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STABILITY OF RADICAL QUADRATIC FUNCTIONAL EQUATION IN RB-SPACE

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ABSTRACT. The main purpose of this research article is to prove the stability of radical quadratic functional equation in Random Banach Space using direct and fixed point methods in sense of "Ulam, Hyers Rassias".

1. INTRODUCTION

The idea of stability of a functional equation stand up when one change a functional equation by an inequality which acts as a perturbation of the equation. In 1940, the main stability problem concerning group homomorphisms was elevated by Ulam [11] and affirmatively solved by Hyers [3] in 1941. Later the result of Hyers was generalized by several mathematicicans one can see [1, 2, 6–8] in countless settings.

Now, we will recall the fundamental result in fixed point theory.

Theorem 1.1. *[5] (The alternative of fixed point) Suppose that for a complete generalized metric space (X, d) and a strictly contractive mapping $T : X \to X$ with Lipschitz constant L. Then, for each given element $x \in X$, either*
(F_1) $d(T^n x, T^{n+1}x) = \infty$*, for all* $n \geq 0$,
or
(F_2) *there exists a natural number* n_0 *such that:*

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$(FPC1)$ $d(T^n x, T^{n+1}x) < \infty$ *for all* $n \geq n_0$ *;*
$(FPC2)$ *The sequence* $(T^n x)$ *is convergent to a fixed point* y^* *of* T
$(FPC3)$ y^* *is the unique fixed point of* T *in the set* $Y = \{y \in X : d(T^{n_0}x, y) < \infty\}$;
$(FPC4)$ $d(y^*, y) \leq \frac{1}{1-L}\, d(y, Ty)$ *for all* $y \in Y$.

In this paper, we establish the generalized Ulam-Hyers stability of radical quadratic functional equation

$$q\left(\sqrt{mx^2+ny^2}\right) = mq\,(x) + nq\,(y) \tag{1.1}$$

in Random Banach Space using direct and fixed point methods. To prove stability results, we assume that $(\mathcal{E}, \mathcal{R})$ and $(\mathcal{F}, \mathcal{R}', \mathcal{T})$ are linear space and Random Banach space.

The usual terminology, notations and conventions of the theory of random normed spaces one can see [9, 10].

From now on, Δ^+ is the space of distribution functions, that is, the space of all mappings $F : R \cup \{-\infty, \infty\} \longrightarrow [0, 1]$, such that F is leftcontinuous and nondecreasing on $R, F(0) = 0$ and $F(+\infty) = 1$. D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^-F(+\infty) = 1$, where $l^-f(x)$ denotes the left limit of the function f at the point x, that is, $l^-f(x) = \lim\limits_{t\to x^-} f(t)$. The space Δ^+ is partially ordered by the usual pointwise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function ϵ_0 given by $\epsilon_0(t) = \begin{cases} 0, & if\ t \leq 0, \\ 1, & if\ t > 0. \end{cases}$

Definition 1.1 ([10]). *A random normed space (briefly, RN-space) is a triple* (X, μ, T)*, where* X *is a vector space,* T *is a continuous* $t-$*norm and* μ *is a mapping from* X *into* D^+ *satisfying the following conditions:*

(RN1) $\mu_x(t) = \varepsilon_0(t)$ *for all* $t > 0$ *if and only if* $x = 0$*;*
(RN2) $\mu_{\alpha\, x}(t) = \mu_x(t/|\alpha|)$ *for all* $x \in X$*, and* $\alpha \in \mathbb{R}$ *with* $\alpha \neq 0$*;*
(RN3) $\mu_{x+y}(t+s) \geq T\left(\mu_x(t), \mu_y(s)\right)$ *for all* $x, y \in X$ *and* $t, s \geq 0$.

2. Random Stability: Hyers Method

Theorem 2.1. *Let* $\mathcal{R}' : \mathcal{E}^2 \to D^+$ *be a function and* $p = \pm 1$ *such that*

$$\lim_{t\to\infty} \mathcal{R}'_{\sqrt{2}^{pr}x, \sqrt{2}^{pr}y}\left(2^{pr}s\right) = 1 = T_{r=0}^{\infty} \mathcal{R}'^{D}_{\sqrt{2}^{pr}\ x, \sqrt{2}^{pr}\ x}\left(2^{pr)}s\right) \tag{2.1}$$

for all $x \in \mathcal{E}$ *and all* $s > 0$*. Let* $q : \mathcal{E} \to \mathcal{F}$ *be a function fulfilling the inequality*

$$\mathcal{R}'_{x,y}(s) \leq \mathcal{R}_{q\left(\sqrt{mx^2+ny^2}\right)-m\ q(x)-n\ q(y)}(s) \tag{2.2}$$

