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EXTREMAL STABILIZATION ALGORITHM OF A COOPERATIVE GAME

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ABSTRACT. The analysis of the core of a cooperative game V is classically linked to linear programming (see [8]). However, the natural structure of a cooperative game is the disjoint over-additive, so an internal probability if we normalize $(V(\Omega) = 1)$, and the elements of the core then appear as increasing probabilities are given a probability internal.

Led A. Fougers, al. (see [5]) to introduce the notion of stability. The goal of this article is to establish a significant improvement of the algorithm, so it will be limited to the eligible "candidate" points and we will play a relatively "time efficient" first generation algorithm when it succeeds, which happens in in most cases. A more precise analysis of the critical experimental cases will undoubtedly lead to combine it with other methods being processed, to end up with a "second generation" algorithm, which always converges in the case with a non-empty core.

1. INTRODUCTION

The analysis of the heart of a cooperative game V is classically linked to linear programming (see [8]). However, the natural structure of a cooperative game is disjoint over-additivity, therefore an internal probability if we normalize $(V(\Omega) = 1)$, and the elements of the heart then appear as the probabilities increasing a given internal probability.

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This is what led A. Fougeres and his Perpignan team "JADE" to introduce the notion of stability (V is said to be "stable" if its dual game $V^{\times}$ is sub-additive, therefore an external probability in the normalized case), as well as its algorithmic approach, stabilization. The elements of the heart are then stable self-dualgames. Stabilization is a necessary condition for the emptiness of the heart; unfortunately, it is not "completely" sufficient.

One of the ways to algorithmically reach an element of the heart is to add additional nodes to the initial game (that is to say to search for elements of the heart passing through its points) then to stabilize it; in most cases, we thus very quickly arrive at an element of the heart which, if we have respected linear independence with the previous nodes (notion of "admissible point"), is extreme. Hence the name of the extreme stabilization algorithm given to the method.

The difficulty of the initial algorithm stems from the fact that when we "force" a node to the value of V, we are not sure that stabilization is possible (notion of "regular point", which prefigures a point of contact between V and the final extremal element).

Thus, a significant improvement in the algorithm is to be limited to the "candidate" admissible points, that is to say, verifying a family of inequalities necessary for the emptiness of the heart.

This gives a relatively "time-efficient" first generation algorithm when it succeeds, which happens in most cases. A more precise analysis of critical experimental cases will undoubtedly lead to combining it with other methods being processed, to arrive at a "second generation"algorithm, which always converges in the case with a non-empty heart (an empty heart must be detected quickly).

For a fairly low cardinality of Ω, all the methods (linear programming, nodes with average values) give results quickly.

On the other hand, when the cardinality of Ω increases, we quickly come up against a problem of memory size for linear programming methods. This is the field of the future for stabilization algorithms, of which the present algorithm is only a first step.

Other useful references are [1,2,4,6,7,9].

2. NOTES, DEFINITIONS, REMINDERS AND RESULTS

We call "cooperative play" the data of a defined function V of the set of parts of a set Ω of finite cardinal N with positive values verifying the property of over-additivity, that is to say:$V(AB) \geq V(A) + V(B) \quad \forall (A, B) \in (\mathcal{P}(\Omega))^2$ such that $A \cap B = \emptyset$.

Here AB designates the disjoint meeting of A and B.

Remark 2.1.

(i) *V is an increasing function on $\mathcal{P}(\Omega)$;*

(ii) *$V(\emptyset) = 0$;*

(iii) *Since Ω is a set of finite cardinalities and V is increased by a positive real number $V(\Omega)$; to make the analogy with probability theory, we will assume that: $V(\Omega) = 1$.*

Definition 2.1. *Let V a cooperative game, the function defined on $\mathcal{P}(\Omega)$ with positive values denoted by $V^{\times}$ such that,*

$$V^{\times}(A) := V(\Omega) - V(A^c) = 1 - V(A^c) \textit{for all} A \in \mathcal{P}(\Omega),$$

a cooperative game V is said to be "self-dual" if $V^{\times}(A) = V(A)$for all$A \in \mathcal{P}(\Omega)$. Note that for any cooperative game V_1 increasing V, if $V(\Omega) = V_1(\Omega)$, we have,

$$V \leq V_1 \Longrightarrow V \leq V_1 \leq V \leq V_1^{\times} \leq V_1,$$

which implies the immediate (but important) property $V \leq V^{\times}$.

