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PROJECTIVE SYNCHRONIZATION IN HYPERCHAOTIC SYSTEMS USING ADAPTIVE CONTROL METHOD

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ABSTRACT. In this manuscript, a systematic procedure has been designed for investigating the projective synchronization (PS) technique between two identical hyperchaotic systems. Based on Lyapunov stability theory (LST) and adaptive control method (ACM), PS has been achieved. The discussed method determines asymptotic stability of the error dynamics globally along with identification of parameters. To illustrate our results, numerical simulations in MATLAB are performed to visualize and validate the effectiveness and superiority of the discussed technique.

1. INTRODUCTION

Chaos theory is a fascinating and an intriguing field of applied mathematics that deals with the behavioural analysis of nonlinear dynamical systems. This field is applicable in several branches of applied science and engineering, including meteorology, cryptography, physics, computer science, environmental science, biomedical engineering, chemistry and so on. As a result, chaos synchronization and control have acquired a significant attention in numerous research fields. An interesting characteristic of chaotic systems, known as "butterfly Effect", is basically the sensitivity dependency on initial conditions and is firstly introduced by E.N. Lorenz [8] in 1963 while studying weather prediction

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model. Synchronization in chaotic systems was first described by Pecora and Caroll [10] in 1990 using master-slave configuration. In chaos synchronization, the state trajectories of two or more chaotic systems (identical or nonidentical) are made to follow similar dynamics. Currently, chaos synchronization and chaos control of chaotic systems have been an active area of study. Various techniques have been introduced for synchronization and control [1, 3–5, 7, 11] of chaos phenomenon.

A hyperchaotic system is defined as a chaotic system having at least two positive Lyapunov exponents. In the year 1979, Rossler advocated the first classic hyperchaotic system. During the last decades, various typical hyperchaotic systems have been reported, namely, Lorenz system, Nikolov system, Chen system, Liu system, Qi system, Lu systems etc. Interestingly, Hubler in 1989 introduced adaptive control technique to synchronize chaotic systems. In 1999, Mainieri and Rehacek [9] initiated the idea of projective synchronization in chaotic systems. Since then, researchers analyzed numerous control techniques in detail among chaotic/hyperchaotic systems.

Considering the above literature review and discussions, our primal goal here is to investigate projective synchronization (PS) between two identical hyperchaotic systems using adaptive control technique. The paper is organized as: Section 2 consists of some preliminaries to be used within the paper. Section 3 describes the basic features of the considered system. Section 4 investigates the projective synchronization via ACM by designing proper controllers along with a parameter estimation update laws. Section 5 contains the numerical simulations which verify our theoretical results. Further, a comparative analysis is also done. Finally, Section 6 concludes the paper.

2. Preliminaries

Considering the master system and the corresponding slave system as:

$$\dot{u}_m = f(u_m) \tag{2.1}$$

$$\dot{u}_s = g(u_s) + W, \tag{2.2}$$

where $u_m = (u_{m1}, u_{m2}, \ldots, u_{mn})^T$, $u_s = (u_{s1}, u_{s2}, \ldots, u_{sn})^T$ are the state vectors of (2.1) and (2.2) respectively, $f, g : R^n \to R^n$ are two nonlinear continuous vector functions and $W = (W_1, W_2, \ldots, W_3) \in R^n$ is the proper controller to be designed.

Definition 2.1. *The master system* (2.1) *and slave system* (2.2) *achieve projective synchronization (PS) if*

$$\lim_{t\to\infty} \|e(t)\| = \lim_{t\to\infty} \|u_s(t) - Au_m(t)\| = 0 \tag{2.3}$$

for some $A = diag(\beta, \beta, \ldots, \beta)$ *and* $\|\cdot\|$ *represents vector norm.*

Remark 2.1. *For* $\beta = 1$*, complete synchronization among systems* (2.1) *and* (2.2) *is achieved.*

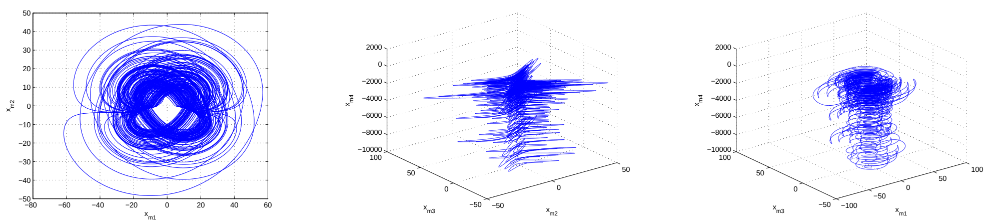
Remark 2.2. *For* $\beta = -1$*, anti-synchronization among systems* (2.1) *and* (2.2) *is attained.*