for all $x, y \in \mathcal{E}$ *and all* $s > 0$*. Then there exists a unique Radical Quadratic function* $\mathcal{Q}(x) : \mathcal{E} \to \mathcal{F}$ *which satisfies (1.1) and*

$$T_{r=0}^{\infty} \mathcal{R}'^{D}_{\sqrt{2}^{pr}\ x, \sqrt{2}^{pr}\ x}\left(2^{p(r+1)} \cdot m\ s\right) \leq \mathcal{R}_{\mathcal{Q}(x)-q(x)}(s) \tag{2.3}$$

where $\mathcal{R}'^{D}_{x,x}$ *and* $\mathcal{Q}(x)$ *are defined by*

$$\begin{aligned} \mathcal{R}'^{D}_{\sqrt{2}^{p}\ x, \sqrt{2}^{p}\ x}\Big(s\Big) = \mathcal{T}^2\ \Big(&\mathcal{R}'_{0, \sqrt{\frac{m}{n}}\sqrt{2}^{p}\ x}\Big(s\Big), \mathcal{R}'_{\sqrt{2}^{p}\ x, 0}\Big(s\Big), \\ &\mathcal{R}'_{\sqrt{2}^{p}\ x, \sqrt{\frac{m}{n}}\sqrt{2}^{p}\ x}\Big(s\Big), \mathcal{R}'_{\sqrt{2}\ \sqrt{2}^{p}\ x,\ 0}\Big(s\Big)\Big) \end{aligned} \tag{2.4}$$

and

$$\mathcal{R}_{\mathcal{Q}(x)}(s) = \lim_{t \to \infty} \mathcal{R}_{\frac{q(\sqrt{2}^{pr} x)}{2^{pr}}}(s) \tag{2.5}$$

for all $x \in \mathcal{E}$ *and all* $s > 0$*, respectively.*

Proof. If we change (x, y) by $\left(\frac{x}{\sqrt{m}}, \frac{y}{\sqrt{n}}\right)$ in (2.2), we get

$$\mathcal{R}'_{\frac{x}{\sqrt{m}}, \frac{x}{\sqrt{n}}}\Big(s\Big) \leq \mathcal{R}_{q\left(\sqrt{x^2+y^2}\right)-m\ q\left(\frac{x}{\sqrt{m}}\right)-n\ q\left(\frac{y}{\sqrt{n}}\right)}\Big(s\Big) \tag{2.6}$$

for all $x \in \mathcal{E}$ and all $s > 0$. Replacing (x, y) by $(0, \sqrt{m}\ x)$ in (2.6), we obtain

$$\mathcal{R}'_{0, \sqrt{\frac{m}{n}}\ x}\Big(s\Big) \leq \mathcal{R}_{q\left(\sqrt{m}\ x\right)-n\ q\left(\sqrt{\frac{m}{n}}\ x\right)}\Big(s\Big) \tag{2.7}$$

for all $x \in \mathcal{E}$ and all $s > 0$. Setting (x, y) by $(\sqrt{m}\ x, 0)$ in (2.6), we get

$$\mathcal{R}'_{x,0}\Big(s\Big) \leq \mathcal{R}_{q\ \left(\sqrt{m}\ x\right)-m\ q(x)}\Big(s\Big) \tag{2.8}$$

for all $x \in \mathcal{E}$ and all $s > 0$. Combining (2.7) and (2.8) with the help of (RNS3), we arrive

$$\mathcal{T}\ \Big(\mathcal{R}'_{0, \sqrt{\frac{m}{n}}\ x}\Big(s\Big), \mathcal{R}'_{x,0}\Big(s\Big)\Big) \leq \mathcal{R}_{m\ q(x)-n\ q\left(\sqrt{\frac{m}{n}}\ x\right)}\Big(2s\Big) \tag{2.9}$$

for all $x \in \mathcal{E}$ and all $s > 0$. If we set (x, y) by $\left(\frac{x+y}{\sqrt{2m}}, \frac{x-y}{\sqrt{2n}}\right)$ in (2.2), we observe

$$\mathcal{R}'_{\frac{x+y}{\sqrt{2m}}, \frac{x-y}{\sqrt{2n}}}(s) \leq \mathcal{R}_{q\left(\sqrt{x^2+y^2}\right)-m\ q\left(\frac{x+y}{\sqrt{2m}}\right)-n\ q\left(\frac{x-y}{\sqrt{2n}}\right)}(s) \tag{2.10}$$

for all $x \in \mathcal{E}$ and all $s > 0$. Combining (2.6) and (2.10) with the help of (RNS3), we arrive