Definition 2.2. *Let V be a cooperative game,*

(i) *The set $C_V := \{W \textit{probability such that} V \leq W\}$ is called the heart of game V.*

(ii) *The cooperative game V is said to be stable if its dual game $V^{\times}$ is sub-additive on $\mathcal{P}(\Omega)$ ($i.e. V^{\times}(AB) \leq V^{\times}(A) + V^{\times}(B)$) for all A, B in $\mathcal{P}(\Omega)$ such that $A \cap B = \emptyset$.*

Remark 2.2.

(i) *$V^{\times}$ is an increasing function, but in general is not sub-additive.*

(ii) *The probabilities are therefore the stable self-dual cooperative games, and the elements of the core of V are the stable self-dual majorants of V.*

Definition 2.3. *We call stability or "stabilized" closure of V, the smallest stable cooperative game increasing V if it exists, which it is denoted by $\widehat{V}$.*

Lemma 2.1. *If C_V is non-empty then the stability closure of V exists and has the same heart as V.*

Proof. There is an element W of the heart of V such that $V \leq W \leq V^{\times}$. The set S of stable upper bound cooperative games S of V is therefore not empty (because it contains W) and it is inf-complete, because, $\widehat{V} = \inf_{S\in\mathbb{S}} S$ is over-additive and for dual $\widehat{V^{\times}} = \inf_{S\in\mathbb{S}} S^{\times}$ which is sub-additive. On the other hand, $V_0 = \inf_{W\in C_V} W$ is the largest stable cooperative game increasing V by the same heart. □

Lemma 2.2. *(Lemma of the measure theory (see [3])) Any set function μ admits a lower bound $\underline{\mu}$ sub-additive and an upper bound $\overline{\mu}$ super-additive given by,*

$$\underline{\mu}(A) = \inf_{A=A_1A_2..A_k} \sum_{i=1}^{k} \mu(A_i) \quad \textit{and} \quad \overline{\mu}(A) = \sup_{A=A_1A_2..A_k} \sum_{i=1}^{k} \mu(A_i).$$

So μ is sub-additive (respectively over-additive) if and only if $\mu = \overline{\mu}$ (respectively $\mu = \underline{\mu}$).

Definition 2.4. *We call " polar V^* of V " the sub–additive lower bound of $V^{\times}$ ($i.e. V^* = \underline{V^{\times}}$).*

If $V^*(\Omega) = 1$, we define "the bipolarity V^{**} of V" as the supper-additive upper bound of $V^{*\times}$, and "the bipolarity V^{**} of V" the supper-additive upper bound of $V^{*\times}$ $\left(i.e., V^{*\times} = \overline{V^{*\times}}\right)$ and by recurrence, if, $V^{(2k)^*}(\Omega) = 1$, for any integer $p \leq k$, we have,

$$V^{(2p+1)^*} := \left(V^{(2p)^*}\right)^* = \underline{V^{(2p)^{*\times}}}.$$

On the other hand, if $V^{(2p+1)^*}(\Omega) = 1$, we have, $V^{(2p+2)^*} := \left(V^{(2p+1)^*}\right)^* = \overline{V^{(2p+1)^{*\times}}}$.

The introduction of these concepts is natural for the following reasons.

Let V be a cooperative game with a non-empty core then for all probability W in C_V, we have, $V \leq W \leq V_{\times}$. Say that V is not stable, that is to say that $V^{\times}$ is not a sub-additive function on $P(\Omega)$ then we will have, $V \leq W \leq V^* \leq V^{\times}$, with the polar V^* of V is different from the dual $V^{\times}$ of V.