3. System Description

Introduced by Dong et al. [2], the discussed hyperchaotic system has been described as:

$$\begin{cases} \dot{u}_{m1} = p_1 u_{m1} - q_1 u_{m2} u_{m3} \\ \dot{u}_{m2} = -r_1 u_{m2} + u_{m1} u_{m3} \\ \dot{u}_{m3} = l_1 u_{m1} - a_1 u_{m3} + u_{m1} u_{m2} \\ \dot{u}_{m4} = n_1 u_{m4} + u_{m1} u_{m2}, \end{cases} \tag{3.1}$$

where $(u_{m1}, u_{m2}, u_{m3}, u_{m4})^T \in R^4$ is the state vector and p_1, q_1, r_1, a_1, l_1 and n_1 are positive parameters. For $p_1 = 4.55$, $q_1 = 1.532$, $r_1 = 10.1$, $a_1 = 5.5$, $l_1 = 3.5$ and $n_1 = 0.04$, the given system (3.1) displays hyperchaos. Also, Figure 1(a-c) exhibits the phase diagrams of (3.1). For more details, one may refers [6].



(A) (B) (C)

FIGURE 1. Phase diagrams of hyperchaotic system in (A) $u_{m1} - u_{m2}$ plane, (B) $u_{m2} - u_{m3} - u_{m4}$ space, (C) $u_{m1} - u_{m3} - u_{m4}$ space

4. Illustrative Example

Conveniently, the system (3.1) has been considered as master system and the corresponding slave system is defined as:

$$
\begin{cases}
\dot{u}_{s1} = p_1 u_{s1} - q_1 u_{s2} u_{s3} + W_1 \\
\dot{u}_{s2} = -r_1 u_{s2} + u_{s1} u_{s3} + W_2 \\
\dot{u}_{s3} = l_1 u_{s1} - a_1 u_{s3} + u_{s1} u_{s2} + W_3 \\
\dot{u}_{s4} = n_1 u_{s4} + u_{s1} u_{s2} + W_4,
\end{cases}
\tag{4.1}
$$

where W_1, W_2, W_3 and W_4 are adaptive controllers that are to be designed.

State errors are defined as:

$$
\begin{cases}
E_{11} = u_{s1} - \beta_1 u_{m1} \\
E_{12} = u_{s2} - \beta_1 u_{m2} \\
E_{13} = u_{s3} - \beta_1 u_{m3} \\
E_{14} = u_{s4} - \beta_1 u_{m4}.
\end{cases}
\tag{4.2}
$$

The primary aim here is to design controllers W_i, $(i = 1, 2, 3, 4)$ so that the state errors given in (4.2) must satisfy

$$\lim_{t \to \infty} E_{1i}(t) = 0 \quad \text{for } (i = 1, 2, 3, 4).$$

Then, error dynamics takes the form

$$
\begin{cases}
\dot{E}_{11} = p_1 E_{11} - q_1 (u_{s2} u_{s3} - \beta_1 u_{m2} u_{m3}) + W_1 \\
\dot{E}_{12} = -r_1 E_{12} + u_{s1} u_{s3} - \beta_1 u_{m1} u_{m3} + W_2 \\
\dot{E}_{13} = l_1 E_{11} - a_1 E_{13} + u_{s1} u_{s2} - \beta_1 u_{m1} u_{m2} + W_3 \\
\dot{E}_{14} = n_1 E_{14} + u_{s1} u_{s2} - \beta_1 u_{m1} u_{m2} + W_4,
\end{cases}
\tag{4.3}
$$

Next, we describe the adaptive controllers by the rule:

$$
\begin{cases}
W_1 = -\hat{p}_1 E_{11} + \hat{q}_1 (u_{s2} u_{s3} - \beta_1 u_{m1} u_{m3}) - K_1 E_{11} \\
W_2 = \hat{r}_1 E_{12} - u_{s1} u_{s3} + \beta_1 u_{m1} u_{m3} - K_2 E_{12} \\
W_3 = -\hat{l}_1 E_{11} + \hat{a}_1 E_{13} - (u_{s1} u_{s2} - \beta_1 u_{m1} u_{m2}) - K_3 E_{13} \\
W_4 = -\hat{n}_1 E_{14} - u_{s1} u_{s2} + \beta_1 u_{m1} u_{m2} - K_4 E_{14},
\end{cases}
\tag{4.4}
$$

where K_1, K_2, K_3, K_4 are positive gain constants.

