ADV MATH SCI JOURNAL

Advances in Mathematics: Scientific Journal **9** (2020), no.10, 7977–7984
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.10.32 Special Issue on AMABDA-2020

GENERALIZED NARAYANA SEQUENCES AND FIGURATE NUMBERS

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ABSTRACT. Among several amusing sequences that exist in mathematics, Fibonacci sequence is the most common and famous sequence that is known to everyone. An equally absorbing sequence was described by Indian mathematician Narayana Panditha. In this paper, we try to generalize Narayana sequence using the coefficients which are Figurate numbers and try to explore the limiting ratios of such sequences. In this way, this paper provides the interesting relationship between Generalized Narayana sequences and Figurate numbers.

1. INTRODUCTION

Leonardo Fibonacci of Italy introduced the most famous Fibonacci sequence using his immortal rabbits in his wonderful book "Liber Abaci" published in 1202 CE. Nearly after a century, Indian notable Indian mathematician Narayana Panditha introduced a wonderful sequence using immortal cows resembling Fibonacci sequence. The behavior of Narayana sequence and the ratio of its successive terms is well known. Also some generalizations of Narayana sequence is also dealt by several authors. In this paper, we shall consider generalizations of Narayana sequence using Figurate numbers as coefficients. In particular, we consider natural numbers, triangular numbers and square numbers as coefficients in the recurrence relations describing the Narayana sequence and try to analyze such generalized sequences. The main objective of this paper

2020 *Mathematics Subject Classification.* 11B50, 11B83, 11B37, 65Q30, 47A48.

Key words and phrases. Narayana sequence, Generalized Narayana sequence, Figurate numbers, Recurrence Relations, Charateristic Equations, Limiting Ratios.

is to obtain interesting results regarding the limiting ratios of such generalized Narayana sequences. First, we begin with some definitions.

2. Definition

Narayana Panditha described the Narayana sequence in such a way that the number of cows present each year is equal to the number of cows in previous year plus the number of cows three years ago. Using this convention, we form the following Recurrence Relation describing Narayana Sequence.

$$N_{n+1} = N_n + N_{n-2}; n \geq 2, N_0 = 0, N_1 = 1, N_2 = 1 \tag{2.1}$$

3. Generalizing Narayana Sequence

We will consider the following three cases in meeting our objective of generalizing Narayana sequence.

3.1. Let k be any positive integer. We define a new sequence whose terms are given by the recurrence relation:

$$N_{k,n+1} = kN_{k,n} + \frac{k(k+1)}{2}N_{k,n-2};\ n \geq 2, N_{k,0} = 0, N_{k,1} = 1, N_{k,2} = k. \tag{3.1}$$

We note that for $k = 1$, equation (3.1) reduces to the classic Narayana sequence defined in (2.1). The coefficients in the right hand side of the recurrence relation (3.1) are natural numbers and triangular numbers respectively. If we assume that the limiting ratio of generalized Narayana sequence is λ then by definition we have $\lim\left(\frac{N_{k,n+1}}{N_{k,n}}\right) = \lambda$ as $n \to \infty$. Now for any integer r, we have the following equation:

$$\begin{aligned}
&\lim_{n\to\infty}\left(\frac{N_{k,n+r}}{N_{k,n}}\right)\\
&\quad = \lim_{n\to\infty}\left(\frac{N_{k,n+r}}{N_k n + r - 1} \times \frac{N_{k,n+r-1}}{N_{k,n+r-2}} \times \frac{N_{k,n+r-2}}{N_{k,n+r-3}} \times \cdots \times \frac{N_{k,n+1}}{N_{k,n}}\right)\\
&\quad = \lambda \times \ldots \times \lambda = \lambda^r
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
&\lim_{n\to\infty}\left(\frac{N_{k,n+1}}{N_{k,n}}\right)\\
&\quad = \lim_{n\to\infty}\left(\frac{kN_{k,n} + \frac{k(k+1)}{2}N_{k,n-2}}{N_{k,n}}\right) = \lim_{n\to\infty}\left(k + \frac{k(k+1)}{2}\frac{N_{k,n-2}}{N_{k,n}}\right).
\end{aligned}$$

