \eta \in \Omega \end{aligned}$$

$$\Rightarrow \xi \supseteq \psi$$

Therefore $\supseteq$ is symmetric.

Let $\xi \supseteq \psi$ and $\psi \supseteq \tau$.

$\xi \supseteq \psi$

$$\begin{aligned} \Rightarrow \models_{\lambda} ((\zeta, \xi, \eta) \in \Upsilon^*) &\rightarrow ((\zeta, \psi, \eta) \in \Upsilon^*) \\ \models_{\lambda} ((\zeta, \psi, \eta) \in \Upsilon^*) &\rightarrow ((\zeta, \xi, \eta) \in \Upsilon^*) \quad \forall \zeta, \eta \in \Omega \end{aligned}$$

$\psi \supseteq \tau$

$$\begin{aligned} \Rightarrow \models_{\lambda} ((\zeta, \psi, \eta) \in \Upsilon^*) &\rightarrow ((\zeta, \tau, \eta) \in \Upsilon^*) \\ \models_{\lambda} ((\zeta, \tau, \eta) \in \Upsilon^*) &\rightarrow ((\zeta, \psi, \eta) \in \Upsilon^*) \quad \forall \zeta, \eta \in \Omega \end{aligned}$$

Now,

$$\begin{aligned} \models_{\lambda} ((\zeta, \xi, \eta) \in \Upsilon^*) &\rightarrow ((\zeta, \psi, \eta) \in \Upsilon^*) \\ &\rightarrow ((\zeta, \tau, \eta) \in \Upsilon^*) \quad \forall \zeta, \eta \in \Omega \\ \models_{\lambda} ((\zeta, \tau, \eta) \in \Upsilon^*) &\rightarrow ((\zeta, \psi, \eta) \in \Upsilon^*) \\ &\rightarrow ((\zeta, \xi, \eta) \in \Upsilon^*) \quad \forall \zeta, \eta \in \Omega \end{aligned}$$

implying $\xi \triangleright \tau$. Thus $\triangleright$ is transitive. Therefore $\triangleright$ is an equivalence relation on Δ^* .

Now let $\tau \in \Delta^*$ and $\xi \triangleright \psi$ for all $\zeta, \eta \in \Omega$. Then

$$\begin{aligned} \models_{\lambda} ((\zeta, \xi \odot \tau, \eta) \in \Upsilon^*) &\rightarrow (\exists \gamma \{((\zeta, \xi, \gamma) \in \Upsilon^*) \wedge ((\gamma, \tau, \eta) \in \Upsilon^*)\}; \gamma \in \Omega) \\ &\rightarrow (\exists \gamma \{((\zeta, \psi, \gamma) \in \Upsilon^*) \wedge ((\gamma, \tau, \eta) \in \Upsilon^*)\}; \gamma \in \Omega) \\ &\rightarrow ((\zeta, \psi \odot \tau, \eta) \in \Upsilon^*) \end{aligned}$$

Also

$$\begin{aligned} \models_{\lambda} ((\zeta, \psi \odot \tau, \eta) \in \Upsilon^*) &\rightarrow (\exists \gamma \{((\zeta, \psi, \gamma) \in \Upsilon^*) \wedge ((\gamma, \tau, \eta) \in \Upsilon^*)\}; \gamma \in \Omega) \\ &\rightarrow (\exists \gamma \{((\zeta, \xi, \gamma) \in \Upsilon^*) \wedge ((\gamma, \tau, \eta) \in \Upsilon^*)\}; \gamma \in \Omega) \\ &\rightarrow ((\zeta, \xi \odot \tau, \eta) \in \Upsilon^*) \end{aligned}$$

implying $\xi \odot \tau \triangleright \psi \odot \tau$.

Similarly we can prove that $\tau \odot \xi \triangleright \tau \odot \psi$.

Therefore $\triangleright$ is a congruence relation on Δ^* . □

Definition 3.2. Define a relation $\trianglelefteq$ on Δ^* by for all $\xi, \psi \in \Delta^*$ as follows $\xi \trianglelefteq \psi$ if and only if $\forall \zeta, \eta \in \Omega$.

$$\begin{aligned} \models_{\lambda} (\epsilon) &\rightarrow ((\zeta, \xi, \eta) \in \Upsilon^*) \text{ if and only if} \\ \models_{\lambda} (\epsilon) &\rightarrow ((\zeta, \psi, \eta) \in \Upsilon^*) \end{aligned}$$

where ϵ is a very small value such that $\epsilon > 0$.