By putting the expressions of controllers (4.4) in error dynamics (4.3), we obtain

$$\begin{cases} \dot{E}_{11} = (p_1 - \hat{p}_1)E_{11} - (q_1 - \hat{q}_1)(u_{s2}u_{s3} - \beta_1 u_{m2}u_{m3}) - K_1 E_{11} \\ \dot{E}_{12} = -(r_1 - \hat{r}_1)E_{12} - K_2 E_{12} \\ \dot{E}_{13} = (l_1 - \hat{l}_1)E_{11} - (a_1 - \hat{a}_1)E_{13} - K_3 E_{13} \\ \dot{E}_{14} = (n_1 - \hat{n}_1)E_{14} - K_4 E_{14}, \end{cases} \tag{4.5}$$

where $\hat{p}_1$, $\hat{q}_1$, $\hat{r}_1$, $\hat{a}_1$, $\hat{n}_1$, $\hat{l}_1$ are estimation values for unknown parameter p_1, q_1, r_1, a_1, n_1, l_1 respectively.

Parameter estimation error is defined as:

$$\tilde{p}_1 = p_1 - \hat{p}_1, \tilde{q}_1 = q_1 - \hat{q}_1, \tilde{r}_1 = r_1 - \hat{r}_1, \tilde{a}_1 = a_1 - \hat{a}_1, \tilde{n}_1 = n_1 - \hat{n}_1, \tilde{l}_1 = l_1 - \hat{l}_1 \tag{4.6}$$

Using (4.6), the error dynamics (4.5) turns into

$$\begin{cases} \dot{E}_{11} = \tilde{p}_1 E_{11} - \tilde{q}_1(u_{s2}u_{s3} - \beta_1 u_{m1}u_{m3}) - K_1 E_{11} \\ \dot{E}_{12} = -\tilde{r}_1 E_{12} - K_2 E_{12} \\ \dot{E}_{13} = \tilde{l}_1 E_{11} - \tilde{a}_1 E_{13} - K_3 E_{13} \\ \dot{E}_{14} = \tilde{n}_1 E_{14} - K_4 E_{14}. \end{cases} \tag{4.7}$$

The derivative of parameter estimation error (4.6) with respect to time is given by

$$\dot{\tilde{p}}_1 = -\dot{\hat{p}}_1, \dot{\tilde{q}}_1 = -\dot{\hat{q}}_1, \dot{\tilde{r}}_1 = -\dot{\hat{r}}_1, \dot{\tilde{a}}_1 = -\dot{\hat{a}}_1, \dot{\tilde{n}}_1 = -\dot{\hat{n}}_1, \dot{\tilde{l}}_1 = -\dot{\hat{l}}_1 \tag{4.8}$$

Define the Lyapunov function as:

$$V = \frac{1}{2}[E_{11}^2 + E_{12}^2 + E_{13}^2 + E_{14}^2 + \tilde{p}_1^2 + \tilde{q}_1^2 + \tilde{r}_1^2 + \tilde{a}_1^2 + \tilde{n}_1^2 + \tilde{l}_1^2] \tag{4.9}$$

which show that V is positive definite.

Derivative of Lyapunov function V can be written as:

$$\begin{aligned} \dot{V} = {} & E_{11}\dot{E}_{11} + E_{12}\dot{E}_{12} + E_{13}\dot{E}_{13} + E_{14}\dot{E}_{14} - \tilde{p}_1\dot{\hat{p}}_1 \\ & - \tilde{q}_1\dot{\hat{q}}_1 - \tilde{r}_1\dot{\hat{r}}_1 - \tilde{a}_1\dot{\hat{a}}_1 - \tilde{n}_1\dot{\hat{n}}_1 - \tilde{l}_1\dot{\hat{l}}_1. \end{aligned} \tag{4.10}$$

In view of (4.10), we prescribe parameter estimates laws as:

$$\text{(4.11)} \qquad \begin{cases} \dot{\hat{p}}_1 = (u_{s1} - \beta_1 u_{m1})E_{11} + K_5(p_1 - \hat{p}_1) \\ \dot{\hat{q}}_1 = -(u_{s2}u_{s3} - \beta_1 u_{m2}u_{m3})E_{11} + K_6(q_1 - \hat{q}_1) \\ \dot{\hat{r}}_1 = -(u_{s2} - \beta_1 u_{m2})E_{12} + K_7(r_1 - \hat{r}_1) \\ \dot{\hat{a}}_1 = -(u_{s3} - \beta_1 u_{m3})E_{13} + K_8(a_1 - \hat{a}_1) \\ \dot{\hat{n}}_1 = (u_{s4} - \beta_1 u_{m4})E_{14} + K_9(n_1 - \hat{n}_1) \\ \dot{\hat{l}}_1 = (u_{s1} - \beta_1 u_{m1})E_{13} + K_{10}(l_1 - \hat{l}_1), \end{cases}$$

where K_5, K_6, K_7, K_8, K_9 and K_{10} are positive gain constants.