$$\begin{aligned} &\mathcal{T}\left(\mathcal{R}'_{\frac{x}{\sqrt{m}},\frac{x}{\sqrt{n}}}(s), \mathcal{R}'_{\frac{x+y}{\sqrt{2m}},\frac{x-y}{\sqrt{2n}}}(s)\right) \\ &\leq \mathcal{R}_{m\ q\left(\frac{x+y}{\sqrt{2m}}\right)+n\ q\left(\frac{x-y}{\sqrt{2n}}\right)-m\ q\left(\frac{x}{\sqrt{m}}\right)-n\ q\left(\frac{y}{\sqrt{n}}\right)}(2s) \end{aligned} \tag{2.11}$$

for all $x \in \mathcal{E}$ and all $s > 0$. Again set (x, y) by $(\sqrt{m}\ x, \sqrt{m}\ x)$ in (2.11), we find

$$\mathcal{T}\left(\mathcal{R}'_{x,\sqrt{\frac{m}{n}}\ x}(s), \mathcal{R}'_{\sqrt{2}\ x,\ 0}(s)\right) \leq \mathcal{R}_{m\ q\left(\sqrt{2}\ x\right)-m\ q(x)-n\ q\left(\sqrt{\frac{m}{n}}\ x\right)}(2s) \tag{2.12}$$

for all $x \in \mathcal{E}$ and all $s > 0$. Combining (2.9) and (2.12) with the help of (RNS3), we realize

$$\begin{aligned} &\mathcal{T}^2\left(\mathcal{R}'_{0,\sqrt{\frac{m}{n}}\ x}(s), \mathcal{R}'_{x,0}(s), \mathcal{R}'_{x,\sqrt{\frac{m}{n}}\ x}(s), \mathcal{R}'_{\sqrt{2}\ x,\ 0}(s)\right) \\ &\leq \mathcal{R}_{m\ q\left(\sqrt{2}\ x\right)-2m\ q(x)}(4s) \end{aligned} \tag{2.13}$$

for all $x \in \mathcal{E}$ and all $s > 0$. Using (RNS2) in (2.13), we land

$$\begin{aligned} &\mathcal{T}^2\left(\mathcal{R}'_{0,\sqrt{\frac{m}{n}}\ x}(s), \mathcal{R}'_{x,0}(s), \mathcal{R}'_{x,\sqrt{\frac{m}{n}}\ x}(s), \mathcal{R}'_{\sqrt{2}\ x,\ 0}(s)\right) \\ &\leq \mathcal{R}_{q\left(\sqrt{2}\ x\right)-2q(x)}\left(\frac{4s}{m}\right) \end{aligned} \tag{2.14}$$

for all $x \in \mathcal{E}$ and all $s > 0$. Now, let us decide

$$\mathcal{T}^2\left(\mathcal{R}'_{0,\sqrt{\frac{m}{n}}\ x}(s), \mathcal{R}'_{x,0}(s), \mathcal{R}'_{x,\ \sqrt{\frac{m}{n}}\ x}(s), \mathcal{R}'_{\sqrt{2}\ x,\ 0}(s)\right) = \mathcal{R}'^{D}_{x,x}(s) \tag{2.15}$$

for all $x \in \mathcal{E}$ and all $s > 0$. It follows from (2.14) and (2.15), it is clear that

$$\mathcal{R}'^{D}_{x,x}(s) \leq \mathcal{R}_{q\left(\sqrt{2}\ x\right)-2q(x)}\left(\frac{4s}{m}\right) \tag{2.16}$$

for all $x \in \mathcal{E}$ and all $s > 0$. Using (RNS2) in (2.16), we achieve

$$\mathcal{R}'^{D}_{x,x}(s) \leq \mathcal{R}_{\frac{q(\ \sqrt{2}\ x)}{2}-q(x)}\left(\frac{s}{2\cdot m}\right) \tag{2.17}$$

for all $x \in \mathcal{E}$ and all $s > 0$. Putting x by $\sqrt{2}^t\ x$ in (2.17), it earn that

$$\mathcal{R}'^{D}_{\sqrt{2}^t\ x,\sqrt{2}^t\ x}(s) \leq \mathcal{R}_{\frac{q(\sqrt{2}^{t+1}x)}{2}-q(\sqrt{2}^t x)}\left(\frac{s}{2\cdot m}\right) \tag{2.18}$$

for all $x \in \mathcal{E}$ and all $s > 0$. It is easy to verify from (2.18), that

$$\mathcal{R}'^{D}_{\sqrt{2}^t\ x,\sqrt{2}^t\ x}(s) \leq \mathcal{R}_{\frac{q(\sqrt{2}^{t+1}x)}{2^{t+1}}-\frac{q(\sqrt{2}^t x)}{2^t}}\left(\frac{s}{2^{t+1}\cdot m}\right) \tag{2.19}$$