Theorem 3.2. Let $\Theta = (\Omega, \Delta, \Upsilon)$ be an implication-based fuzzy semiautomaton over the finite group Ω . Let $\xi, \psi \in \Delta^*$. Define a relation $\trianglelefteq$ on Δ^* by $\xi \trianglelefteq \psi$ if and only if

$$\begin{aligned} \models_{\lambda} (\epsilon) &\rightarrow ((\zeta, \xi, \eta) \in \Upsilon^*) \text{ if and only if} \\ \models_{\lambda} (\epsilon) &\rightarrow ((\zeta, \psi, \eta) \in \Upsilon^*) \end{aligned}$$

where ϵ is a very small value such that $\epsilon > 0$. Then $\trianglelefteq$ is a congruence relation on Δ^* .

Proof. Let $\xi, \psi, \tau \in \Delta^*$ and $\zeta, \eta \in \Omega$.

(i) Clearly $\xi \trianglelefteq \xi$ by the definition. Therefore $\trianglelefteq$ is reflexive.

(ii) Let $\xi \trianglelefteq \psi$. Then

$$\begin{aligned} \Rightarrow \quad & \models_{\lambda} (\epsilon) \rightarrow ((\zeta, \xi, \eta) \in \Upsilon^*) \Leftrightarrow \models_{\lambda} (\epsilon) \rightarrow ((\zeta, \psi, \eta) \in \Upsilon^*) \\ \Rightarrow \quad & \models_{\lambda} (\epsilon) \rightarrow ((\zeta, \psi, \eta) \in \Upsilon^*) \Leftrightarrow \models_{\lambda} (\epsilon) \rightarrow ((\zeta, \xi, \eta) \in \Upsilon^*) \quad \forall \zeta, \eta \in \Omega \end{aligned}$$

and $\psi \trianglelefteq \xi$. Therefore $\trianglelefteq$ is symmetric.

(iii) Let $\xi \trianglelefteq \psi$ and $\psi \trianglelefteq \tau$. Then

$$\begin{aligned} \xi \trianglelefteq \psi & \Rightarrow \models_{\lambda} (\epsilon) \rightarrow ((\zeta, \xi, \eta) \in \Upsilon^*) \Leftrightarrow \models_{\lambda} (\epsilon) \rightarrow ((\zeta, \psi, \eta) \in \Upsilon^*) \quad \forall \zeta, \eta \in \Omega \\ \psi \trianglelefteq \tau & \Rightarrow \models_{\lambda} (\epsilon) \rightarrow ((\zeta, \psi, \eta) \in \Upsilon^*) \Leftrightarrow \models_{\lambda} (\epsilon) \rightarrow ((\zeta, \tau, \eta) \in \Upsilon^*) \quad \forall \zeta, \eta \in \Omega \end{aligned}$$

Now

$$\begin{aligned} \models_{\lambda} (\epsilon) \rightarrow ((\zeta, \xi, \eta) \in \Upsilon^*) & \Leftrightarrow \models_{\lambda} (\epsilon) \rightarrow ((\zeta, \psi, \eta) \in \Upsilon^*) \\ & \Leftrightarrow \models_{\lambda} (\epsilon) \rightarrow ((\zeta, \tau, \eta) \in \Upsilon^*) \end{aligned}$$

Therefore $\models_{\lambda} (\epsilon) \rightarrow ((\zeta, \xi, \eta) \in \Upsilon^*) \Leftrightarrow \models_{\lambda} (\epsilon) \rightarrow ((\zeta, \tau, \eta) \in \Upsilon^*) \forall \zeta, \eta \in \Omega$, implying $\xi \trianglelefteq \tau$. Therefore $\trianglelefteq$ is transitive.

Thus, $\trianglelefteq$ is an equivalence relation on Δ^* .