In the same way, we have, $V \leq V^{*\times} \leq W = W^* = W^{\times} \leq V^* \leq V^{\times}$ with $V \neq V^{*\times}$. If $V^{*\times}$ is an over-additive function on $P(\Omega)$, then the cooperative game V has for stabilized,

$$\widehat{V} = V^{*\times} \text{ because } \widehat{V}^{\times} = V^{*\times\times} = V^*.$$

If not, we have $V \le V^{*\times} \le V^{**} \le W = W^{*} = W^{\times} \le V^{*} \le V^{\times}$, with $V \neq V^{*\times} \neq V^{**}$ and $V^{*} \neq V^{\times}$ where V^{**} is the upper-additive upper bound of $V^{*\times}$ (because $V^{**} \le V^{*}$). By recurrence, we will have the same,

$$V \le V^{**} \le ... \le V^{(2p)*} \le W \le V^{(2p+1)*} \le \cdots \le V^{*} \le V^{\times}.$$

Definition 2.5. *We call "stabilization of V", the operation of calculating $V^{(2p)*}$ until the sequence becomes stationary. We have reached stabilized V.*

Proposition 2.1. *(see [5]) The stabilization of V is a necessary condition of non-emptiness of the heart in the case where V is with rational values.*

Let's start with the following technical lemma.

Lemma 2.3. *If V is integer then $V^{(2p)*}$ is a stationary series.*

Proof. (proof of Lemma 2.3) We consider the general term series,

$$\alpha_k = \sum_{A \in \mathcal{P}(\Omega)} \left(V^{(2k+2)^{*}}(A) - V^{(2k)^{*}}(A)\right) \quad \text{with} \quad V^{0*} = V.$$

So $\alpha_k \ge 2$ for all k such that $V^{(2k+2)^{*}} \neq V^{(2k)^{*}}$. Or for all p, we have,

$$\sum_{k=1}^{p} \alpha_k = \sum_{A \in \mathcal{P}(\Omega)} \left(V^{(2p+2)^{*}}(A) - V(A)\right) \le \sum_{A \in \mathcal{P}(\Omega)} \left(V^{\times}(A) - V(A)\right).$$

Then α_k are therefore zero from a rank p_0. □

Proof. (proof of Proposition 2.1) First, note that any game with rational values is reduced by homothetic to a game with whole values. The proposition will therefore be a consequence of Lemma 2, since, $V^{(2p_0)^{*}} = V^{2(p_0+1)^{*}} = V^{(2p_0+1)^{*\times}}$ it is therefore stable since, $V^{(2p_0)^{*\times}} = V^{(2p_0+1)^{*}}$. □

Lemma 2.4. $(\mathbf{CN})_{\mathbf{1}}$. *$V \le V^{*}$ if only if $V(\Omega) = V^{*}(\Omega)$.*

Proof. The condition is obviously necessary, since,

$$V(\Omega) \le V^{*}(\Omega) \le V^{\times}(\Omega) \le V(\Omega).$$

Conversely, the existence of a part A of Ω satisfying, $V(A) > V^{*}(A)$
Is from, $V(\Omega) > V^{*}(\Omega)$ since

$$V(\Omega) = 1 = V(A) + V^{\times}(A^{c}) > V^{*}(A) + V^{\times}(A^{c}) \ge V^{*}(A) + V^{*}(A^{c}) \ge V^{*}(\Omega).$$

□

We establish in the same way the following Lemma.

Lemma 2.5. $(\mathbf{CN})_2$ $V_{**} \leq V^*$ *if only if* $V(\Omega) = V^{**}(\Omega)$.

So, $V \leq V^{*\times} \leq V^{**} \leq V^* \leq V^\times$.

In the same way, we get (CN_{2k-1}) and (CN_{2k}).

Remark 2.3. *In practice, we calculate the successive polar of V, as long as*

$$V^{(2p)^*} \neq V^{(2p-1)^*} \quad \textit{with} \quad V^{(2p)^*}(\Omega) = V(\Omega).$$

If there exist a q such as we have $V^{q^*}(\Omega) \neq V(\Omega)$, then the heart is empty.