for all $x \in \mathcal{E}$ and all $s > 0$. Replacing s by $2^{(t+1)}\ m\ s$ in (2.19), one finds that

$$\mathcal{R'}^{D}_{\sqrt{2}^t\ x, \sqrt{2}^t\ x}\left(2^{t+1} \cdot m\ s\right) \leq \mathcal{R}_{\frac{q(\sqrt{2}^{t+1}x)}{2^{t+1}} - \frac{q(\sqrt{2}^t x)}{2^t}}(s) \tag{2.20}$$

for all $x \in \mathcal{E}$ and all $s > 0$. It is easy to see that

$$\frac{q(\sqrt{2}^t x)}{2^t} - f(x) = \sum_{r=0}^{t-1}\left[\frac{q(\sqrt{2}^{r+1}x)}{2^{r+1}} - \frac{q(\sqrt{2}^r x)}{2^r}\right] \tag{2.21}$$

for all $x \in \mathcal{E}$ and all $s > 0$. From equations (2.20) and (2.21), we gain

$$\begin{aligned} T_{r=0}^{t-1}\mathcal{R'}^{D}_{\sqrt{2}^r\ x, \sqrt{2}^r\ x}\left(2^{(r+1)} \cdot m\ s\right) &\leq \mathcal{R}_{\sum_{r=0}^{q-1}\frac{q(\sqrt{2}^{r+1}x)}{2^{r+1}} - \frac{q(\sqrt{2}^r x)}{2^r}}(s) \\ &\leq \mathcal{R}_{\frac{q(\sqrt{2}^r x)}{2^r} - f(x)}(s) \end{aligned} \tag{2.22}$$

for all $x \in \mathcal{E}$ and all $s > 0$. Replacing x by $2^{t_0}x$ in (2.22) and using $(RNS2)$, we attain

$$T_{r=t_0}^{t+m-1}\mathcal{R'}^{D}_{\sqrt{2}^r\ x, \sqrt{2}^r\ x}\left(2^{(r+1)} \cdot m\ s\right) \leq \mathcal{R}_{\frac{f(\sqrt{2}^{t+t_0}x)}{2^{(t+t_0)}} - \frac{f(\sqrt{2}^{t_0}x)}{2^{t_0}}}(s) \quad \text{or} \tag{2.23}$$

$$\mathcal{R}_{\frac{f(\sqrt{2}^{t+t_0}x)}{2^{(t+t_0)}} - \frac{f(\sqrt{2}^{t_0}x)}{2^{t_0}}} \to 1 \quad \text{as} \quad t_0 \to \infty \tag{2.24}$$

for all $x \in \mathcal{E}$ and all $s > 0$ and all $t > t_0 > 0$, which implies that $\left\{\dfrac{f(\sqrt{2}^t x)}{2^t}\right\}$ is a Cauchy sequence. Using completeess of $\mathcal{F}$ we assume the mapping $\mathcal{Q} : \mathcal{E} \to \mathcal{F}$ by

$$\mathcal{R}_{\mathcal{Q}(x)}(s) = \lim_{t\to\infty}\mathcal{R}_{\frac{f(\sqrt{2}^t x)}{2^t}}(s) \tag{2.25}$$

for all $x \in \mathcal{E}$ and all $s > 0$. Letting $t_0 = 0$ and $t \to \infty$in (2.23), we get

$$T_{r=0}^{\infty}\mathcal{R'}^{D}_{\sqrt{2}^r\ x, \sqrt{2}^r\ x}\left(2^{(r+1)} \cdot m\ s\right) \leq \mathcal{R}_{\mathcal{Q}(x)-f(x)}(s)\,(s)$$

for all $x \in \mathcal{E}$ and all $s > 0$. To prove $\mathcal{Q}$ satisfies the (1.1), replacing (x, y) by $(\sqrt{2}^t\ x, \sqrt{2}^t\ y)$ in (2.2), we obtain

$$\mathcal{R'}^{D}_{\sqrt{2}^t\ x, \sqrt{2}^t\ y}\left(2^t s\right) \leq \mathcal{R}_{\frac{1}{2^t}q\left(\sqrt{2}^t\sqrt{(mx^2+ny^2)}\right) - \frac{m}{2^t}\ q\left(\sqrt{2}^t\ x\right) - \frac{n}{2^t}\ q\left(\sqrt{2}^t\ y\right)}(s) \tag{2.26}$$