Now let $\xi \trianglelefteq \psi$ and $\tau \in \Delta^*$

$\xi \trianglelefteq \psi \Rightarrow \models_{\lambda} (\epsilon) \rightarrow ((\zeta, \xi, \eta) \in \Upsilon^*) \Leftrightarrow \models_{\lambda} (\epsilon) \rightarrow ((\zeta, \psi, \eta) \in \Upsilon^*) \quad \forall \zeta, \eta \in \Omega$. We have

$$\models_{\lambda} ((\zeta, \xi \odot \tau, \eta) \in \Upsilon^*) \rightarrow (\exists \gamma \{((\zeta, \xi, \gamma) \in \Upsilon^*) \wedge ((\gamma, \tau, \eta) \in \Upsilon^*)\}; \gamma \in \Omega).$$

So,

$$\begin{aligned} & \models_{\lambda} (\epsilon) \rightarrow ((\zeta, \xi \odot \tau, \eta) \in \Upsilon^*) \\ \Leftrightarrow & \models_{\lambda} (\epsilon) \rightarrow (\exists \gamma \{((\zeta, \xi, \gamma) \in \Upsilon^*) \wedge ((\gamma, \tau, \eta) \in \Upsilon^*)\}; \gamma \in \Omega) \\ \Leftrightarrow & \text{if there exists } \gamma \in \Omega \text{ such that} \\ & \models_{\lambda} (\epsilon) \rightarrow (((\zeta, \xi, \gamma) \in \Upsilon^*) \wedge ((\gamma, \tau, \eta) \in \Upsilon^*)) \\ \Leftrightarrow & \models_{\lambda} (\epsilon) \rightarrow (((\zeta, \psi, \gamma) \in \Upsilon^*) \wedge ((\gamma, \tau, \eta) \in \Upsilon^*)) \\ \Leftrightarrow & \models_{\lambda} (\epsilon) \rightarrow (\exists \gamma \{((\zeta, \psi, \gamma) \in \Upsilon^*) \wedge ((\gamma, \tau, \eta) \in \Upsilon^*)\}; \gamma \in \Omega) \\ \Leftrightarrow & \models_{\lambda} (\epsilon) \rightarrow ((\zeta, \psi \odot \tau, \eta) \in \Upsilon^*) \end{aligned}$$

$$\Rightarrow \quad \xi \odot \tau \trianglelefteq \psi \odot \tau.$$

Similarly we can prove that $\tau \odot \xi \trianglelefteq \tau \odot \psi$.

Therefore $\trianglelefteq$ is a congruence relation on Δ^* . □

Definition 3.3. Let $\xi \in \Delta^*$. Define $[\xi] = \{\psi \in \Delta^* / \xi \triangleright \psi\}$; $[\xi] = \{\psi \in \Delta^* / \xi \trianglelefteq \psi\}$ and $C(\Theta) = \{[\xi] / \xi \in \Delta^*\}$; $\widehat{C(\Theta)} = \{[\xi] / \xi \in \Delta^*\}$.

Theorem 3.3. Let $\Theta = (\Omega, \Delta, \Upsilon)$ be an implication-based fuzzy semiautomaton over the finite group Ω . Define a binary operation $\perp$ on $C(\Theta)$ by $\forall [\xi], [\psi] \in C(\Theta)$, $[\xi] \perp [\psi] = [\xi \odot \psi]$. Then $(C(\Theta), \perp)$ is a finite semigroup with identity.

Proof. Clearly $\perp$ is well-defined. Let $[\xi], [\psi], [\tau] \in C(\Theta)$ and let $\kappa \in ([\xi] \perp [\psi]) \perp [\tau]$.

$$\Rightarrow \kappa \in [\xi \odot \psi] \perp [\tau]$$

$$\Rightarrow \kappa \in [(\xi \odot \psi) \odot \tau]$$

$$\Rightarrow (\xi \odot \psi) \odot \tau \triangleright \kappa$$

$$(i) \models_{\lambda} ((\zeta, (\xi \odot \psi) \odot \tau, \eta) \in \Upsilon^*) \rightarrow ((\zeta, \kappa, \eta) \in \Upsilon^*)$$

$$(ii) \models_{\lambda} ((\zeta, \kappa, \eta) \in \Upsilon^*) \rightarrow ((\zeta, (\xi \odot \psi) \odot \tau, \eta) \in \Upsilon^*) \quad \forall \zeta, \eta \in \Omega$$

$$(i) \models_{\lambda} ((\zeta, (\xi \odot \psi) \odot \tau, \eta) \in \Upsilon^*) \rightarrow ((\zeta, \kappa, \eta) \in \Upsilon^*)$$

