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CHARACTERIZATION OF PARTIAL WORDS USING GRAPH PARAMETERS

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ABSTRACT. Partial words representable graph is one such notion which has been introduced in the recent times. Let $\Gamma(h,n)$ represent the collection of all partial words with two periods of length n and h holes. In this paper, the authors prove a characterization result that constructs the requisite conditions on $\pi \in \Gamma(h,n)$ which make its associated bipartite graph to be complete. Further, a partition of the set $\Gamma(h,n)$ is established based on the partial words which yield complete bipartite graphs.

1. INTRODUCTION

The area of graph grammars and graph transformations generalizes formal language theory based on strings and it is considered as a fundamental computation where it includes specification, programming and implementation [3]. A string may not have all its elements known. There may be ambiguity at some positions. The need for partial words arise mainly due to this allowance of ambiguity in the grammar. Partial words or finite sequences that may contain a number of "do not know"symbols or holes, appear in a very natural way in the fields of both pure mathematics and computer science. Formally a partial word π may be defined as a sequence of characters in which some positions are uncertain. These positions are filled with placeholders (called holes). The collection of all possible partial words of length n and h holes is denoted by $\Gamma(h,n)$. A

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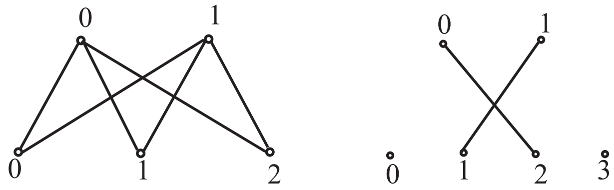
word can be described by a total function whereas a partial word is described by a partial function [2, 4, 5]. Partial words have repeatedly been researched in literature [1].

Partial words representable graph is one such notion which has been introduced in the recent times. F.Blanchet-Sadri et.al extended the Fine and Wilf's theorem in the framework of partial words and their proofs are based on connectivity in graphs associated with bounds and pairs of periods [6]. Zoltan Kása accentuated the role of using graphs instead of transition functions in the definition of different types of automata [9]. Lidia A. Idiatulina et.al described the interaction property for periodic partial words in probabilistic terms and further the threshold functions for the edge connectedness of random bipartite graphs and multigraphs are found [7]. Some properties of periodic partial words are exhibited and classification of the interaction of different periods of a word is established in [8]. Motivated by the work on [7], we characterize the periodic partial words with respect to certain graph parameters. In section 2 deals with the basic preliminaries which is required to study the rest of the paper and to prove our results. In section 3 results and discussions are generalized and followed by a conclusion in section 4.

2. Preliminaries

In this section, we review basic concepts on partial words, periodicity and bipartite graph.

Let Σ be a non-empty finite set of symbols called an alphabet. Symbols in Σ are called letters and any finite sequence of letters is called word. Let $\Sigma = a \cup \Diamond$ is an Unary word with a one-symbol input alphabet. A Partial word π of length n over Σ is a partial function $\pi : \{1, \cdots, n\} \rightarrow \Sigma$. For $1 \leq i \geq n$, if $\pi(i)$ is defined,then i belongs to the domain D of π. Periodicity is an important concept related to partial words. If π is p-periodic, then it can interpreted by writing in order, the letters of π into p-columns in which each column will contain the same letter (if any) in it. The study on various properties of a partial word can be done by comparing it to some parameters of a graph. To do so, a proper construction of a graph, for the given partial word π should be defined. This was done by Lidia A. et.al in [7] using perioidicity of π.



(a). p=3,q=2. (b).p=4,q=2.

FIGURE 1. For the same $\pi = aa\Diamond\Diamond aa\Diamond\Diamond aa$, the graph is connected

Let $\pi = a_1a_2\cdots a_n$ be a partial word of length n and h holes. Suppose p and q are two periods of π with $p > q$. Then a corresponding graph $G = G(\pi_{p,q})$ is constructed as follows: the graph G has a vertex set $V = (X, Y)$ with bipartitions $X = \{0, 1, \cdots, p-1\}$ and $Y = \{0, 1, \cdots, q-1\}$; the edge set E is formed by $E = \{(u, v)|u \in X, v \in Y, u \equiv i(\mod p), v \equiv i(\mod q)\}$. The resulting graph G is thus a bipartite, simple, undirected graph. The graph need not be connected in general. For example, given a partial word $\pi = aa\Diamond\Diamond aa\Diamond\Diamond aa$, construction of two different bipartite graphs based on their periodicity is shown in Figure 1. The first graph (Figure 1(a)) is obtained when the periods are $p = 3, q = 2$ while the second graph (Figure 1(b)) is when $p = 4, q = 2$.

