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MODULAR COLORINGS OF CORONA PRODUCT OF C_m WITH C_n

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ABSTRACT. For $\ell \geq 2$, a modular ℓ-coloring of a graph $\mathcal{G}$ without isolated vertices is a coloring of the vertices of $\mathcal{G}$ with the elements in $\mathbb{Z}_\ell$ having the property that for every two adjacent vertices of $\mathcal{G}$, the sums of the colors of their neighbors are different in $\mathbb{Z}_\ell$. The minimum ℓ for which $\mathcal{G}$ has a modular ℓ-coloring is the modular chromatic number of $\mathcal{G}$. In this paper, we determine the modular chromatic number of corona product of cycles.

1. INTRODUCTION

For graph-theoretical terminology and notation, we in general follow [1]. For a vertex v of a graph $\mathcal{G}$, let $N_{\mathcal{G}}(v)$, the *neighborhood of* v, denote the set of vertices adjacent to v in $\mathcal{G}$. For a graph $\mathcal{G}$ without isolated vertices, let $c : V(\mathcal{G}) \to \mathbb{Z}_\ell$, $\ell \geq 2$, be a vertex coloring of $\mathcal{G}$ where adjacent vertices may be colored the same. The *color sum* $\mathcal{S}(v) = \sum\limits_{u \in N_{\mathcal{G}}(v)} c(u)$ of a vertex v of $\mathcal{G}$ is the sum of the colors of the vertices in $N_{\mathcal{G}}(v)$. The coloring c is called a *modular ℓ-coloring* of $\mathcal{G}$ if $\mathcal{S}(x) \neq \mathcal{S}(y)$ in $\mathbb{Z}_\ell$ for all pairs x, y of adjacent vertices in $\mathcal{G}$. The *modular chromatic number* $Mc(\mathcal{G})$ of $\mathcal{G}$ is the minimum ℓ for which $\mathcal{G}$ has a modular ℓ-coloring. This concept was introduced by Zhang et. al. [2].

Okamoto, Salehi and Zhang proved, in [2], they proved that: every nontrivial connected graph $\mathcal{G}$ has a modular ℓ-coloring for some integer $\ell \geq 2$ and $Mc(\mathcal{G}) \geq \chi(\mathcal{G})$, where $\chi(\mathcal{G})$ denotes the chromatic number of $\mathcal{G}$; for the cycle C_n of length n, $Mc(C_n)$ is 2 if $n \equiv 0 \ mod\ 4$ and it is 3 otherwise; every nontrivial

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tree has modular chromatic number 2 or 3; for the complete multipartite graph $\mathcal{G}$, $Mc(\mathcal{G}) = \chi(\mathcal{G})$; for the cartesian product $\mathcal{G} = K_r \Box K_2$, $Mc(\mathcal{G})$ is r if $r \equiv 2 \ mod\ 4$ and it is $r+1$ otherwise; for the wheel $W_n = C_n \vee K_1, n \geq 3$, $Mc(W_n) = \chi(W_n)$, where $\vee$ denotes the join of two graphs; for $n \geq 3$, $Mc(C_n \vee K_2^c) = \chi(C_n \vee K_2^c)$, where $\mathcal{G}^c$ denotes the complement of $\mathcal{G}$; and for $n \geq 2$, $Mc(P_n \vee K_2) = \chi(P_n \vee K_2)$, where P_n denotes the path of length $n-1$; and in [3] proved that: for $m, n \geq 2$, $Mc(P_m \Box P_n) = 2$.

Paramaguru and Sampthkumar proved, in [5], that: $Mc(C_3 \Box P_2) = 4$; except some special cases, for $m \geq 3$ and $n \geq 2$, $Mc(C_m \Box P_n) = \chi(C_m \Box P_n)$; if $m \equiv 2 \ mod\ 4$ and $n \equiv 1 \ mod\ 4$, then $Mc(C_m \Box P_n) \leq 3$; if $n \equiv 1 \ mod\ 4$, then $Mc(C_6 \Box P_n) = 3$. In [6], they proved that: if $m \geq 4$ and $n \geq 4$ are even integers and at least one of m, n is congruent to $0 \ mod\ 4$, then $Mc(C_m \Box C_n) = \chi(C_m \Box C_n)$; if $n \geq 3$ is an integer, then $Mc(C_3 \Box C_n) = \chi(C_3 \Box C_n)$; if at least one of m, n is congruent to $1 \ mod\ 2$, except some special cases, $m \geq 4$, $n \geq 4$, then $Mc(C_m \Box C_n) = \chi(C_m \Box C_n)$; if $n \equiv 2 \ mod\ 4$, and $n \geq 6$, then $Mc(C_6 \Box C_n) = 3$, where $\Box$ denotes the Cartesian product of two graphs.