$$\Rightarrow \models_{\lambda} (\exists \gamma \{((\zeta, \xi \odot \psi, \gamma) \in \Upsilon^*) \wedge ((\gamma, \tau, \eta) \in \Upsilon^*)\}; \gamma \in \Omega) \rightarrow ((\zeta, \kappa, \eta) \in \Upsilon^*)$$

$$\Rightarrow \models_{\lambda} (\exists \gamma \{(\exists \delta \{((\zeta, \xi, \delta) \in \Upsilon^*) \wedge ((\delta, \psi, \gamma) \in \Upsilon^*)\}; \delta \in \Omega) \wedge ((\gamma, \tau, \eta) \in \Upsilon^*)\};$$

$$\gamma \in \Omega) \rightarrow ((\zeta, \kappa, \eta) \in \Upsilon^*)$$

$$\Rightarrow \models_{\lambda} (\exists \gamma, \delta \{((\zeta, \xi, \delta) \in \Upsilon^*) \wedge ((\delta, \psi, \gamma) \in \Upsilon^*) \wedge ((\gamma, \tau, \eta) \in \Upsilon^*)\}; \gamma, \delta \in \Omega)$$

$$\rightarrow ((\zeta, \kappa, \eta) \in \Upsilon^*)$$

$$\Rightarrow \models_{\lambda} (\exists \delta \{((\zeta, \xi, \delta) \in \Upsilon^*) \wedge (\exists \gamma \{((\delta, \psi, \gamma) \in \Upsilon^*) \wedge ((\gamma, \tau, \eta) \in \Upsilon^*)\}; \gamma \in \Omega)\};$$

$$\delta \in \Omega) \rightarrow ((\zeta, \kappa, \eta) \in \Upsilon^*)$$

$$\Rightarrow \models_{\lambda} (\exists \delta \{((\zeta, \xi, \delta) \in \Upsilon^*) \wedge ((\delta, \psi \odot \tau, \eta) \in \Upsilon^*)\}; \delta \in \Omega) \rightarrow ((\zeta, \kappa, \eta) \in \Upsilon^*)$$

$$\Rightarrow \models_{\lambda} ((\zeta, \xi \odot (\psi \odot \tau), \eta) \in \Upsilon^*) \rightarrow ((\zeta, \kappa, \eta) \in \Upsilon^*) \quad \forall \zeta, \eta \in \Omega$$

Similarly we can show that from (ii)

$$\models_{\lambda} ((\zeta, \kappa, \eta) \in \Upsilon^*) \rightarrow ((\zeta, \xi \odot (\psi \odot \tau), \eta) \in \Upsilon^*) \quad \forall \zeta, \eta \in \Omega$$

$$\Rightarrow \xi \odot (\psi \odot \tau) \triangleright \kappa$$

$$\Rightarrow \kappa \in [\xi \odot (\psi \odot \tau)]$$

$$\Rightarrow \kappa \in [\xi] \perp [\psi \odot \tau]$$

$$\Rightarrow \kappa \in [\xi] \perp ([\psi] \perp [\tau]).$$

Similarly we can prove that $\kappa \in [\xi] \perp ([\psi] \perp [\tau]) \Rightarrow \kappa \in ([\xi] \perp [\psi]) \perp [\tau]$

Therefore $([\xi] \perp [\psi]) \perp [\tau] = [\xi] \perp ([\psi] \perp [\tau])$. Thus $\perp$ is associative. Moreover the identity element is $[0]$.

Also let $\xi \in \Delta^*$ and let $\xi = \xi_1 \odot \xi_2 \odot \cdots \odot \xi_n$ where $\xi_1, \xi_2, \dots, \xi_n \in \Delta$
 $\models_{\lambda} ((\zeta, \xi, \eta) \in \Upsilon^*) \rightarrow (\exists \gamma_i \{((\zeta, \xi_1, \gamma_1) \in \Upsilon) \wedge ((\gamma_1, \xi_2, \gamma_2) \in \Upsilon) \wedge \cdots \wedge$
 $((\gamma_{n-1}, \xi_n, \eta) \in \Upsilon)\}; \gamma_i \in \Omega, i = 1, 2, \dots, n)$

Since image of Υ is finite, we have image of Υ^* also to be finite. Therefore $(C(\Theta), \perp)$ is a finite semigroup with identity. $\square$

Theorem 3.4. Let $\Theta = (\Omega, \Delta, \Upsilon)$ be an implication-based fuzzy semiautomaton over the finite group Ω . Define a binary operation $\vdash$ on $\widehat{C(\Theta)}$ by $\forall [\xi], [\psi] \in \widehat{C(\Theta)}, [\xi] \vdash [\psi] = [\xi \odot \psi]$. Then $(\widehat{C(\Theta)}, \vdash)$ is a finite semigroup with identity.