for all $x, y \in \mathcal{E}$ and all $s > 0$. Now,

$$T^3\Big\{\mathcal{R}_{\mathcal{Q}\left(\sqrt{(mx^2+ny^2)}\right)-\frac{1}{2^t}q\left(\sqrt{2}^t\sqrt{(mx^2+ny^2)}\right)}\left(\frac{s}{4}\right), \mathcal{R}_{-m\mathcal{Q}(x)+\frac{m}{2^t}\,q\left(\sqrt{2}^t\,x\right)}\left(\frac{s}{4}\right), \mathcal{R}_{-n\mathcal{Q}(y)+\frac{n}{2^t}\,q\left(\sqrt{2}^t\,y\right)}\left(\frac{s}{4}\right), \mathcal{R}_{\frac{1}{2^t}q\left(\sqrt{2}^t\sqrt{(mx^2+ny^2)}\right)-\frac{m}{2^t}\,q\left(\sqrt{2}^t\,x\right)-\frac{n}{2^t}\,q\left(\sqrt{2}^t\,y\right)}\left(\frac{s}{4}\right)\Big\} \leq \mathcal{R}_{\mathcal{Q}\left(\sqrt{(mx^2+ny^2)}\right)-m\mathcal{Q}(x)-n\mathcal{Q}(y)}(s) \tag{2.27}$$

for all $x, y \in \mathcal{E}$ and all $s > 0$. Using (2.25), (2.26) in (2.27), we reach

$$T^3\left\{1, 1, 1, \mathcal{R'}^{D}_{\sqrt{2}^t\,x, \sqrt{2}^t\,y}\left(2^t s\right)\right\} \leq \mathcal{R}_{\mathcal{Q}\left(\sqrt{(mx^2+ny^2)}\right)-m\mathcal{Q}(x)-n\mathcal{Q}(y)}(s) \tag{2.28}$$

for all $x, y \in \mathcal{E}$ and all $s > 0$. Approaching t tends to infinity in (2.28), using (2.1) and $(RNS1)$, which gives $\mathcal{Q}\left(\sqrt{(mx^2+ny^2)}\right) = m\mathcal{Q}(x) + n\mathcal{Q}(y)$ for all $x, y \in \mathcal{E}$ and all $s > 0$. Hence $\mathcal{Q}$ satisfies the radical quadratic functional equation (1.1). The existence $\mathcal{Q}(x)$ is unique. Indeed, if $\mathcal{Q}'(x)$ be another radical quadratic function satisfying (1.1) and (2.5). Hence,

$$\mathcal{R}_{\mathcal{Q}(x)-\mathcal{Q}'(x)}(s) \geq T\Big\{T_{r=0}^{\infty}\mathcal{R'}^{D}_{\sqrt{2}^{r+t}y, \sqrt{2}^{r+t}y}\left(2^{(r+t+1)}s\right), T_{r=0}^{\infty}\mathcal{R'}^{D}_{\sqrt{2}^{r+t}y, \sqrt{2}^{r+t}y}\left(2^{(r+t+1)}s\right)\Big\} \to 1 \text{ as } q \to \infty$$

for all $x \in \mathcal{E}$ and all $s > 0$ which implies $\mathcal{Q}(x)$. Therefore $\mathcal{Q}(x) - \mathcal{Q}'(x)$ is unique. Hence for $p = 1$ the theorem holds. Replacing x by $\dfrac{x}{\sqrt{2}}$ in (2.17), we arrive

$$\mathcal{R'}^{D}_{\frac{x}{\sqrt{2}}, \frac{x}{\sqrt{2}}}(s) \leq \mathcal{R}_{q(x)-2q\left(\frac{x}{\sqrt{2}}\right)}\left(\frac{s}{m}\right) \tag{2.29}$$

for all $x \in \mathcal{E}$ and all $s > 0$. The rest of proof is similar to that of the case $p = 1$. This completes the proof of the theorem. □

Corollary 2.1. *If there exist real numbers* a *and* b *with* $q : \mathcal{E} \to \mathcal{F}$ *be a mapping satisfying*

$$\left.\begin{array}{l} \mathcal{R}'_{a}(s); \\ \mathcal{R}'_{a\left\{|x|^b+|y|^b\right\}}(s); \\ \mathcal{R}'_{a|x|^b|y|^b}(s); \\ \mathcal{R}'_{a\left\{|x|^b|y|^b+\left\{|x|^{2b}+|y|^{2b}\right\}\right\}}(s); \end{array}\right\} \leq \mathcal{R}_{q\left(\sqrt{mx^2+ny^2}\right)-m\,q(x)-n\,q(y)} \tag{2.30}$$

for all $x, y \in \mathcal{E}$, then there exists a unique Radical Quadratic function $\mathcal{Q} : \mathcal{E} \to \mathcal{F}$ such that