3. Role of the nodes of a stable cooperative game

Definition 3.1. *We call"node" of a stable cooperative game V any element A_0 of $\mathcal{P}(\Omega)$ if we have that $V(A_0) = V^\times(A_0)$.*

3.1. Characterization of a Node of a stable cooperative game.

Let's start by recalling the following elementary lemma which will be useful for deducing the characteristic properties of the nodes of a cooperative game (the equivalence of properties (b), (c) and (d) of Proposition 2.2).

Lemma 3.1. *(see [5])*

(i) *V is a super-additive if only if two element A and B of $\mathcal{P}(\Omega)$, we have,*

$$V(A) + V^\times(B) \leq V^\times(AB),$$

(ii) *is a sub–additive if only if for two A and B two disjoint element of $\mathcal{P}(\Omega)$,we have,*

$$V(AB) \leq V(A) + V^\times(B).$$

Using Lemma 3.1 it is easy to verify the following proposition.

Proposition 3.1. *Let V be a stable cooperative game and A_0 an element of $\mathcal{P}(\Omega)$, the following properties are equivalent,*

$a)$ *A_0 is a node of V;*

$b)$ *$V(A_0) + V(A_0^c) = 1$;*

$c)$ *For all $B \in \mathcal{P}(\Omega)$, $A \cap B = \varnothing$ we have $V(A_0 B) = V(A_0) + V(B)$.*

$d)$ *For all $B \in \mathcal{P}(\Omega)$, $A \cap B = \varnothing$ we have $V^\times(A_0 B) = V^\times(A_0) + V^\times(B)$.*

Remark 3.1. *According to the property (b), the complementary of a node is also a node, and any element W of the heart of V passes by the value "forced" $W(A_0) = V(A_0)$ which justifies the name of "knot".*

Let V be a stable cooperative game and A_0 a non-empty element of $\mathcal{P}(\Omega)$.

Definition 3.2. *Let V' be a cooperative game such that $V \leq V'$, we say that V' is an exact major in "A_0" if we have $V(A_0) = V'(A_0)$. In addition, we say that A_0 is the contact point of V' with V.*

Remark 3.2. *Ω is always a point of contact.*

Proposition 3.2. *The function V_{A_0} defined by,*

$$V_{A_0} = \begin{cases} V^{\times}(A_0^c) + V(A \cap A_0) & \text{if } A \cup A_0 = \Omega \\ V(A) & \text{if } A \cup A_0 \neq \Omega. \end{cases}$$

is the smallest upper bound V' of exact over-additive V in A_0 admitting a node in A_0.

Proof.

a) To prove the following inequality, $V_{A_0} \geq V$, from Lemma 5 (ii) with, $(B = A \cap A_0$ and $C = A_0^c)$ where $(A = BC$ with $B = A \cap A_0$ and $C = A_0^c)$.

b) $V_{A_0}(A_0) = V(A_0)$ since $A_0 \cup A_0 \neq \Omega$ (Clearly, we have the equality where $A = \Omega$).

c) Let A and B be two disjoint elements of $\mathcal{P}(\Omega)$, two cases arise.

   i) If neither A, nor B contain A_0^c then,
   $V_{A_0}(A) + V_{A_0}(B) = V(A) + V(B) \leq sdV(AB) \leq V_{A_0}(AB)$.

   ii) if $A = A_0^c C$ then,

$$\begin{aligned} V_{A_0}(A) + V_{A_0}(B) &= V^{\times}(A_0^c) + V(B) + V(C) \\ \leq& V^{\times}(A_0^c) + V(BC) = V_{A_0}(AB). \end{aligned}$$

d) If V' is an over-additive upper bound of V equal to A_0 such that, $V'(A_0) = V'^{\times}(A_0)$. We have, $V'(A) \geq V_{A_n}(A)$ for all $A \in \mathcal{P}(\Omega)$, such that $A \cup A_0 \neq \Omega$,and if $A = A_0^c B$, we have, $V'(A_0^c) := 1 - V'(A_0) = 1 - V(A_0) = V^{\times}(A_0)$. So, we have, $V'(A) \geq V'(A_0^c) + V'(B) \geq V^{\times}(A_0^c) + V(B) = V_{A_0}(A)$.

