ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **9** (2020), no.10, 8883–8889
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.10.111

MODIFIED CYCLE NEIGHBOR POLYNOMIAL OF GRAPHS

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ABSTRACT. Let G be a simple graph. The cycle neighbor polynomial of G is defined as $CN[G,z] = c_0(G) + \Sigma_{k=g(G)}^{c(G)} c_k(G)z^k$, where $c_0(G)$ is the number of vertices of G which does not belong to any cycle of G and $c_k(G)$ is the number of cycles of length k in G for $3 \leq g(G) \leq k \leq c(G) \leq n$. Here $g(G)$ and $c(G)$ are respectively the girth and circumference of G. This paper deals with an improvement of cycle neighbor polynomial and a brief comparitive study of these two polynomials.

1. INTRODUCTION

Many graph polynomials are introduced and studied in graph theory. Chromatic polyomial [9], Tutte polynomial [6], clique polynomial [7], etc., are some examples. These polynomials are studied because some of them are generating functions of some graph properties, some count the number of occurrences of certain graph features and some others make an attempt to find complete graph invariants and so on. In [2] A S Paul and R Pilakkat introduced one such univariate graph polynomial called cycle neighbor polynomial of a graph. For any simple graph G, this polynomial is defined as $CN[G,z] = c_0(G)+\Sigma_{k=g(G)}^{c(G)} c_k(G)z^k$, where $c_0(G)$ is the number vertices which do not belong to any cycle of G called the cycle neighbor free vertices, and $c_k(G)$ is the number of k-cycles of length k for $3 \leq g(G) \leq k \leq c(G) \leq n$. Here $g(G)$ and $c(G)$ are respectively the girth

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2020 *Mathematics Subject Classification.* 05C38.

Key words and phrases. MCNP, cyn^* equivalence, cyn^* uniqueness.

and circumference of G. This cycle generating polynomial reveals many graph properties like girth [5], circumference [5], hamiltonicity [5], pacyclicity [4], whether the graph is bipartite or not etc., of a graph.

2. Modified cycle neighbor polynomial of graphs

Motivated from the interpretation of simple cycles of lengths one and two [10], we improve the definition of cycle neighbor polynomial [2] of a graph by taking into account the isolated vertices, non isolated cycle neighbor free vertices and bridges which were not considered in the original cycle neighbor polynomial.

Definition 2.1. *Modified cycle neighbor polynmial (in short MCNP) of a graph is denoted by $CN^*[G; z]$ and it is defined as $CN^*[G; z] = \Sigma_{k=0}^{c(G)} c_k(G) z^k$, where $c_0(G)$ is the number of isolated vertices, $c_1(G)$ is the number of non isolated cycle neighbor free vertices, $c_2(G)$ is the number of bridges and $c_k(G)$ is the number of k-cycles in G for $g(G) \leq k \leq c(G)$, where $g(G)$ and $c(G)$ are respectively the girth and the circumference of G.*

The zeros of MCNP of G are the roots of $CN^*[G; z]$. From the definition of $CN^*[G; z]$ of G it follows that:

Proposition 2.1.

(1) *Consider a simple graph G. Then $CN[G; z] = CN^*[G; z]$ of G if and only if G contains no non isolated cycle neighbor free vertices.*
(2) *For a graph G, $CN^*[G; z]$ is a constant polynomial if and only if $G \cong \overline{K_n}$, the empty graph on $n = 1, 2, 3, \ldots$ vertices.*
(3) *If G is connected and $|V(G)| \geq 3$, then $CN^*[G; z]$ contains exactly one term if and only if G is a cactus graph [3] in which every cycle has the same length and there are no bridges in G.*
(4) *Let $G(V, E)$ be an acyclic graph with $CN^*[G; z] = a_0 + a_1 z + a_2 z^2$. Then $a_0 + a_1 = |V(G)|$ and $a_2 = |E(G)|$. Moreover, when G is a connected acyclic graph, then $a_0 = 0$, $a_1 = |V(G)|$ and $a_2 = a_1 - 1$.*

Proposition 2.2.

(1) *The degree of MCNP of a graph G is two if and only if G is a forest. In particular, degree of MCNP of a connected graph G is two if and only if G is a tree.*
(2) *No polynomial of degree one can be the MCNP of a graph.*

Proof.
(1) follows from the fact that trees and forests are acyclic.