Now using (3.2), as $n \to \infty$ we get:

$$\lambda = k + \frac{k(k+1)}{2}\frac{1}{\lambda^2}. \tag{3.3}$$

This leads to the equation:

$$2\lambda^3 - 2k\lambda^2 - k(k+1) = 0. \tag{3.4}$$

We call equation (3.4) as the Characteristic Equation, which is a cubic polynomial in limiting ratio λ corresponding to the generalized Narayana sequence defined in (3.1). First we note that if:

$$P_1(\lambda) = 2\lambda^3 - 2k\lambda^2 - k(k+1) \tag{3.5}$$

is the cubic polynomial corresponding to equation (3.4), we see that $P_1(k) = -k(k+1) < 0$ (since k is a positive integer) and $P_1(k+1) = (k+1)(k+2) > 0$.Moreover by Descarte's Rule of signs, we observe that the polynomial in (3.5) possess only one positive real root. Thus, the positive real root of equation (3.4) lies in the interval $(k, k+1)$. So the limiting ratio is of order k. That is:

$$\lambda = O(k). \tag{3.6}$$

Hence using (3.6), we get:

$$\frac{k(k+1)}{2}\frac{1}{\lambda^2} \to \frac{k(k+1)}{2}\frac{1}{k^2} = \frac{1}{2} \tag{3.7}$$

as $k \to \infty$. Hence as $k \to \infty$, equation (3.3) becomes $\lambda = k + \frac{1}{2}$. So, the limiting ratio of generalized Narayana sequence defined in (3.1) is:

$$\lambda = k + \frac{1}{2} \tag{3.8}$$

for very large values of k.

3.2. Let k be any positive integer. We define a new sequence whose terms are given by the recurrence relation:

$$N_{k,n+1} = kN_{k,n} + k^2 N_{k,n-2};\ n \geq 2, N_{k,0} = 0, N_{k,1} = 1, N_{k,2} = k. \tag{3.9}$$

We note that for $k = 1$, equation (3.9) reduces to the classic Narayana sequence defined in (2.1). The coefficients in the right hand side of the recurrence relation (3.9) are natural numbers and square numbers respectively. In similar way, if we try to compute the limiting ratio corresponding to the equation (3.9) then we get:

$$\lim_{n\to\infty}\left(\frac{N_{k,n+1}}{N_{k,n}}\right) = \lim_{n\to\infty}\left(\frac{kN_{k,n}+k^2N_{k,n-2}}{N_{k,n}}\right) = \lim_{n\to\infty}\left(k+k^2\frac{N_{k,n-2}}{N_{k,n}}\right).$$

Now using (3.2) (which is applicable generally) and taking the limit as $n \to \infty$, we have:

$$\lambda = k + \frac{k^2}{\lambda^2}. \tag{3.10}$$

Thus the characteristic equation of (3.9) is:

$$\lambda^3 - k\lambda^2 - k^2 = 0. \tag{3.11}$$

As k is positive, if we consider:

$$P_2(\lambda) = \lambda^3 - k\lambda^2 - k^2, \tag{3.12}$$

then by Descarte's Rule of signs, we find that there is only one positive real root for the cubic polynomial (3.12). Further, we note that $P_2(k) = -k^2 < 0, P_2(k+2) = k^2+8k+8 > 0$. Thus the positive real root of (3.11) lies in the interval $(k, k+2)$. So, the limiting ratio is of order k. That is:

$$\lambda = O(k). \tag{3.13}$$

Hence using (3.13) in (3.10), we get:

$$\lambda = k + \frac{k^2}{\lambda^2} \to k + \frac{k^2}{k^2} = k+1, \tag{3.14}$$

as $k \to \infty$. Thus as $k \to \infty$ equation (3.10) becomes $\lambda = k+1$. Hence, the limiting ratio of generalized Narayana sequence defined in (3.9) is:

$$\lambda = k+1, \tag{3.15}$$

for very large values of k.