(2.31)

$$\left.\begin{array}{ll} \mathcal{R}'_a\left(|2|\cdot m\ s\right), & \\ \mathcal{R}'_{\left(2\sqrt{\frac{m}{n}}^b+\sqrt{2}^b+2\right)\ |x|^b}\left(\dfrac{\sqrt{2}^b\ 8m\ s}{|\sqrt{2}^b-2|}\right), & b\neq 2; \\ \mathcal{R}'_{\left(\sqrt{\frac{m}{n}}^b\right)\ |x|^{2b}}\left(\dfrac{\sqrt{2}^{2b}\ 8m\ s}{|\sqrt{2}^{2b}-2|}\right), & 2b\neq 2; \\ \mathcal{R}'_{\left(2\sqrt{\frac{m}{n}}^{2b}+\sqrt{\frac{m}{n}}^b+\sqrt{2}^{2b}+2\right)\ |x|^{2b}}\left(\dfrac{\sqrt{2}^{2b}\ 8m\ s}{|\sqrt{2}^{2b}-2|}\right), & 2b\neq 2; \end{array}\right\} \leq \mathcal{R}_{q(x)-\mathcal{Q}(x)}\left(s\right)$$

for all $x \in \mathcal{E}$.

Theorem 2.2. *Let $p = \pm 1$ and $\mathcal{R}' : \mathcal{E}^2 \to D^+$ be a function such that*

$$\begin{aligned} &\lim_{t\to\infty} \mathcal{R}'_{\sqrt{(m+n)}^{pt}x,\sqrt{(m+n)}^{pt}y}\left((m+n)^{pt}s\right) = 1 \\ &= T_{r=0}^{\infty}\mathcal{R}'^{D}_{\sqrt{(m+n)}^{pr}\ x,\sqrt{(m+n)}^{pr}\ x}\left((m+n)^{p(r+1)}s\right) \end{aligned} \tag{2.32}$$

for all $x \in \mathcal{E}$ and all $s > 0$. Let $q : \mathcal{E} \to \mathcal{F}$ be a function fulfilling the inequality

$$\mathcal{R}'_{x,y}\left(s\right) \leq \mathcal{R}_{q\left(\sqrt{mx^2+ny^2}\right)-m\ q(x)-n\ q(y)}\left(s\right) \tag{2.33}$$

for all $x, y \in \mathcal{E}$ and all $s > 0$. Then there exists a unique Radical Quadratic function $\mathcal{Q}(x) : \mathcal{E} \to \mathcal{F}$ which satisfies (1.1) and

$$T_{r=0}^{\infty}\mathcal{R}'^{D}_{\sqrt{(m+n)}^{pr}\ x,\sqrt{(m+n)}^{pr}\ x}\left((m+n)^{p(r+1)}s\right) \leq \mathcal{R}_{\mathcal{Q}(x)-q(x)}\left(s\right) \tag{2.34}$$

where $\mathcal{Q}(x)$ is defined by

$$\mathcal{R}_{\mathcal{Q}(x)}\left(s\right) = \lim_{t\to\infty} \mathcal{R}_{\frac{q(\sqrt{(m+n)}^{pt}x)}{(m+n)^{pt}}}\left(s\right) \tag{2.35}$$

for all $x \in \mathcal{E}$ and all $s > 0$, respectively.

Proof. If we change (x, y) by (x, x) in (2.33), we get

$$\mathcal{R}'_{x,x}(s) \leq \mathcal{R}_{q\left(\sqrt{(m+n)}\ x\right)-(m+n)q(x)}(s) \tag{2.36}$$

for all $x \in \mathcal{E}$ and all $s > 0$. The rest of the proof is similar ideas to that of Theorem 2.1. □

Corollary 2.2. *Let $q : \mathcal{E} \to \mathcal{F}$ be a mapping. If there exist real numbers a and b satisfying (2.30) for all $x, y \in \mathcal{E}$, then there exists a unique Radical Quadratic function $\mathcal{Q} : \mathcal{E} \to \mathcal{F}$ such that*

$$\left.\begin{array}{ll} \mathcal{R}'_a\left(\frac{(m+n)\ s}{|(m+n)-1|}\right), & \\ \mathcal{R}'_{2|x|^b}\left(\frac{\sqrt{(m+n)}\ s}{|\sqrt{(m+n)}^b-(m+n)|}\right), & b \neq 2; \\ \mathcal{R}'_{|x|^{2b}}\left(\frac{\sqrt{(m+n)}\ s}{|\sqrt{(m+n)}^{2b}-(m+n)|}\right), & 2b \neq 2; \\ \mathcal{R}'_{3|x|^{2b}}\left(\frac{\sqrt{(m+n)}\ s}{|\sqrt{(m+n)}^{2b}-(m+n)|}\right), & 2b \neq 2; \end{array}\right\} \leq \mathcal{R}_{q(x)-\mathcal{Q}(x)}(s) \tag{2.37}$$

for all $x \in \mathcal{E}$.