Nicholas and Sanma discussed in [4], that: the modular chromatic number of Fan, Helm graph, Friendship graph and gear graph.

The *corona* of two graphs $\mathcal{G}$ and $\mathcal{H}$ is the graph $\mathcal{G} \circ \mathcal{H}$ formed from one copy of $\mathcal{G}$ and $|V(\mathcal{G})|$ copies of $\mathcal{H}$, where the ith vertex of $\mathcal{G}$ is adjacent to every vertex in the ith copy of $\mathcal{H}$. Such type of graph products was introduced by Frucht and Harary in 1970.

2. Corona of C_m with C_n

Define $V(C_m) = \{u_1, u_2, u_3, \ldots, u_m\}$; $V(C_n) = \{v_1, v_2, v_3, \ldots, v_n\}$; $E(C_m) = \{u_1u_2, u_2u_3, u_3u_4, \ldots, u_{m-1}u_m, u_mu_1\}$; $E(C_n) = \{v_1v_2, v_2v_3, v_3v_4, \ldots, v_{n-1}v_n, v_nv_1\}$; $V(C_m \circ C_n) = V(C_m) \bigcup \{v_j^i : i \in \{1,2,3,\ldots,m\}$ and $j \in \{1,2,3,\ldots,n\}\}$; $E(C_m \circ C_n) = E(C_m) \cup \{v_j^i v_{j+1}^i : i \in \{1,2,3,\ldots,m\}$ and $j \in \{1,2,3,\ldots,n-1\}\} \cup \{u_i v_j^i : i \in \{1,2,3,\ldots,m\}$ and $j \in \{1,2,3,\ldots,n\}\} \cup \{v_n^i v_1^i : i \in \{1,2,3,\ldots,m\}\}$.

Theorem 2.1. *For m even and n even, $m \geq 4$, $n \geq 4$, $Mc(C_m \circ C_n) = 3$.*

Proof. Let $c : V(C_m \circ C_n) \to \mathbb{Z}_3$.
Case 1. $n \equiv 4 \ mod\ 6$.
Define c as follows: $c(u_i) = 0$ if i is even; $c(u_i) = 1$ if i is odd; $c(v_j^i) = 0$ if

$i \in \{1,2,3,\ldots,m\}$, j is even; $c(v_j^i) = 1$ if $i \in \{1,2,3,\ldots,m\}$, j is odd; then $\mathcal{S}(u_i) = 1$ if i is even; $\mathcal{S}(u_i) = 2$ if i is odd; $\mathcal{S}(v_j^i) = 1$ if i,j odd; $\mathcal{S}(v_j^i) = 2$ if i,j even; $\mathcal{S}(v_j^i) = 0$ if i is odd, j is even; $\mathcal{S}(v_j^i) = 0$ if i is even, j is odd.

Case 2. $n \equiv 2 \ mod\ 6$.

Define c as follows: $c(u_i) = 0$ if $i \in \{1,2,3,\ldots,m\}$; $c(v_j^i) = 0$ if $i \in \{1,2,3,\ldots,m\}$, j is even; $c(v_j^i) = 1$ if i,j odd; $c(v_j^i) = 2$ if i is even, j is odd; then $\mathcal{S}(u_i) = 1$ if i is odd; $\mathcal{S}(u_i) = 2$ if i is even; $\mathcal{S}(v_j^i) = 1$ if i,j even; $\mathcal{S}(v_j^i) = 2$ if i is odd, j is even; $\mathcal{S}(v_j^i) = 0$ if $i \in \{1,2,3,\ldots,m\}$, j is odd.