□

Definition 3.3. *Any point C where V_C stabilizes is called a "regular point" of V.*

We will note $V_{\widehat{C}}$ his stabilized.

Lemma 3.2. *Let V be a stable cooperative game, N_V is the set of its nodes, if C is a regular point of V, then, $N_V \subseteq N_{V_C} \subseteq N_{V_{\widehat{C}}}$. And for S in N_V, we have, $V(S) = V_C(S) + V_{\widehat{C}}(S)$.*

Proof. Based on the following equivalence, $S \in N_V \Leftrightarrow V(S) + V(S^c) = 1$, we have, $1 \leq V_{\widehat{C}}(S) + V_{\widehat{C}}(S^c) \leq V_{\widehat{C}}(\Omega)$. Then, $V(S) \leq V_C(S) \leq V_{\widehat{C}}(S)$ and $V(S^c) \leq V_C(S^c) \leq V_{\widehat{C}}(S^c)$. So, we have, $V(S) = V_C(S) = V_{\widehat{C}}(S)$. □

Lemma 3.3. *Let C be a regular point of V, and $V_{\widehat{C}}$ the stability of node C, the heart of $V_{\widehat{C}}$ is the set of elements of the heart of V exact in C.*

Proof. Any element W of $V_{\widehat{C}}(C)$ goes through $V_{\widehat{C}}(C) = V(C)$. Since C is a node of $V_{\widehat{C}}$.

Conversely, if W is an element of C_V such that $W(C) = V(C)$ and $W(C^c) = V^{\times}(C^c)$ and for all B such that $B \cap C^c = \emptyset$, we have, $V(B) + V^{\times}(C^c) = V_C(BC^c) \leq W(BC^c) = W(B) + W(C^c)$, it follows that $V_{\widehat{C}} \leq W$. □

3.2. **Vector representation.**

In all that follows, we will use the classical vector representation of the parts A of Ω by the vertices $\vec{A}$ of the cube $[0,1]^N$ of $\mathbb{R}^N$ where N is the cardinal of Ω, defined as,

$$\vec{A} = (a_i)_{1 \leq i \leq N} \quad \text{where} \quad a_i = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{if } i \notin A. \end{cases}$$

We will therefore say that k parts $(A_1, A_2, \ldots, A_k)$ are linearly independent when the family $\left(\vec{A}_1, \vec{A}_2, \ldots, \vec{A}_k\right)$ of the associated vectors is independent.

Similarly, we recall that the set of real measurements on $\mathcal{P}(\Omega)$ is identified with $\left(\mathbb{R}^N\right)^*$ by,

$$\mathcal{M} \longrightarrow \overleftarrow{\mathcal{M}} = (\mathcal{M}(1), \mathcal{M}(2), \ldots, \mathcal{M}(N))$$

we will have for any real measurement M and any element A of $\mathcal{P}(\Omega)$,

$$\mathcal{M}(A) = \left\langle \overleftarrow{\mathcal{M}}, \vec{A} \right\rangle.$$

Definition 3.4. *The "extremal element" of the heart of V is any point W of C_V which cannot be expressed as a convex combination of other points of C_V.*

Under these conditions, we can adapt the classic linear programming results as follows,

Lemma 3.4. *An element W of the heart of V is extremal if and if the set of its points of contact with V is of rank N.*

The proof of Lemma 3.4 is a direct consequence of Theorem B (see [10]) (because W is extremal of the heart implies that W is an extremal point in the classical sense of the term).

Remark 3.3.

i) *All the contact points between V and an extremal of the heart are regular points.*

ii) *The progressive formation of an admissible base within the meaning of linear programming justifies the following designation.*

Definition 3.5. *The "admissible point" of V is any part A of Ω which is linearly independent of the nodes of V.*

4. Extremal stabilization algorithm

4.1. **Initial diagram.**

Let V be a stable cooperative game. The algorithm consists in determining an extremal element of the heart of V (supposed not empty) by forming successive nodes in admissible points of V expected regular.

