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DOMATIC NUMBER OF AN UNDIRECTED GRAPH $G_{m,n}$

SYEDA ASMA KAUSER[1] AND M. SIVA PARVATHI[2]

ABSTRACT. An undirected graph $G_{m,n}$ is defined as a graph whose vertex set $V = I_n = \{1, 2, 3, \ldots, n\}$, where u, v $\in$ V are adjacent if and only if u$\neq$v and $u + v$ is not divisible by m where m belongs to Natural numbers greater than 1. Let $G(V, E)$ be a graph, a set S $\subset$V , is said to be a dominating set of G if for every vertex a$\in$V is an element of S or adjacent to an element of S. The domatic number $d(G)$ of a graph G is the maximum positive integer k such that V can be partitioned into k pair wise disjoint dominating sets $D_1, D_2, \ldots, D_k$. A partition of V into pair wise disjoint dominating sets is called a domatic partition. We find some results on dominating parameters of an undirected graph $G_{m,n}$ .After studying several properties of this graph we derived some bounds for domatic number for different values of m, n.

1. INTRODUCTION

In this paper we discuss the domatic number. The concept of domination in graph theory originated from a chess board problem. In 1850,Chess players were interested in minimizing the number of queens such that every square on the chess board either contains a queen or is attacked by a queen. On a chess board a queen can move either vertically or horizontally or diagonally and other moves of queen are invalid. C. F. deJaenisch [6] in the year 1862 described this problem mathematically.

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[1]*corresponding author*

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Apart from chess, Domination arises in facility location problems, where the number of facilities (e.g.,hospitals, fire stations) is fixed and one attempts to minimize the distance that a person needs to travel to get to the closest facility. A similar problem occurs when the maximum distance to a facility is fixed and one attempts to minimize the number of facilities necessary so that everyone is serviced. Concepts from domination also appear in problems involving finding sets of representatives, in monitoring communication or electrical networks, and in land surveying (e.g., minimizing the number of places a surveyor must stand in order to take height measurements for an entire region). Part of what motivates so much research into domination is the multitude of varieties of domination. In [4] [5], the authors cite more than 75 variations of domination.

There can be conditions on the dominating set D (D is connected, independent, a clique, a path, $G[D]$ contains a perfect matching, etc.) or conditions on $V - D$ (each vertex in $V - D$ is dominated exactly once, or at least k times, or the number of vertices that is dominated multiply is minimized, etc.).

The graphs considered in this paper are simple and without isolated vertices. A vertex $v \in V$ is called a pendant vertex, if $deg(v) = 1$ and is an isolated vertex if $deg(v) = 0$.A vertex which is adjacent to a pendant vertex is called a support vertex.

Let $G(V, E)$ be a graph, a set $S \subset V$, is said to be a dominating set of G if for every vertex $a \in V$ is an element of S or adjacent to an element of S. The domination number $\gamma(G)$ of the graph G is the minimum cardinality of a dominating set in G [1].

The domatic number $d(G)$ of a graph G is the maximum positive integer k such that V can be partitioned into k pair wise disjoint dominating sets D_1, $D_2, \ldots, D_k$. A partition of V into pair wise disjoint dominating sets is called a domatic partition. The concept of a domatic number was introduced in [3].Undirected graph and its properties were given in [7].

2. DOMATIC NUMBER OF AN UNDIRECTED GRAPH $G_{m,n}$

The concept of domatic number was introduced by Cockayne and Hedetniemi in [3]. In [3, 8-10] lower bounds, upper bounds and many propositions for domatic numbers were studied extensively.

Theorem 2.1. *[3] For any graph $G_{m,n}$,$d(G) \leq \delta(G) + 1$, where $\delta(G)$ is the minimum degree of a vertex of G.*

Theorem 2.2. *Any graph $G_{m,n}$ is domatically full if $m \geq 2n$.*

Proof. From [3] any graph is domatically full if it is a complete graph. Also from [2] theorem [2.6] $G_{m,n}$ is complete. Hence the theorem. □

Theorem 2.3. *If $G_{m,n}$ is an undirected graph and $n < m < 2n$ where m is even and n is odd with condition that $n > 3$ then $d(G_{m,n}) = m/2$.*