Case 3. $n \equiv 0 \ mod\ 6$.

Define c as follows:

$c(u_i) = 0$ if i is odd; $c(u_i) = 2$ if i is even; $c(v_j^i) = 0$ if $i \in \{1,2,3,\ldots,m\}$, j is even; $c(v_j^i) = 1$ if $i \in \{1,2,3,\ldots,m\}$, j is odd; then $\mathcal{S}(u_i) = 1$ if i is odd; $\mathcal{S}(u_i) = 0$ if i is even; $\mathcal{S}(v_j^i) = 1$ if i,j even; $\mathcal{S}(v_j^i) = 2$ if i is odd, j is even; $\mathcal{S}(v_j^i) = 2$ if i is even, j is odd; $\mathcal{S}(v_j^i) = 0$ if i,j odd. Clearly, $\chi(C_m \circ C_n) = 3$. Hence, $Mc(C_m \circ C_n) = 3$. This completes the proof. $\square$

Theorem 2.2. *For m even and n odd, $m \geq 4$, $n \geq 3$, $Mc(C_m \circ C_n) = 4$.*

Proof. Let $c : V(C_m \circ C_n) \to \mathbb{Z}_4$.

Case 1. $n \equiv 1 \ mod\ 8$.

Define c as follows: $c(u_i) = 0$ if i is even; $c(u_i) = 1$ if i is odd; $c(v_j^i) = 0$ if $i \in \{1,2,3,\ldots,m\}$, $j \equiv 0,2,3 \ mod\ 4$; $c(v_n^i) = c(v_{n-4}^i) = 1$ if i is odd; $c(v_n^i) = 1$ if i is even; $c(v_j^i) = 2$ if i is odd, $j \equiv 1 \ mod\ 4$; $j \notin \{n, n-4\}$; $c(v_j^i) = 2$ if i is even, $j \equiv 1 \ mod\ 4$, $j \neq n$; then $\mathcal{S}(u_i) = 0$ if i is odd; $\mathcal{S}(u_i) = 3$ if i is even; $\mathcal{S}(v_j^i) = 0$ if i is even, $j \in \{3,5,7,\ldots,n-2\}$; $\mathcal{S}(v_j^i) = 1$ if i is odd, $j \in \{3,5,7,\ldots,n-2\}$; $\mathcal{S}(v_j^i) = 1$ if i is even, $j \in \{1, n-1\}$; $\mathcal{S}(v_j^i) = 2$ if i is odd, $j \in \{1, n-1, n-3, n-5\}$; $\mathcal{S}(v_j^i) = 2$ if i is even, $j \in \{2,4,6,\ldots,n-3,n\}$; $\mathcal{S}(v_j^i) = 3$ if i is odd, $j \in \{2,4,6,\ldots,n-7,n\}$.

Case 2. $n \equiv 3 \ mod\ 8$ and $n \neq 3$.

Define c as follows: $c(u_i) = 0$ if i is even; $c(u_i) = 1$ if i is odd; $c(v_j^i) = 0$ if $i \in \{1,2,3,\ldots,m\}$, $j \equiv 0,2,3 \ mod\ 4$; $c(v_{n-2}^i) = c(v_{n-6}^i) = 1$ if i is odd; $c(v_{n-2}^i) = 1$ if i is even; $c(v_j^i) = 2$ if i is odd, $j \equiv 1 \ mod\ 4$; $j \notin \{n-2, n-6\}$; $c(v_j^i) = 2$ if i is even, $j \equiv 1 \ mod\ 4$, $j \neq n-2$; then $\mathcal{S}(u_i) = 0$ if i is odd; $\mathcal{S}(u_i) = 3$ if i is even; $\mathcal{S}(v_j^i) = 0$ if i is even, $j \in \{1,3,5,\ldots,n-2\}$; $\mathcal{S}(v_j^i) = 1$ if i is odd, $j \in \{1,3,5,\ldots,n-2\}$; $\mathcal{S}(v_j^i) = 1$ if i is even, $j \in \{n-1, n-3\}$; $\mathcal{S}(v_j^i) = 2$ if i is odd, $j \in \{n-1, n-3, n-5, n-7\}$; $\mathcal{S}(v_j^i) = 2$ if i is even, $j \in \{2,4,6,\ldots,n-5,n\}$;

$\mathcal{S}(v_j^i) = 3$ if i is odd, $j \in \{2, 4, 6, \ldots, n-9, n\}$.