(2) Suppose $P(z)$ is a polynomial of degree one say $P(z) = a_0 + a_1 z$, $a_1 \neq 0$, which is the MCNP of a graph G. Hence there are a_0 isolated vertices and a_1 non isolated vertices which do not belong to any cycle of G. Since $a_1 \neq 0$, the induced subgraph of these non isolated cycle neighbor free vertices is a non trivial forest. Hence it contains at least one bridge, a contradiction. □

Corollary 2.1. *Let G be an acyclic connected graph of order n. Then the zeros of MCNP of G are 0 and $\frac{-n}{n-1}$.*

Corollary 2.2. *Let G be a forest of order n, which does not contain any isolated vertices. Let $G_1, G_2, \ldots, G_k$ be the components of G of order $n_1, n_2, ..., n_k$ respectively. Then the set of zeros of MCNP of G is $\{0, \frac{-n}{n-k}\}$.*

Theorem 2.1. *Let G be connected and $|V(G)| \geq 4$. Then the MCNP of G contains maximum number of terms if and only if $G \cong H_{n-1,1}$, where $H_{n-1,1}$ is a graph consisting of a pancyclic graph on $(n-1)$ vertices and a vertex connected to any one of the vertices of H by a bridge.*

Proof. Let the MCNP of G be $CN^*[G; z] = a_0 + a_1 z + a_2 z^2 + \cdots + a_k z^k$, where k is the circumference $c(G)$ of G. Since G is connected, $a_0 = 0$. Also when $a_1 \neq 0$, then $a_n = 0$ since circumference of G is less than or equal to $n-1$ whenever G contains cycle neighbor free vertices. Hence the number of terms in $CN^*[G; z]$ is less than or equal to $n-1$. Thus for a graph with its MCNP containing maximum number of terms, we have $a_1 \neq 0$, $a_2 \neq 0$,..., $a_{n-1} \neq 0$, and $a_n = 0$. But this is possible only when $a_1 = 1$ and $a_2 = 1$. Otherwise $a_1 \geq 2$, $a_2 \geq 2$ and then $k = c(G) \leq n-2$, so that $a_{n-1} = 0$. Hence G contains a subgraph H containing cycles of all lengths k, for $3 \leq k \leq n-1$ and a cycle neighbor free vertex attached to H by a bridge. That is $G \cong H_{n-1,1}$.

Conversely, when $G \cong H_{n-1,1}$, $CN^*[G; z]$ contains $n-1$ terms, which is the maximum possible number of terms in the MCNP of any graph G. This completes the proof. □

Note that when G is a connected graph of order $n < 3$, that is when G is isomorphic to K_1 or K_2, the MCNP of G contains exactly n terms. And the number of terms in the MCNP of a connected graph G is minimum, that is $CN^*[G; z]$ contains exactly one term that is if and only if $G \cong K_1$ or G is a cactus graph [3] in which each edge of G belongs to a k-cycle in G, where $3 \leq k \leq n$. Empty graphs and graphs G with all of its components are cactus graphs of the above type are examples of disconnected graphs whose modified cycle neighbor polynomial contains exactly one term.

Cycle neighbor equivalence and cycle neighbor uniqueness are introduced in [1]. A similar concept with respect to MCNP of graphs is defined as follows.

Definition 2.2. *Let G and H be any two graphs which are said to be cycle neighbor equivalent with respect to MCNP if $CN^*[G; z] = CN^*[H; z]$ and G and H are called cycle neighbor unique with respect to MCNP if $CN^*[G; z] = CN^*[H; z]$ then $G \cong H$.*

We use the abbereviations cyn^*-equivalent graphs and cyn^*-unique graphs respectively to denote cycle neighbor equivalence and cycle neighbor uniqueness of graphs with respect to MCNP.

Theorem 2.2. *Let T be a tree of order n and let $\overline{T}$ denotes its complement. Then $CN^*[T; z] = CN^*[\overline{T}; z]$ if only if $T \cong P_n$ where $n = 1$ or 4.*

Proof. Consider a tree T. First let T be a path P_n. Then for $n = 2$ and 3, $\overline{P}_n$ contains isolated vertices while P_n does not and $P_4 \cong \overline{P}_4$. Therefore $CN^*[P_4; z] = CN^*[\overline{P}_4; z]$. For $n \geq 5$, P_n is acyclic and $\overline{P}_n$ contains cycles. Now let T is not a path. Then the order of T is greater than or equal to four. Since T is acyclic and it is not a path, there are more than two pendant vertices in T. In $\overline{T}$ the pendant vertices of T forms a cycle. Hence $\overline{T}$ is not acyclic. Therefore in this case, $CN^*[T; z] \neq CN^*[\overline{T}; z]$. Conversely when $T \cong P_n$ with $n = 1$ or 4, $CN^*[T; z] = CN^*[\overline{T}; z]$. Hence the proof. □

Remark 2.1. *The only acyclic graphs G either connected or disconnected such that $CN^*[G, z] = CN^*[\overline{G}, z]$ are paths P_n, with $n = 1$ or 4.*