3.3. In this case, we consider a more general situation, in which the coefficients are natural numbers and Figurate numbers of order m ($m \geq 3$) respectively. Let k be a positive integer. The recurrence relation for this situation is given by:
(3.16)

$$N_{k,n+1} = kN_{k,n} + F_m(k)N_{k,n-2}; m \geq 3, n \geq 2, N_{k,0} = 0, N_{k,1} = 1, N_{k,2} = k.$$

We note that if $k = 1$, equation (3.16) reduces to the classic Narayana sequence defined in (2.1). As in the previous two cases, if we assume that if λ is the limiting ratio of the generalized Narayana sequence defined in (3.16), then we get:

$$\lim_{n\to\infty}\left(\frac{N_{k,n+1}}{N_{k,n}}\right)=\lim_{n\to\infty}\left(\frac{kN_{k,n}+F_m(k)N_{k,n-2}}{N_{k,n}}\right)=\lim_{n\to\infty}\left(k+F_m(k)\frac{N_{k,n-2}}{N_{k,n}}\right).$$

Now using (3.2), and taking the limit as $n\to\infty$, we get:

$$\lambda=k+\frac{F_m(k)}{\lambda^2}. \tag{3.17}$$

Thus the characteristic equation of (3.16) is given by:

$$\lambda^3-k\lambda^2-F_m(k)=0. \tag{3.18}$$

Since k is positive, we see that the polynomial:

$$P_3(\lambda)=\lambda^3-k\lambda^2-F_m(k) \tag{3.19}$$

by Descarte's Rule of signs has only one positive real root. Thus the equation (3.18) has only one only one positive real root in the interval $(k, k+m-2)$. So, the limiting ratio is of order k. That is:

$$\lambda=O(k). \tag{3.20}$$

Hence using (3.20) in (3.17), and from equation (2.3), we get:

$$\begin{aligned}\lambda&=k+\frac{F_m(k)}{\lambda^2}=k+\frac{(m-2)k^2-(m-4)k}{2\lambda^2}\\ &\to\quad k+\frac{(m-2)k^2-(m-4)k}{2k^2}=k+\frac{m-2}{2},\end{aligned} \tag{3.21}$$

as $k\to\infty$. Thus as $k\to\infty$, equation (3.17) becomes $\lambda=k+\frac{m-2}{2}$. Hence, the limiting ratio of generalized Narayana sequence defined in (3.16) is:

$$\lambda=k+\frac{m-2}{2} \tag{3.22}$$

for very large values of k.

We note that the values of λ obtained in sections 3.1 and 3.2 through equations (3.8), (3.15) are special cases of equation (3.22) for $m=3,4$ respectively.

4. Generalized Narayana Sequences and Figurate Numbers

In this section, we consider two cases by considering two interesting recurrence relations and try to determine the limiting ratios for each case.

4.1. For positive integer k, we consider the recurrence relation given by:

$$N_{k,n+1} = F_m(k)N_{k,n} + k\mathrm{N}_{k,n-2}; m \geq 3, n \geq 2, N_{k,0} = 0, N_{k,1} = 1, N_{k,2} = k. \tag{4.1}$$

We note that if $k = 1$, equation (3.16) reduces to the classic Narayana sequence defined in (2.1). If λ is the limiting ratio of the generalized Narayana sequence defined in (4.1), then we get:

$$\lim_{n\to\infty}\left(\frac{N_{k,n+1}}{N_{k,n}}\right) = \lim_{n\to\infty}\left(\frac{F_m(k)N_{k,n} + kN_{k,n-2}}{N_{k,n}}\right) = \lim_{n\to\infty}\left(F_m(k) + k\frac{N_{k,n-2}}{N_{k,n}}\right).$$