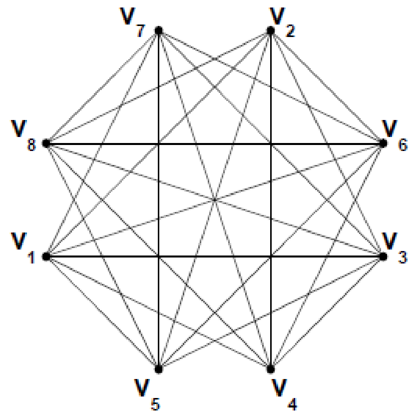
Proof. Let $n < m < 2n$. Let m be even and n be odd with vertex set V_i, j=$\{1, 2, 3, \ldots, n\}$.
Case 1: Let $n = m - 1$.
By theorem [2.1] [2] the vertex $v_i = m/2$ is adjacent to all other vertices $V_{i,j}$=$\{1, 2, 3, \ldots, n\}$ because $m \nmid (m/2+j)$. So D_i=$\{v_i\}$ is a dominating set. From remaining vertices in V we have degree of each vertex equals to $n - 2$. Now we take D_r=$\{(v_i, v_j) : i + j = m\}$ where $r = 2, 3, \ldots.$ and $deg(v_i)$=$n - 2$=$deg(v_j)$.It implies that every vertex in $V - D_r$ is adjacent to some vertex in D_r . Hence D_r is a dominating set with $\gamma(G_{m,n}) = 2$. Hence we get $d(G_{m,n}) = m/2$.
Case 2: Let n$\neq m - 1$.
By theorem 2.1 [2] the vertex $v_i = m/2$ is adjacent to all other vertices v_j,$j = 1, 2, \ldots, n$ because $m \nmid (m/2 + j)$. So D_1=$\{V_i\}$ is a dominating set.Also the vertices v_j= 1,2 are adjacent to all other vertices for any $G_{m,n}$.Now we take D_r=$\{v_j : j = 1, 2\}$ where $r = 2, 3$. Clearly D_r is a dominating set.Also we take D_s=$\{(v_p, v_q) : p + q = m\}$, where $s = 4, 5, \ldots.$ and $deg(v_p)$=$n - 2$=$deg(v_q)$. It implies that every vertex in V-D_s is adjacent to some vertex in D_s .Hence D_s is a dominating set with $\gamma(G_{m,n}) = 2$. In general we get $d(G_{m,n}) = m/2$.Hence from case 1 and case 2 $d(G_{m,n}) = m/2$.

□

Theorem 2.4. *If $G_{m,n}$ is an undirected graph and $n < m < 2n$ where m is odd and n is even with condition that $n > 3$ then $d(G_{m,n}) = (m - 1)/2$.*

Proof. Let $n < m < 2n$. Let m be odd and n be even with vertex set V=$\{1, 2, 3, \ldots, n\}$.
Case 1: Let $n = m - 1$.
Clearly, in this type of graphs each vertex has degree equals to $n - 2$. Let us

consider a set D_r=$\{(v_i, v_j); (i+j) = m\}$. Moreover D_r is a dominating set since every vertex in V-D_r is adjacent to some vertex in D_r with $\gamma(G_{m,n}) = 2$. So we get $(m-1)/2$ distinct disjoint dominating sets. Therefore $d(G_{m,n}) = (m-1)/2$.
Case 2: Let n$\neq m-1$.
By Theorem [2.1] [2] vertex v_i=1 is adjacent to all other vertices v_j where $j = 2, 3, \ldots, n$ .So $D_1 = \{v_1\}$ be a dominating set.Now from V-D_1,$\exists$ a vertex v_i=2 such that it is adjacent to all other vertices in $G_{m,n}$.Let D_2=$\{V_2\}$ and every vertex in V-D_2 is also adjacent to v_2. Hence D_2 is a dominating set .Now from remaining vertices we chooseD_r=$\{(v_p, v_q) : p+q = m\}$.Clearly D_r is a dominating set with $\gamma(G_{m,n}) = 2$.So we get $(m-1)/2$ distinct disjoint dominating sets. Therefore $d(G_{m,n}) = (m-1)/2$. Hence from case 1 and case 2 $d(G_{m,n}) = (m-1)/2$. □



$G_{9,8}$

Theorem 2.5. *If $G_{m,n}$ is a complete k-partite graph with $n = m-1$,m is odd and $k = n/2$ then $d(G_{m,n}) = k$.*

Proof. From [2]Theorem[2.3] $G_{m,n}$ is a complete k-partite graph. Let the vertex set of $G_{m,n}$ is V=$\{v_1, v_2, v_3, \ldots, v_n\}$. Let D be a dominating set such that D=$\{(v_i, v_j) : (i+j)\ is\ divisible\ by\ m\}$. Also by definition of $G_{m,n}$, v_i,v_j are not adjacent to each other. Hence we have k number of partitions of the vertex set V. Hence $d(G_{m,n}) = k$.