Proof. Can be proved as in previous theorem. $\square$

Theorem 3.5. Let $\Theta = (\Omega, \Delta, \Upsilon)$ be an implication-based fuzzy semiautomaton over the finite group Ω . Define $C(\Theta) = \{[\xi]/\xi \in \Delta^*\}$ and $\widehat{C(\Theta)} = \{[\xi]/\xi \in \Delta^*\}$. Then $\phi : [\xi] \rightarrow [\xi]$ is a homomorphism of $C(\Theta)$ onto $\widehat{C(\Theta)}$.

Proof. Define $\phi : C(\Theta) \rightarrow \widehat{C(\Theta)}$ by $\phi([\xi]) = [\xi]$ for all $[\xi] \in C(\Theta)$. Let $\xi, \psi \in \Delta^*$ such that $[\xi] = [\psi]$ and let ϵ be a very small value such that $\epsilon > 0$.

$$\begin{aligned} \Rightarrow & (i) \models_{\lambda} ((\zeta, \xi, \eta) \in \Upsilon^*) \rightarrow ((\zeta, \psi, \eta) \in \Upsilon^*) \\ & (ii) \models_{\lambda} ((\zeta, \psi, \eta) \in \Upsilon^*) \rightarrow ((\zeta, \xi, \eta) \in \Upsilon^*) \quad \forall \zeta, \eta \in \Omega \\ \Rightarrow & (i) \models_{\lambda} (\epsilon) \rightarrow ((\zeta, \psi, \eta) \in \Upsilon^*) \\ & (ii) \models_{\lambda} (\epsilon) \rightarrow ((\zeta, \xi, \eta) \in \Upsilon^*) \quad \forall \zeta, \eta \in \Omega \\ \Rightarrow & [\xi] = [\psi] \\ \Rightarrow & \phi([\xi]) = \phi([\psi]) \end{aligned}$$

Thus, ϕ is well-defined. Clearly ϕ is onto. Now

$$\begin{aligned} \phi([\xi] \perp [\psi]) &= \phi([\xi \odot \psi]) \\ &= [\xi \odot \psi] \\ &= [\xi] \vdash [\psi] \\ &= \phi([\xi]) \vdash \phi([\psi]) \end{aligned}$$

ϕ is a homomorphism. Moreover, $\widehat{C(\Theta)}$ is finite since $C(\Theta)$ is finite. Thus $\phi : C(\Theta) \rightarrow \widehat{C(\Theta)}$ is an onto homomorphism. $\square$

Definition 3.4. Let $\Theta = (\Omega, \Delta, \Upsilon)$ be an implication-based fuzzy semiautomaton over the finite group Ω . For all $\xi \in \Delta^*$, define the fuzzy set ξ^{Θ} of $\Omega \times \Omega$ (i.e.,) $\xi^{\Theta} : \Omega \times \Omega \rightarrow [0, 1]$ by $\models_{\lambda} ((\delta, \xi, \kappa) \in \Upsilon^*) \rightarrow ((\delta, \kappa) \in \xi^{\Theta})$ for all $(\delta, \kappa) \in \Omega \times \Omega$.

Theorem 3.6. *Let $\Theta = (\Omega, \Delta, \Upsilon)$ be an implication-based fuzzy semiautomaton over the finite group Ω . Then ξ^Θ is an implication-based fuzzy subgroup for each $\xi \in \Delta^*$.*

Proof. Let $\xi \in \Delta^*$.

(i) Let $(\delta_1, \kappa_1), (\delta_2, \kappa_2) \in \Omega \times \Omega$.

$$\begin{aligned} \models_\lambda (((\delta_1, \xi, \kappa_1) \in \Upsilon^*) \wedge ((\delta_2, \xi, \kappa_2) \in \Upsilon^*)) &\rightarrow (((\delta_1\delta_2, \xi, \kappa_1\kappa_2) \in \Upsilon^*)) \\ &\rightarrow ((\delta_1\delta_2, \kappa_1\kappa_2) \in \xi^\Theta) \end{aligned}$$

(ii) Let $(\delta, \kappa) \in \Omega \times \Omega$.