Now using (3.2), and taking the limit as $n \to \infty$ we get:

$$\lambda = F_m(k) + \frac{k}{\lambda^2}. \tag{4.2}$$

As discussed in section 3 for the three cases, we see that:

$$\lambda = O(k). \tag{4.3}$$

Hence using (4.3) in (4.2) and considering the limit as $k \to \infty$, we get:

$$\lambda = F_m(k) + \frac{k}{\lambda^2} \to F_m(k) + \frac{k}{k^2} = F_m(k). \tag{4.4}$$

Thus as $k \to \infty$, equation (4.2) becomes $\lambda = F_m(k)$. Hence the limiting ratio of generalized Narayana sequence defined in (4.1) is:

$$\lambda = F_m(k), \tag{4.5}$$

for large values of k. We observe that value of the limiting ratio obtained in (4.5) is precisely the Figurate numbers of order m defined in (2.3). This gives the intimate connection between Generalized Narayana sequence and Figurate numbers of order m.

4.2. In this case, we consider generalized Narayana sequence through a recurrence relation constructed using figurate numbers with two different orders say p and q. Let k, p, q be positive integers such that $p, q \geq 3$. We define the Generalized Narayana sequence using Figurate numbers coefficients given by the following recurrence relation:

$$N_{k,n+1} = F_p(k)N_{k,n} + F_q(k)N_{k,n-2}, \tag{4.6}$$

$p, q \geq 3, n \geq 2, N_{k,0} = 0, N_{k,1} = 1, N_{k,2} = k$. If $k = 1$, then we get the classic Narayana sequence as defined in (2.1). If is the limiting ratio of the recurrence relation defined in (4.6) then we get:

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{N_{k,n+1}}{N_{k,n}}\right) &= \lim_{n\to\infty}\left(\frac{F_p(k)N_{k,n} + F_q(k)N_{k,n-2}}{N_{k,n}}\right)\\ &= \lim_{n\to\infty}\left(F_p(k) + F_q(k)\frac{N_{k,n-2}}{N_{k,n}}\right)\end{aligned}$$

Now using (3.2), and taking the limit as $n \to \infty$ we get:

$$\lambda = F_p(k) + \frac{F_q(k)}{\lambda^2}. \tag{4.7}$$

Thus the characteristic equation corresponding to (4.6) is given by:

$$\lambda^3 - F_p(k)\lambda^2 - F_q(k) = 0. \tag{4.8}$$

Using Descarte's rule of signs, we see that there is only one positive real root for (4.8). Moreover, we also find that such a root should be such that:

$$\lambda = O(k). \tag{4.9}$$

Hence using (4.9) in (4.2) and considering the limit as $k \to \infty$ we get:

$$\begin{aligned}\lambda &= F_p(k) + \frac{F_q(k)}{\lambda^2} \to F_p(k) + \frac{(q-2)k^2 - (q-4)k}{2k^2}\\ &= F_p(k) + \frac{q-2}{2} - \frac{q-4}{2k} \quad \to \quad F_p(k) + \frac{q-2}{2}.\end{aligned} \tag{4.10}$$

Thus as $k \to \infty$, the unique positive real root of (4.8) is $\lambda = F_p(k) + \frac{q-2}{2}$. Hence the limiting ratio of generalized Narayana sequence defined in (4.6) is given by:

$$\lambda = F_p(k) + \frac{q-2}{2}. \tag{4.11}$$

for very large values of k.

Conclusion

In this paper, we have generalized the usual Narayana sequence in variety of ways using different recurrence relations defined through equations (3.1), (3.9), (3.16), (4.1), (4.6). We notice in all these five cases we get the recurrence relation of Narayana sequence defined in (2.1) for the choice of $k = 1$. Thus, by generalizing the Narayana sequence using recurrence relations using natural numbers and figurate numbers as coefficients we could produce various limiting ratios corresponding to each of the five cases. The limiting ratio obtained in (4.10) generalizes all other limiting ratios obtained in equations (3.8), (3.15), (3.22), (4.5). Thus the recurrence relation defined in (4.6) and the associated limiting ratio in (4.10) provides the explicit connection between Generalized Narayana sequence and Figurate numbers thereby serving the main purpose of this article. But the limiting ratio values make sense only if k is very large that is as $k \to \infty$.

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