The number of edges in a bipartite graph G plays a key role in many of its characteristic properties. The determination of whether G exhibits properties like complete, Eulerian and Hamiltonian, existence of a perfect matching all can be made using this edge cardinality as a bound. If a bipartite graph is complete, then the determination of many graph properties like domination number, split domination number, independence number, covering number are all very easy. These graph parameters may be associated to the base partial word to form partition among its elements. Hence the fundamental work is to characterize the criterion for a partial word w to yield a complete bipartite graph $G\left(\pi_{p,q}\right)$. This paper aims to bring out one such characterization for a partial word with fixed periodicity p, q.

3. Results and Discussion

In this section, we define a binary function on partial word using this we partition the set of all partial word with respect to their corresponding complete bipartite.

Lemma 3.1. *Let $\pi = a_1 a_2 \cdots a_n$ be a unary partial word with periods p and q. Then there exists a partition of the position of elements of π.*

Proof. Given $\pi = a_1 a_2 a_3 \cdots a_n$. Relabel the elements of π as follows:

$$\pi = b_1^1 b_2^1 b_3^1 \cdots b_{pq}^1 b_1^2 b_2^2 b_3^2 \cdots b_{pq}^2 \cdots b_k^i,$$

where $a_1 = b_1^1, a_2 = b_2^1, \cdots$ *i.e.*, the order of the elements of π is not changed. Also if $n = (pq)r + s$, then $i = r + 1$ and $s = k$. Now group the elements with respect to their suffixes. Define pq sets $B_i,\ i = 1, 2, \cdots, pq$ such that $B_i = b_i^j, \forall j \geq 1$ for $i = 1, 2, \cdots, pq$. Clearly $\bigcup_{i=1}^{pq} B_i$ will have all the entries of π and as we only aim on the positions; any two sets B_i and B_j will be disjoint. Thus $B_i, i = 1, 2, \cdots, pq$ form a partition of the position of elements of π. □

We now define a binary function on the partition constructed in Lemma 3.1. Let π be a partial word with partition $B_i; i = 1, 2, \cdots, pq$. Define a function $\gamma : B_i \to \{0, 1\}$ such that

$$\gamma(B_i) = \gamma_i = \begin{cases} 1 & \text{if atleast one entry in } B_i \text{ is not a hole} \\ 0 & \text{otherwise.} \end{cases}.$$

For example if $q = 2$ and $p = 3$, then $B_1 = \{a_1, a_7, a_{13}, a_{19}, \cdots\}$. If all the elements in B_1 are just holes then $\gamma_1 = 0$ for π. If at least one of these elements is not a hole then $\gamma_1 = 1$ for π.

Lemma 3.2. *The above defined binary function is well defined.*

Proof. The proof follows from the fact that the definition of γ clearly shows that a set B_i can have exactly one of those values $0, 1$. Hence every set B_i will have a unique image under γ. □

Theorem 3.3. *Let π be a non-trivial unary partial word. Then the bipartite graph $G = G(\pi_{p,q})$ is complete bipartite if and only if the corresponding binary function γ is not surjective.*

Proof. As the given partial word π is non-trivial, there exists at least one entry in π which is not a hole. The associated partitioned set will then be mapped to 1 under γ. Hence if γ is not surjective, then all the partitions will be mapped only to 1. By the construction of the bipartite graph G, we know that if two partitions B_i and B_j have the same image 1, then their corresponding entries in the domain D have the same congruent value at p and q at some positions and hence they will be adjacent in G. As all are mapped only to 1, this will make G to be complete bipartite and sufficiency of the theorem is proved.

To prove the necessary condition, assume that G is complete bipartite. Then by the previous argument, to have an edge between $u \in U$ and $v \in V$, there should exists a domain element $x \in D$ such that $x \equiv u(\mod p)$ and $x \equiv v(\mod p)$. If $x \in B_j$, then $\gamma(B_j) = 1$. As G is complete bipartite, this scenario is true for all $x \in D$ and hence $\gamma(B_i) = 1 \forall\ i$. Hence 0 does not have a preimage under γ and hence γ is not surjective. □

Corollary 3.4. *Suppose $\Gamma(h, n)$ denote the collection of all unary partial words π with h holes and a total length n. Then $\Gamma(h, n)$ can be partitioned into exactly two sets with respect to the binary function γ.*

Proof. Theorem 3.3 demonstrated that the binary function γ is not onto if and only if the corresponding the bipartite graph is complete. We know that there are $|\Gamma(h, n)| = 2^n$ unary partial word. Thus γ explicitly separates those partial words (among the 2^n collection) which are complete and which are not by its one property : surjectivity. □

4. Conclusion

We have thus defined a partition of the positions of the elements of a given binary word through which we defined a binary function. Through this binary function we characterized those partial words which have a complete bipartite equivalent structure. As a consequence, the collection $\Gamma(h, n)$, of all possible partial words of length n and h holes, has been split-up into exaclty two disjoint groups: one which have complete bipartite graph structure and the other which does not. To determine this split, the function γ plays a key role.

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