$$\begin{aligned} \models_\lambda ((\delta, \xi, \kappa) \in \Upsilon^*) &\rightarrow ((\delta^{-1}, \xi, \kappa^{-1}) \in \Upsilon^*) \\ &\rightarrow ((\delta^{-1}, \kappa^{-1}) \in \xi^\Theta) \\ &\rightarrow ((\delta, \kappa)^{-1} \in \xi^\Theta) \\ \models_\lambda ((\delta, \kappa) \in \xi^\Theta) &\rightarrow ((\delta, \kappa)^{-1} \in \xi^\Theta) \end{aligned}$$

Therefore ξ^Θ is an implication-based fuzzy subgroup for each $\xi \in \Delta^*$. $\square$

Definition 3.5. *Let Ω be a finite group with the identity element ' ε '. Let Υ and Φ be two implication-based fuzzy subgroups on $\Omega \times \Omega$. Then*

$$\models_\lambda (\exists \psi \{((\xi, \psi) \in \Upsilon) \wedge ((\psi, \tau) \in \Phi)\}; \psi \in \Omega) \rightarrow ((\xi, \tau) \in \Upsilon \circ \Phi),$$

for all $\xi, \tau \in \Omega$.

Theorem 3.7. *Let Ω be a finite group. Let Υ and Φ be two implication-based fuzzy subgroups on $\Omega \times \Omega$. Then $\Upsilon \circ \Phi$ product is also an implication-based fuzzy subgroup on $\Omega \times \Omega$.*

Proof. Let $(\xi_1, \tau_1), (\xi_2, \tau_2) \in \Omega \times \Omega$.

$$\begin{aligned} \models_\lambda (((\xi_1, \tau_1) \in \Upsilon \circ \Phi) \wedge ((\xi_2, \tau_2) \in \Upsilon \circ \Phi)) \\ \rightarrow (\exists \psi_1 \{((\xi_1, \psi_1) \in \Upsilon) \wedge ((\psi_1, \tau_1) \in \Phi)\}; \psi_1 \in \Omega) \wedge \\ (\exists \psi_2 \{((\xi_2, \psi_2) \in \Upsilon) \wedge ((\psi_2, \tau_2) \in \Phi)\}; \psi_2 \in \Omega) \end{aligned}$$

$$\begin{aligned}
&\rightarrow (\exists \psi_1, \psi_2 \{((\xi_1, \psi_1) \in \Upsilon) \wedge ((\psi_1, \tau_1) \in \Phi)) \wedge \\
&\quad (((\xi_2, \psi_2) \in \Upsilon) \wedge ((\psi_2, \tau_2) \in \Phi))\}; \psi_1, \psi_2 \in \Omega) \\
&\rightarrow (\exists \psi_1, \psi_2 \{((\xi_1, \psi_1) \in \Upsilon) \wedge ((\xi_2, \psi_2) \in \Upsilon)) \wedge \\
&\quad (((\psi_1, \tau_1) \in \Phi) \wedge ((\psi_2, \tau_2) \in \Phi))\}; \psi_1, \psi_2 \in \Omega) \\
&\rightarrow (\exists \psi_1, \psi_2 \{(\xi_1 \xi_2, \psi_1 \psi_2) \in \Upsilon) \wedge ((\psi_1 \psi_2, \tau_1 \tau_2) \in \Phi)\}; \psi_1, \psi_2 \in \Omega) \\
&\quad \Upsilon \text{ and } \Phi \text{ are implication-based fuzzy subgroups} \\
&\rightarrow ((\xi_1 \xi_2, \tau_1 \tau_2) \in \Upsilon \circ \Phi)
\end{aligned}$$

Let $(\xi, \tau) \in \Omega \times \Omega$.

$$\begin{aligned}
&\models_{\lambda} ((\xi, \tau) \in \Upsilon \circ \Phi) \\
&\rightarrow (\exists \psi \{((\xi, \psi) \in \Upsilon) \wedge ((\psi, \tau) \in \Phi)\}; \psi \in \Omega) \\
&\rightarrow (\exists \psi \{((\xi, \psi)^{-1} \in \Upsilon) \wedge ((\psi, \tau)^{-1} \in \Phi)\}; \psi \in \Omega) \\
&\rightarrow (\exists \psi^{-1} \{((\xi^{-1}, \psi^{-1}) \in \Upsilon) \wedge ((\psi^{-1}, \tau^{-1}) \in \Phi)\}; \psi^{-1} \in \Omega) \\
&\rightarrow ((\xi^{-1}, \tau^{-1}) \in \Upsilon \circ \Phi)
\end{aligned}$$