Step 1. Let C_1 be an admissible point for V, we calculate V_{C_1} and we analyze the regularity of C by calculating the stability fence $V_{\widehat{C}_1}$ of V_{C_1} if it exists, we have the following equality,

$$V_{\widehat{C}_1}(C_1) = V(C_1)$$

and we go to the second step (by replacing the cooperative game V by the cooperative game $V_{\widehat{C}_1}$).

Otherwise, you change the eligible point until you get a regular point.

Step 2. Let C_2 be an admissible point for $V_{\widehat{C}_1}$, we calculate $V_{\widehat{C}_1 C_2}$ and stabilize it, if C_2 is regular, even if it means changing the admissible point, we will have,

$$V_{\widehat{C}_1 \widehat{C}_2}(C_i) = V(C_i) \quad \text{for all} \quad i = 1, 2.$$

In a recurring way, for all $p \geq 3$, we have:

Step (p-1). Let $C_{(p-1)}$ an admissible point for $V_{\widehat{C}_1 \widehat{C}_2 \ldots \widehat{C}_{(p-2)}}$, we calculate $V_{\widehat{C}_1 \widehat{C}_2 \ldots \widehat{C}_p C_{(p-1)}}$ and we stabilize it if $C_{(p-1)}$ is regular, even if it means changing the admissible point, we will have,

$$V_{\widehat{C}_1 \widehat{C}_2 \ldots \widehat{C}_p \widehat{C}_{(p-1)}}(C_i) = V(C_i) \quad \text{for} \quad i = 1, 2, \ldots, (p-1).$$

Step p. Let C_p an admissible point for $V_{\widehat{C}_1 \widehat{C}_2 .. \widehat{C}_{(p-1)}}$, we calculate $V_{\widehat{C}_1 \widehat{C}_2 .. \widehat{C}_p C_p}$ and we stabilize it if C_p is regular, even if it means changing the admissible point, we will have,

$$V_{\widehat{C}_1 \widehat{C}_2 .. \widehat{C}_p \widehat{C}_p}(C_i) = V(C_i) \quad \text{for all} \quad i = 1, 2, \ldots, p.$$

Step (p+1). Let $C_{(p+1)}$ an admissible point for $V_{\widehat{C}_1 \widehat{C}_2 .. \widehat{C}_p}$, we calculate $V_{\widehat{C}_1 \widehat{C}_2 .. \widehat{C}_p C_{(p+1)}}$ and we stabilize it if $C_{(p+1)}$ is regular, even if it means changing the admissible point, we will have,

$$V_{\widehat{C}_1 \widehat{C}_2 .. \widehat{C}_p \widehat{C}_{(p+1)}}(C_i) = V(C_i) \quad \text{for all} \quad i = 1, 2, \ldots, (p+1).$$

And so on.

See the algorithm on the next page.

Note: This method is an algorithm because it ends in a finite number of steps (less than or equal to 2^N where $N = \mid \Omega \mid$). Indeed, at each iteration as long as, the cardinality of A drops by at least 1 (if $\overline{V}_C$ stabilizes, C is no longer admissible for $\overline{V}_{\widehat{C}}$).

Proposition 4.1. *When the algorithm ends, we have* $\mid C_{\widehat{V}} \mid \leq 1$.

Proof.
Case 1. If the cardinality of $\mathcal{F}$ is strictly less than N then it is clear that in this case the heart of $\overline{V}$ is empty.

Case 2 If the cardinality of $\mathcal{F}$ is equal to N then the only "probability" W candidate checks,

$$W(A) = \overline{V}(A) \quad \text{for all} \quad A \in \mathcal{F},$$

Algorithm 1 Extremal stabilization algorithm.

```
Beginning
𝓕 ← Base (V nodes)
V̄ ← V
A ← {A/A admissible for V}
while A ≠ ∅ and | 𝓕 |< N do
    Make C choose from A
    if V̄_C then stabilizes,
        𝓕 ← Base (Nodes of V̄_Ĉ)
        A ← {A/A admissible for V̄_Ĉ (A) = V (A)}
        V̄ ← V̄_Ĉ
    else
        A ← A \ {C}
    end if
end while
End
```

and we have the following equivalence, $C_{\overline{V}} \neq \varnothing \Leftrightarrow W \in C_{\overline{V}}$. □