□

Theorem 2.6. *For an undirected graph $G_{m,n}$ if $n = 3$ and m≥6 then $d(G_{m,n}) = 3$.*

Proof. Let $n = 3$ and m≥6 ,then from [2]Theorem[2.2] $G_{m,n} \cong K_3$. Then $V = \{v_1, v_2, v_3\}$ .Also any distinct $v_i, v_j \in$ V are adjacent to each other since $v_i + v_j < 6$ for all m≥6.Moreover $G_{m,n} \cong K_3$.each vertex v_i dominates the other two vertices.So there are 3 dominating sets with $\gamma(G_{m,n}) = 1$.Hence $d(G_{m,n}) = 3$. □

Theorem 2.7. *For an undirected graph $G_{m,n}$ if if $m = 2$ then $d(G_{m,n}) = 2$.*

Proof. Let $V = \{v_1, v_2, v_3, \ldots, v_n\}$. Let D_i be the dominating set of $G_{m,n}$.
Case(i):
Suppose n is even. Let $D_1 = \{v_1, v_3.v_5, \ldots, v_{(n-1)}\}$ and $D_2 = \{v_2, v_4, v_6, \ldots, v_n\}$ such that $V = D_1 \cup D_2$.Now let a$\in D_1$ and the vertex a dominates all the vertices of D_2 .Again for b$\in D_2$ the vertex b dominates all the vertices of D_1.Also no two vertices of D_i are adjacent for $i = 1, 2$. Hence $d(G_{m,n}) = 2$.
Case(ii):
Suppose n is odd Let $V = \{v_1, v_2, v_3, \ldots, v_n\}$.Let D_i be the dominating set of $G_{m,n}$.Let $D_1 = \{v_1, v_3.v_5, \ldots, v_n\}$ and $D_2 = \{v_2, v_4, v_6, \ldots, v_{(n-1)}\}$ such that $V = D_1 \cup D_2$.The order of the set D_1 is $(n + 1)/2$ and the order of the set $D_2 = (n - 1)/2$.Let a$\in D_1$ and b$\in D_2$.The vertex a$\in D_1$ is adjacent to all vertices of b$\in D_2$ and vice versa. Moreover no two vertices of D_i are adjacent for $i = 1, 2$. Hence $d(G_{m,n}) = 2$. □

Theorem 2.8. *For an undirected graph $G_{m,n}$, if $m = 3$ then $d(G_{m,n}) = 2$.*

Proof. Consider a graph $G_{3,n}$ with the vertex set $V = \{v_1, v_2, v_3, \ldots, v_n\}$. Let D_i be the dominating set of $G_{3,n}$.Let $D_1 = \{v_i \colon i = m\}$ and $D_2 = \{v_j \colon \forall j \neq m\}$ and $V = D_1 \cup D_2$. Since the vertex $v_i \in D_1$ is a vertex of $\Delta(G_{m,n})$ and it dominates all other vertices of D_2. Hence $d(G_{m,n}) = 2$. □

Theorem 2.9. *For an undirected graph $G_{m,n}$ if m≥2n and n≥3 then $d(G_{m,n}) = n$.*

Proof. From above property 2.2.5 [7] $G_{m,n}$ is a complete graph. Also we know that each vertex v_i in a complete graph dominates all other vertices v_j where i≠j. Hence $d(G_{m,n}) = n$. □

3. CONCLUSION

In this paper we have computed certain interesting variations of domination parameters such as domatic number of an undirected graph $G_{m,n}$. Research in this field is growing rapidly and achieving tangible results.

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Email address: [1]asmaxlnc@gmail.com

[1,2]DEPARTMENT OF APPLIED MATHEMATICS,
SRI PADMAVATI MAHILA VISVAVIDYALAYAM,
TIRUPATI, ANDHRA PRADESH, INDIA.

Email address: [2]parvathimani2008@gmail.com