Theorem 4.1. *The hyperchaotic systems* (3.1)-(4.1) *are asymptotically projective synchronized for all initial states* $(u_{m1}(0), u_{m2}(0), u_{m3}(0), u_{m4}(0)) \in R^4$ *by the adaptive controller* (4.4) *and the parameter updating law* (4.11).

Proof. The Lyapunov function V as defined in (4.9) is a positive definite function. By solving equations (4.7), (4.10) and (4.11), one finds that

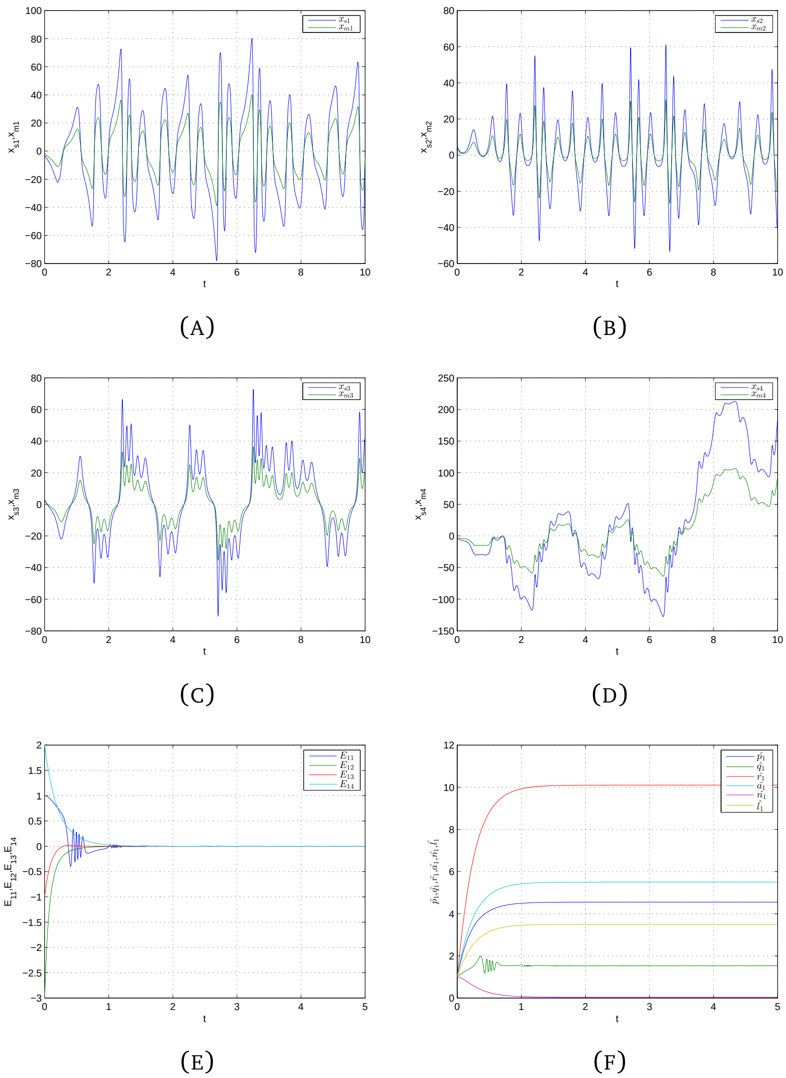
$$\dot{V} = -K_1E_{11}^2 - K_2E_{12}^2 - K_3E_{13}^2 - K_4E_{14}^2 - K_5\tilde{p}_1^2 - K_6\tilde{q}_1^2 - K_7\tilde{r}_1^2 \\ -K_8\tilde{a}_1^2 - K_9\tilde{n}_1^2 - K_{10}\tilde{l}_1^2 < 0$$

ensuring that $\dot{V}$ is negative definite.