3. Random Stability: Fixed Point Method

Using Theorem 1.1, we obtain the generalized Ulam - Hyers stability of (1.1).

Theorem 3.1. *Let $q : \mathcal{E} \to \mathcal{F}$ be a mapping for which there exist a function $\mathcal{R}' : \mathcal{E}^2 \to D^+$ with the condition*

$$\lim_{t\to\infty} \mathcal{R}'_{\gamma_i^t x, \gamma_i^t y}\left(\gamma_i^{2t} s\right) = 1 \tag{3.1}$$

for all $x \in \mathcal{E}$ and all $s > 0$ where

$$\gamma_i = \begin{cases} \sqrt{m+n} & if \quad i = 0, \\ \frac{1}{\sqrt{m+n}} & if \quad i = 1 \end{cases} \tag{3.2}$$

and satisfying the functional inequality

$$\mathcal{R}_{q\left(\sqrt{mx^2+ny^2}\right)-m\ q\ (x)-n\ q\ (y)}(s) \geq \mathcal{R}'_{x\ ,\ y}(s) \tag{3.3}$$

for all $x, y \in \mathcal{E}$ and all $s > 0$. If there exists $L = L(i)$ such that the function $\mathcal{R}'_{x,x}(s) = \mathcal{R}'_{\frac{x}{\sqrt{m+n}}, \frac{x}{\sqrt{m+n}}}(s)$ with the property

$$\mathcal{R}'_{\gamma_i x, \gamma_i x}\left(s\ \gamma_i^2\right) = \mathcal{R}'_{x,x}(Ls), \tag{3.4}$$

for all $x \in \mathcal{E}$ and all $s > 0$. Then there exists unique unique quadratic mapping $\mathcal{Q} : \mathcal{E} \to \mathcal{F}$ satisfying the functional equation (1.1) and

$$\mathcal{R}_{q\ (x)-\mathcal{Q}\ (x)}\left(\frac{L^{1-i}}{1-\ L}\ s\right) \geq \mathcal{R}'_{x,x}(s), \tag{3.5}$$

for all $x \in \mathcal{E}$ and all $s > 0$.

Proof. Take the set $\Lambda = \{h_1/h_1 : \mathcal{E} \to \mathcal{F},\ h\,(0) = 0\}$ and introduce the generalized metric on Λ,

$$d(h_1, h_2) = \inf\{\rho \in (0, \infty) : \mathcal{R}_{h_1(x)-h_2(x)}(s) \geq \mathcal{R}'_{x,x}\Big(\rho\ s\Big), x \in \mathcal{E}, s > 0\}. \tag{3.6}$$

It is easy to see that (3.6) is complete with respect to the defined metric. Define $J : \Lambda \to \Lambda$ by $Jh(x) = \frac{1}{\gamma_i^2} h(\gamma_i x)$, for all $x \in \mathcal{E}$. Now, from (3.6) and $h_1, h_2 \in \Lambda$, we arrive $d(h_1, h_2) \leq \rho d(Jh_1, Jh_2) \leq L\rho$ which gives J is a strictly contractive mapping on Λ with Lipschitz constant L. It follows from (3.6), (2.36) and (3.4) for the case $i = 0$, we reach

$$\mathcal{R}_{Jq(x)-q(x)}\,(s) \geq \mathcal{R}'_{x,x}\left(L^{1-i}\ s\right) \quad (x \in \mathcal{E}, s > 0). \tag{3.7}$$

Again replacing $x = \frac{x}{\sqrt{m+n}}$ in (2.36) and (3.4) for the case $i = 1$, we get

$$\mathcal{R}_{f(x)-Jf(x)}\,(s) \geq \mathcal{R}'_{x,x}\left(L^{1-i}\ s\right), \quad (x \in \mathcal{E}, s > 0). \tag{3.8}$$

From (3.7) and (3.8), we arrive

$$\mathcal{R}_{f(x)-Jf(x)}\,(s) \geq \mathcal{R}'_{x,x}\left(L^{1-i}\ s\right), x \in \mathcal{E}, s > 0. \tag{3.9}$$

Hence property (FPC1) holds. It follows from property (FPC2) that there exists a fixed point $\mathcal{Q}$ of J in Λ such that $\mathcal{R}_{\mathcal{Q}(x)}(s) = \lim_{t\to\infty} \mathcal{R}_{\frac{1}{\gamma_i^{2t}} q(\gamma_i^t x)}\,(s)$ for all $x \in \mathcal{E}$. In order to show that $\mathcal{Q}$ satisfies (1.1), the proof is similar to that of Theorem 2.1. By property (FPC3), $\mathcal{Q}$ is the unique fixed point of J in the set $\Delta = \{\mathcal{Q} \in \Lambda : d(q, \mathcal{Q}) < \infty\}$, such that $\mathcal{R}_{q(x)-\mathcal{Q}(x)}\,(s) \geq \mathcal{R}'_{x,x}\,(\rho s)\,, x \in \mathcal{E}, s > 0$. Finally, by property (FPC4), we obtain $\mathcal{R}_{q(x)-\mathcal{Q}(x)}\,(s) \geq \mathcal{R}'_{q(x)-Jf(x)}\,(s)\,,$ which gives the the proof of the theorem. □