It is clear from the definitions of $CN[G;z]$ and $CN^*[G;z]$ that for two graphs G and H, whenever G is cyn^*-equivalent to H, then G is cyn-equivalent to H. But two cyn-equivalent graphs need not be cyn^*-equivalent. On the other hand, every cyn-unique graph is cyn^*-unique. But the converse need not be. For example, the graph G in figure 1 is cyn^*-unique but it is not cyn-unique

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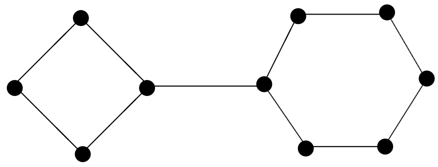

Figure 1 - graph G

Theorem 2.3. *Let G be any graph with $|V(G)| = n$. If G is isomorphic to any of the following graphs,*

(1) *C_n, a cycle on n vertices.*
(2) *$\overline{K}_n$, empty graph on n vertices.*
(3) *P_n, a path on n vertices, where $n = 1, 2$ or 3.*
(4) *K_n, a complete graph of order n, where $n = 1, 2$ or 3.*
(5) *H, where H is a graph containing exactly two cycles joined by a bridge between them.*

then G is cyn^*-unique.

Proof. Let G be any graph which is isomorphic to one of the graphs as in the statement of the theorem. Let us consider the cases one by one.

Case (1): When $G \cong C_n$, G contains no cycle neighbor free vertices or bridges hence by Proposition 2.1, $CN^*[G;z] = CN[G;z] = z^k$, Where k is the length of the cycle in G. Hence as in the case of cyn-uniqueness of cycles C_n [1], $n \geq 3$, it follows that cycles C_n, $n \geq 3$ are cyn^*-unique.

Case (2): $G \cong \overline{K}_n$. Then $CN^*[G;z] = n$, a constant polynomial. If H is any graph other than G with $CN^*[H;z] = n$, it means that H contains n isolated vertices and no edges. That is $H \cong G$. Therefore $\overline{K}_n$ is cyn^*-unique.

Case (3): $G \cong P_n$, where $n = 1, 2$ or 3. Then by Proposition 2.2, $CN^*[G;z] = a_1 z + a_2 z^2$, with $a_2 = a_1 - 1$. Since there is a unique non isomorphic tree on $n \leq 3$ vertices, $CN^*[G;z] = CN^*[H;z] = a_1 z + a_2 z^2$ implies that $H \cong G$.

Case (4): $G \cong K_n$, where $n = 1, 2$ or 3.

When $n = 1$, $CN^*[G;z] = 1$, hence it is clear from case (2) that K_1 is cyn^*-unique.

The only simple graphs of order two are K_2 and $\overline{K}_2$. $\overline{K}_2$ contains isolated vertices while K_2 does not. Hence K_2 is also cyn^*-unique.

When $n = 3$, $K_3 \cong C_3$ hence by case (1), K_3 is cyn^*-unique.

Case (5): $G \cong H$. Then $CN^*[H;z] = z^2 + z^k + z^m$, where k and m are the lengths of the cycles in G with $k \geq 3$, $m \geq 3$ and $m + k = n$. Suppose if possible, H_1 is a graph of order n such that $CN^*[H_1;z] = CN^*[H;z]$ but H_1 not isomorphic to H. Therefore H_1 contains exactly two cycles of lengths k and m and these cycles will be disjoint, otherwise they will have a vertex in common and therefore H_1 will contain a cycle neighbor free vertex contradicting our assumption that $CN^*[H_1;z] = z^2 + z^k + z^m$. By case (1), cycles C_n are cyn^*-unique and since order of H_1 is $m + k = n$, one end point of the bridge in H_1 should be in the k-cycle and the other end is in the m-cycle of H_1. That is $H_1 \cong H$. Therefore H is cyn^*-unique. □

3. Conclusion

The main difference between cycle neighbor polynomial of a graph and its MCNP is that the cut edges in G are also taken into account in the MCNP of a graph. A graph polynomial is complete [8] if it distinguishes all non isomorphic graphs. But a complete graph polynomial which can be easily computed is not yet succeeded mainly due to two reasons. The first one is there are so many indistinguishable non isomorphic graphs. And the second reason is that such a graph polynomial too hard to compute. The two univariate polynomials cycle neighbor polynomial and MCNP of a graph introduced in [2] and in this paper respectively can be compared in terms of this 'completeness' property of graph polynomials. Every cyn- unique graph is cyn^*- unique. But the converse need not hold. That is,MCNP of a graph distinguishes more non isomorphic graphs than that of cycle neighbor polynomial of the graph. As a consequence, MCNP of a graph can be considered to be stronger than the cycle neighbor polynomial of the graph in terms of this completeness property of graph polynomials.

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