Therefore $\Upsilon \circ \Phi$ is an *implication-based fuzzy subgroup* on $\Omega \times \Omega$. □

Theorem 3.8. Let $\Theta = (\Omega, \Delta, \Upsilon)$ be an *implication-based fuzzy semiautomaton* over the finite group Ω . Define $\Psi_{\Theta} = \{\xi^{\Theta}/\xi \in \Delta^*\}$. Then (i) $\xi^{\Theta} \circ \psi^{\Theta} = (\xi \circ \psi)^{\Theta}$ for all $\xi, \psi \in \Delta^*$ (ii) $(\Psi_{\Theta}, \circ)$ is a *semigroup with identity*.

Proof. Let $\Psi_{\Theta} = \{\xi^{\Theta}/\xi \in \Delta^*\}$.

(i) Let $\xi, \psi \in \Delta^*$ and $(\zeta, \eta) \in \Omega \times \Omega$

$$\begin{aligned}
&\models_{\lambda} ((\zeta, \eta) \in \xi^{\Theta} \circ \psi^{\Theta}) \\
&\rightarrow (\exists \omega \{((\zeta, \omega) \in \xi^{\Theta}) \wedge ((\omega, \eta) \in \psi^{\Theta})\}; \omega \in \Omega) \\
&\rightarrow (\exists \omega \{((\zeta, \xi, \omega) \in \Upsilon^*) \wedge ((\omega, \psi, \eta) \in \Upsilon^*)\}; \omega \in \Omega) \\
&\rightarrow ((\zeta, \xi \circ \psi, \eta) \in \Upsilon^*) \\
&\rightarrow ((\zeta, \eta) \in (\xi \circ \psi)^{\Theta})
\end{aligned}$$

Similarly we can prove that

$$\models_{\lambda} ((\zeta, \eta) \in (\xi \circ \psi)^{\Theta}) \rightarrow ((\zeta, \eta) \in \xi^{\Theta} \circ \psi^{\Theta}) \text{ for all } (\zeta, \eta) \in \Omega \times \Omega \text{ and } \xi, \psi \in \Delta^*.$$

Therefore $\xi^{\Theta} \circ \psi^{\Theta} = (\xi \circ \psi)^{\Theta}$ for all $\xi, \psi \in \Delta^*$.

(ii) Let $\xi, \psi, \tau \in \Delta^*$ and $(\zeta, \eta) \in \Omega \times \Omega$.

$$\begin{aligned}
& \models_{\lambda} ((\zeta, \eta) \in \xi^{\Theta} \circ (\psi^{\Theta} \circ \tau^{\Theta})) \\
& \rightarrow (\exists \omega \{((\zeta, \omega) \in \xi^{\Theta}) \wedge ((\omega, \eta) \in \psi^{\Theta} \circ \tau^{\Theta})\}; \omega \in \Omega) \\
& \rightarrow (\exists \omega \{((\zeta, \omega) \in \xi^{\Theta}) \wedge (\exists \kappa \{((\omega, \kappa) \in \psi^{\Theta}) \wedge ((\kappa, \eta) \in \tau^{\Theta})\}; \kappa \in \Omega)\}; \omega \in \Omega) \\
& \rightarrow (\exists \omega, \kappa \{((\zeta, \omega) \in \xi^{\Theta}) \wedge ((\omega, \kappa) \in \psi^{\Theta}) \wedge ((\kappa, \eta) \in \tau^{\Theta})\}; \omega, \kappa \in \Omega) \\
& \rightarrow (\exists \omega, \kappa \{((\zeta, \xi, \omega) \in \Upsilon^*) \wedge ((\omega, \psi, \kappa) \in \Upsilon^*) \wedge ((\kappa, \tau, \eta) \in \Upsilon^*)\}; \omega, \kappa \in \Omega) \\
& \rightarrow (\exists \kappa \{(\exists \omega \{((\zeta, \xi, \omega) \in \Upsilon^*) \wedge ((\omega, \psi, \kappa) \in \Upsilon^*)\}; \omega \in \Omega) \wedge ((\kappa, \tau, \eta) \in \Upsilon^*)\}; \kappa \in \Omega) \\
& \rightarrow (\exists \kappa \{(\exists \omega \{((\zeta, \omega) \in \xi^{\Theta}) \wedge ((\omega, \kappa) \in \psi^{\Theta})\}; \omega \in \Omega) \wedge ((\kappa, \eta) \in \tau^{\Theta})\}; \kappa \in \Omega) \\
& \rightarrow (\exists \kappa \{((\zeta, \kappa) \in (\xi^{\Theta} \circ \psi^{\Theta})) \wedge ((\kappa, \eta) \in \tau^{\Theta})\}; \kappa \in \Omega) \\
& \rightarrow ((\zeta, \eta) \in (\xi^{\Theta} \circ \psi^{\Theta}) \circ \tau^{\Theta})
\end{aligned}$$