Corollary 3.1. *Let $q : \mathcal{E} \to \mathcal{F}$ be a mapping. If there exist real numbers a and b such that the inequality (2.30) for all $x, y \in \mathcal{E}$, then there exists a unique Radical Quadratic function $\mathcal{Q} : \mathcal{E} \to \mathcal{F}$ such that*

$$\left.\begin{array}{ll} \mathcal{R}'_a\left(\frac{(m+n)\ s}{|(m+n)-1|}\right), & \\ \mathcal{R}'_{2||x||^b}\left(\frac{(m+n)\ s}{|\sqrt{(m+n)}^b-(m+n)|}\right), & b \neq 2; \\ \mathcal{R}'_{||x||^{2b}}\left(\frac{(m+n)\ s}{|(m+n)^b-(m+n)|}\right), & b \neq 1; \\ \mathcal{R}'_{3||x||^{2b}}\left(\frac{(m+n)\ s}{|(m+n)^b-(m+n)|}\right), & b \neq 1; \end{array}\right\} \leq \mathcal{R}_{q(x)-\mathcal{Q}(x)}\,(s) \tag{3.10}$$

for all $x \in \mathcal{E}$.

Proof. Let us take $\mathcal{R}'_{x,y}(s)$ as in (2.30) for all $x, y \in \mathcal{E}$. Now its is easy to verify that $\mathcal{R}'_{\frac{1}{\gamma_i^{2t}} h_d(\gamma_i^t x, \gamma_i^t x)}(s) \to 1$ as $t \to \infty$. Thus, (3.1) holds. By definition and property (3.4), the inequality (3.5) holds for $i = 0$, $L = \gamma_i^2; \gamma_i^{2-b}; \gamma_i^{2-2b}; \gamma_i^{2-2b}$ and for and $i = 1$, $L = \frac{1}{\gamma_i^2}; \frac{1}{\gamma_i^{2-b}}; \frac{1}{\gamma_i^{2-2b}}; \frac{1}{\gamma_i^{2-2b}}$, we arrive our result. $\square$

Theorem 3.2. *Let $q : \mathcal{E} \to \mathcal{F}$ be a mapping for which there exist a function $\mathcal{R}' : \mathcal{E}^2 \to D^+$ with the condition $\lim_{t\to\infty} \mathcal{R}'_{\gamma_i^t x, \gamma_i^t y}(\gamma_i^{2t} s) = 1$ for all $x \in \mathcal{E}$ and all $s > 0$ where $\gamma_i = \begin{cases} \sqrt{2} & if \quad i = 0, \\ \frac{1}{\sqrt{2}} & if \quad i = 1 \end{cases}$ and satisfying the functional inequality $\mathcal{R}_{q\left(\sqrt{mx^2+ny^2}\right) - m\ q(x) - n\ q(y)}(s) \geq \mathcal{R}'_{x,y}(s)$ for all $x, y \in \mathcal{E}$ and all $s > 0$. If there exists $L = L(i)$ such that the function $\mathcal{R}'^{D}_{x,x}(s) = \mathcal{R}'^{D}_{\frac{x}{\sqrt{2}}, \frac{x}{\sqrt{2}}}(s)$ with the property $\mathcal{R}'_{\gamma_i x, \gamma_i x}(s\ \gamma_i^2) = \mathcal{R}'_{x,x}(Ls)$, for all $x \in \mathcal{E}$ and all $s > 0$. Then there exists unique unique quadratic mapping $\mathcal{Q} : \mathcal{E} \to \mathcal{F}$ satisfying the functional equation (1.1) and $\mathcal{R}_{f(x)-\mathcal{Q}(x)}\left(\frac{L^{1-i}}{1-L}\ s\right) \geq \mathcal{R}'_{x,x}(s)$, for all $x \in \mathcal{E}$ and all $s > 0$.*

Corollary 3.2. *Let $q : \mathcal{E} \to \mathcal{F}$ be a mapping. If there exist real numbers a and b such that the inequality (2.30) for all $x, y \in \mathcal{E}$, then there exists a unique Radical Quadratic function $\mathcal{Q} : \mathcal{E} \to \mathcal{F}$ such that the inequality (2.31) holds for all $x \in \mathcal{E}$.*

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Dedicated to all mathematicians working in this field.

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