Now, by using Lyapunov stability theory, we find that the projective synchronization error $e(t) \to 0$ exponentially as $t \to \infty$ for each initial conditions $e(0) \in R^4$. The proof is now complete. □

5. Numerical Simulation

This section performs some simulation results to illustrate the effectiveness and feasibility of the proposed PS scheme via ACM. The initial states of the systems (3.1) and (4.1) are $(-2, 4, 2, -3)$ and $(-3, 5, 3, -4)$ respectively. When the scaling matrix A is chosen as $\beta_1 = 2$. In this case, we achieved complete synchronization in master (3.1) and slave (4.1) systems. Control gains are taken as $K_i = 4$, $i = 1, 2, \ldots, 10$. The Figure 2(a-d) display trajectories systems (3.1) and (4.1) and Figure 2(e) shows the synchronization error $(E_{11}, E_{12}, E_{13}, E_{14}) = (1, -3, -1, 2)$ tending to zero for t tends to infinity. Further, Figure 2(f) displays the estimated values $(\hat{p}_1, \hat{q}_1, \hat{r}_1, \hat{a}_1, \hat{n}_1, \hat{l}_1)$ of unknown parameters converging to their real values asymptotically with time. Thus, the proposed PS scheme among master and slave system has been attained computationally. The figure (a-e) shows that projective anti-synchronization in systems (3.1) and (4.1) is achieved numerically if we choose the scaling matrix A with $\beta_1 = -3$.



(A) (B)

(C) (D)

(E) (F)

FIGURE 2. Projective complete synchronization of hyperchaotic system (A) between $u_{m1}(t) - u_{s1}(t)$, (B) between $u_{m2}(t) - u_{s2}(t)$, (C) between $u_{m3}(t) - u_{s3}(t)$, (D) between $u_{m4}(t) - u_{s4}(t)$, (E) synchronization error, (F) Parameter estimation

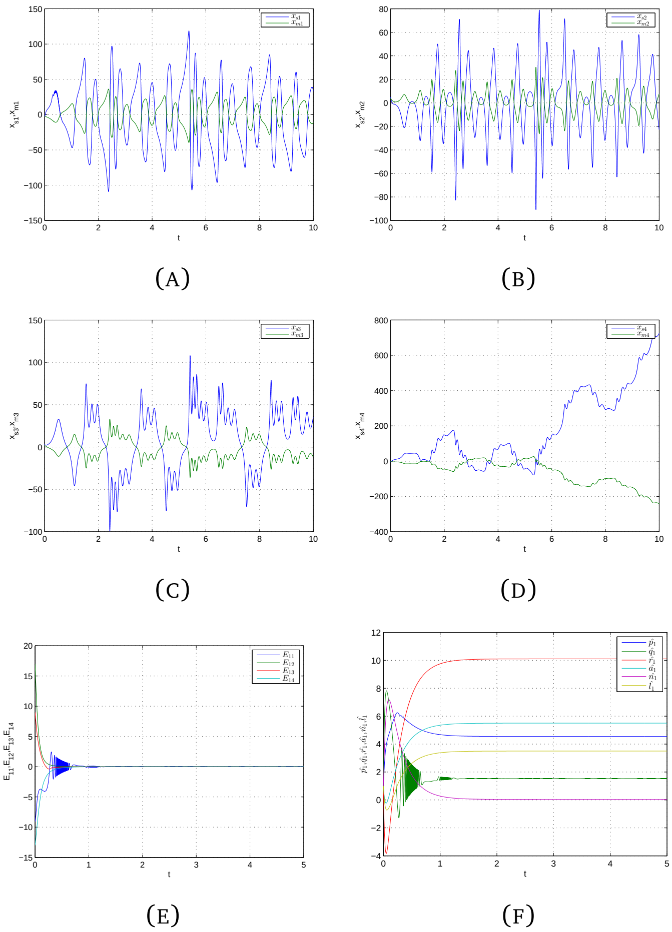


FIGURE 3. Projective anti-synchronization of hyperchaotic system (A) between $u_{m1}(t) - u_{s1}(t)$, (B) between $u_{m2}(t) - u_{s2}(t)$, (C) between $u_{m3}(t) - u_{s3}(t)$, (D) between $u_{m4}(t) - u_{s4}(t)$, (E) synchronization error, (F) Parameter estimation

6. CONCLUSION

In this manuscript, we have investigated the projective synchronization among identical hyperchaotic systems using adaptive control method. By defining proper controllers based on Lyapunov stability theory, the discussed PS scheme has been achieved. The effectiveness of the theoretical results are verified through simulations conducted in MATLAB. Remarkably, the theoretic study and numerical outcomes both are in excellent agreement. In addition, the considered PS technique is very effective since it has various applications in encryption and secure communication.

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