Therefore $\models_{\lambda} ((\zeta, \eta) \in \xi^{\Theta} \circ (\psi^{\Theta} \circ \tau^{\Theta})) \rightarrow ((\zeta, \eta) \in (\xi^{\Theta} \circ \psi^{\Theta}) \circ \tau^{\Theta})$.

Similarly we can prove that

$$\models_{\lambda} ((\zeta, \eta) \in (\xi^{\Theta} \circ \psi^{\Theta}) \circ \tau^{\Theta}) \rightarrow ((\zeta, \eta) \in \xi^{\Theta} \circ (\psi^{\Theta} \circ \tau^{\Theta})).$$

Therefore $\circ$ is associative. Clearly $\mathbf{0}^{\Theta}$ is the identity element of $(\Psi_{\Theta}, \circ)$. Thus $(\Psi_{\Theta}, \circ)$ is a semigroup with identity element. $\square$

Theorem 3.9. Let $\Theta = (\Omega, \Delta, \Upsilon)$ be an implication-based fuzzy semiautomaton over the finite group Ω . Then the semigroups Ψ_{Θ} and $C(\Theta)$ are isomorphic.

Proof. Define $\phi : (\Psi_{\Theta}, \circ) \rightarrow (C(\Theta), \perp)$ as $\phi(\xi^{\Theta}) = \lfloor \xi \rfloor$ for all $\xi^{\Theta} \in \Psi_{\Theta}$. Let $\xi^{\Theta}, \psi^{\Theta} \in \Psi_{\Theta}$ such that $\xi^{\Theta} = \psi^{\Theta}$

$$\begin{aligned}
& \Rightarrow \models_{\lambda} ((\zeta, \eta) \in \xi^{\Theta}) \rightarrow ((\zeta, \eta) \in \psi^{\Theta}) \\
& \models_{\lambda} ((\zeta, \eta) \in \psi^{\Theta}) \rightarrow ((\zeta, \eta) \in \xi^{\Theta}) \text{ for all } \zeta, \eta \in \Omega \\
& \Rightarrow \models_{\lambda} ((\zeta, \xi, \eta) \in \Upsilon^*) \rightarrow ((\zeta, \psi, \eta) \in \Upsilon^*) \\
& \models_{\lambda} ((\zeta, \psi, \eta) \in \Upsilon^*) \rightarrow ((\zeta, \xi, \eta) \in \Upsilon^*) \text{ for all } \zeta, \eta \in \Omega \\
& \Rightarrow \lfloor \xi \rfloor = \lfloor \psi \rfloor \\
& \Rightarrow \phi(\xi^{\Theta}) = \phi(\psi^{\Theta})
\end{aligned}$$

ϕ is single-valued and one - to - one.

$$\begin{aligned}\phi(\xi^\Theta \circ \psi^\Theta) &= \phi((\xi \odot \psi)^\Theta) \\ &= [\xi \odot \psi] \\ &= [\xi] \perp [\psi] \\ &= \phi(\xi^\Theta) \perp \phi(\psi^\Theta)\end{aligned}$$

ϕ is a homomorphism. And clearly, ϕ is onto. Therefore Ψ_Θ is isomorphic to $C(\Theta)$ $\square$

4. CONCLUSION

The structure of a finite group's *implication-based fuzzy semiautomaton* was elaborately discussed in this paper. On this system two more different types of relations are established. The relation thus established proves to be a relation of equivalence and congruence. The study proves that a finite semigroup with identity element is constructed by the equivalence classes generated by this equivalence relation. It is further confirmed that homomorphism or isomorphism connects these semigroups to one another.

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