Case 3. $n \equiv 5 \; mod \; 8$.

Define c as follows: $c(u_i) = 0$ if $i \in \{1, 2, 3, \ldots, m\}$; $c(v_j^i) = 0$ if $i \in \{1, 2, 3, \ldots, m\}$, $j \equiv 0, 2, 3 \; mod \; 4$; $c(v_n^i) = 1$ if i is even; $c(v_n^i) = 3$ if i is odd; $c(v_j^i) = 2$ if $i \in \{1, 2, 3, \ldots, m\}$, $j \equiv 1 \; mod \; 4$, $j \neq n$; then $\mathcal{S}(u_i) = 1$ if i is odd; $\mathcal{S}(u_i) = 3$ if i is even; $\mathcal{S}(v_j^i) = 0$ if $i \in \{1, 2, 3, \ldots, m\}$, $j \in \{3, 5, 7, \ldots, n-2\}$; $\mathcal{S}(v_j^i) = 1$ if i is even, $j \in \{1, n-1\}$; $\mathcal{S}(v_j^i) = 3$ if i is odd, $j \in \{1, n-1\}$; $\mathcal{S}(v_j^i) = 2$ if $i \in \{1, 2, 3, \ldots, m\}$, $j \in \{2, 4, 6, \ldots, n-3, n\}$.

Case 4. $n \equiv 7 \; mod \; 8$.

Define c as follows: $c(u_i) = 0$ if $i \in \{1, 2, 3, \ldots, m\}$; $c(v_j^i) = 0$ if $i \in \{1, 2, 3, \ldots, m\}$, $j \equiv 0, 2, 3 \; mod \; 4$; $c(v_{n-2}^i) = 1$ if i is even; $c(v_{n-2}^i) = 3$ if i is odd; $c(v_j^i) = 2$ if $i \in \{1, 2, 3, \ldots, m\}$, $j \equiv 1 \; mod \; 4$, $j \neq n-2$; then $\mathcal{S}(u_i) = 1$ if i is odd; $\mathcal{S}(u_i) = 3$ if i is even; $\mathcal{S}(v_j^i) = 0$ if $i \in \{1, 2, 3, \ldots, m\}$, $j \in \{1, 3, 5, \ldots, n-2\}$; $\mathcal{S}(v_j^i) = 1$ if i is even, $j \in \{n-1, n-3\}$; $\mathcal{S}(v_j^i) = 3$ if i is odd, $j \in \{n-1, n-3\}$; $\mathcal{S}(v_j^i) = 2$ if $i \in \{1, 2, 3, \ldots, m\}$, $j \in \{2, 4, 6, \ldots, n-5, n\}$.

Case 5. $n = 3$.

Subcase 5.1. $m \equiv 0 \; mod \; 4$.

Define c as follows: $c(u_i) = 0$ if $i \equiv 0, 2, 3 \; mod \; 4$; $c(u_i) = 1$ if $i \equiv 1 \; mod \; 4$; $c(v_1^i) = 0$ if $i \equiv 1 \; mod \; 4$; $c(v_2^i) = 2$ if $i \equiv 1 \; mod \; 4$; $c(v_3^i) = 3$ if $i \equiv 1 \; mod \; 4$; $c(v_1^i) = 0$ if i is even; $c(v_2^i) = 1$ if i is even; $c(v_3^i) = 2$ if i is even; $c(v_1^i) = 1$ if $i \equiv 3 \; mod \; 4$; $c(v_2^i) = 2$ if $i \equiv 3 \; mod \; 4$; $c(v_3^i) = 3$ if $i \equiv 3 \; mod \; 4$; then $\mathcal{S}(u_i) = 1$ if $i \equiv 1 \; mod \; 4$; $\mathcal{S}(u_i) = 0$ if $i \equiv 0, 2 \; mod \; 4$; $\mathcal{S}(u_i) = 2$ if $i \equiv 3 \; mod \; 4$; $\mathcal{S}(v_1^i) = 2$ if $i \equiv 1 \; mod \; 4$; $\mathcal{S}(v_2^i) = 0$ if $i \equiv 1 \; mod \; 4$; $\mathcal{S}(v_3^i) = 3$ if $i \equiv 1 \; mod \; 4$; $\mathcal{S}(v_1^i) = 1$ if $i \equiv 3 \; mod \; 4$; $\mathcal{S}(v_2^i) = 0$ if $i \equiv 3 \; mod \; 4$; $\mathcal{S}(v_3^i) = 3$ if $i \equiv 3 \; mod \; 4$; $\mathcal{S}(v_1^i) = 3$ if i is even; $\mathcal{S}(v_2^i) = 2$ if i is even; $\mathcal{S}(v_3^i) = 1$ if i is even.