Corollary 4.1. *If the algorithm gives a solution W of the heart of $\overline{V}$, W is an extremal element of C_V.*

Proof. To have a solution is to have, $|\mathcal{F}| = N$. W belongs to C_V since $V \leq \overline{V} \leq W$ and W is extremal since we have a base of points of contact (i.e. the elements of F). □

4.2. **Improvement of the Algorithm.**

The efficiency of the algorithm is reduced by the fact that the regular points are only known after stabilization, it is therefore important to limit as much as possible the set of admissible points to be analyzed. We will highlight a family of simple inequalities necessary for the emptiness of the heart.Indeed, for all points C and D and any additive W, we will have,

$$W(C) + W(D) = W((C \cap D) \cup A) + W((C \cap D) \cup B)$$

for any by-partition (A, B) of $C \triangle D$.

Therefore, if W increases a cooperative game V with two contact points C and D, we will necessarily have, $V(C)+V(D) \geq V((C \cap D) \cup A)+V((C \cap D) \cup B)$. If we apply this to a cooperative game V,

i) The preceding inequalities are always checked for any couple of nodes.
ii) A non-node point C which does not check with the nodes D of V of an element W of the heart, candidates eligible for additional nodes must therefore check all these inequalities with any node of V. Hence the following definition.

Definition 4.1. *On will say that a point is "candidate" of the contact if it checks with all the nodes of V the previous inequalities.*

Algorithm 2 Improvement of the algorithm.

Beginning
$\mathcal{F} \leftarrow$ Base (V nodes)
$\overline{V} \leftarrow V$
$A' \leftarrow \{A/A \text{ admissible for} V\}$
while $A' \neq \varnothing$ and $| \mathcal{F} | < N$ **do**
  Make C choose from A'
  if $\overline{V}_C$ **then** stabilizes,
    $\mathcal{F} \leftarrow$ Base (Nodes of $\overline{V}_{\widehat{C}}$)
    $A' \leftarrow \left\{A/A \text{ admissible for } \overline{V}_{\widehat{C}}(A) = V(A)\right\}$
  else
    $A' \leftarrow A' \setminus \{C\}$
  end if
end while
End

Remark on the efficiency of the algorithm. We can solve the problem of the emptiness of the heart (and obtain an extremal element), using for example phase 1 of the 2-phase method. This is equivalent to solving the following problem:

$$\begin{cases} & Max\, W(\Omega), \\ W(S) \leq & V^X(S) \text{ for all } S \in \mathcal{P}(\Omega), \\ & W \geq 0. \end{cases}$$

The heart is not empty if and only if: $Max\, W(\Omega) = V(\Omega)$. W will be an extremal element of the heart. The comparison of the execution times here is made with the dual of the initial problem, that is to say the dual of,

$$\begin{cases} & Min\, W(\Omega) \\ W(S) \geq & V(S) \text{ for all } S \in \mathcal{P}(\Omega), \\ & W \geq 0. \end{cases}$$

Where we choose as a feasible starting base a base which (if the core is not empty) is optimal that is to say the base $\mathcal{B}_1$ composed of the parts: $\mathcal{B}_1 = (\{N\}, \{N-1, N\}, \{N-2, N-1, N\}, \ldots, \{\Omega\})$ or the base $\mathcal{B}_2$ composed of $\mathcal{B}_2 = (\{2\}, \{3\}, \ldots, \{N\}, \{\Omega\})$. We have seen that on non-trivial examples, times become prohibitive.

The interest of the algorithms of the type that we have developed (in so far as the execution time is not too large) is to occupy a memory space of the order 2^N, while that used by the simplex is $N.2^N$. The extremal stabilization algorithm in its current version can be improved; its effectiveness depends on the right choice of eligible candidates.
Any refinement of the candidate concept will bring an improvement in this fact.

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