Subcase 5.2. $m \equiv 2 \; mod \; 4$.

Define c as follows: $c(u_i) = 0$ if $i \equiv 0, 2, 3 \; mod \; 4$; $c(u_i) = 1$ if $i \equiv 1 \; mod \; 4$; $c(v_1^i) = 0$ if $i \equiv 1 \; mod \; 4$; $c(v_2^i) = 2$ if $i \equiv 1 \; mod \; 4$; $c(v_3^i) = 3$ if $i \equiv 1 \; mod \; 4$; $c(v_1^i) = 0$ if $i \in \{2, 4, 6, \ldots, m-2\}$; $c(v_2^i) = 1$ if $i \in \{2, 4, 6, \ldots, m-2\}$; $c(v_3^i) = 2$ if $i \in \{2, 4, 6, \ldots, m-2\}$; $c(v_1^m) = 0$; $c(v_2^m) = 1$; $c(v_3^m) = 3$; $c(v_1^i) = 1$ if $i \equiv 3 \; mod \; 4$; $c(v_2^i) = 2$ if $i \equiv 3 \; mod \; 4$; $c(v_3^i) = 3$ if $i \equiv 3 \; mod \; 4$; then $\mathcal{S}(u_i) = 1$ if $i \equiv 1 \; mod \; 4$; $\mathcal{S}(u_i) = 0$ if $i \in \{2, 4, 6, \ldots, m-2\}$; $\mathcal{S}(u_i) = 2$ if $i \equiv 3 \; mod \; 4$; $\mathcal{S}(u_m) = 2$; $\mathcal{S}(v_1^i) = 2$ if $i \equiv 1 \; mod \; 4$; $\mathcal{S}(v_2^i) = 0$ if $i \equiv 1 \; mod \; 4$; $\mathcal{S}(v_3^i) = 3$ if $i \equiv 1 \; mod \; 4$; $\mathcal{S}(v_1^i) = 1$ if $i \equiv 3 \; mod \; 4$; $\mathcal{S}(v_2^i) = 0$ if $i \equiv 3 \; mod \; 4$; $\mathcal{S}(v_3^i) = 3$ if $i \equiv 3 \; mod \; 4$; $\mathcal{S}(v_1^i) = 3$ if $i \in \{2, 4, 6, \ldots, m-2\}$; $\mathcal{S}(v_2^i) = 2$ if $i \in \{2, 4, 6, \ldots, m-2\}$;

$\mathcal{S}(v_3^i) = 1$ if $i \in \{2, 4, 6, \ldots, m-2\}$; $\mathcal{S}(v_1^m) = 0$; $\mathcal{S}(v_2^m) = 3$; $\mathcal{S}(v_3^m) = 1$. Clearly, $Mc(C_m \circ C_n) \geq \chi(C_m \circ C_n) = 4$. Hence, $Mc(C_m \circ C_n) = 4$. This completes the proof. □

3. CONCLUSION

For some graphs $\mathcal{G}$ and $\mathcal{H}$ considered in this paper, we have seen that $Mc(\mathcal{G} \circ \mathcal{H}) = \chi(\mathcal{G} \circ \mathcal{H})$. Except the case: For $m \geq 1$, $n \geq 1$, $Mc(C_{2m+1} \circ C_{2n+1})$.

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