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RELATION BETWEEN SUMS OF POWERS OF NATURAL NUMBERS

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ABSTRACT. Summation Process have been studied by mathematicians for many centuries. Many notable mathematicians have worked rigorously for finding sum of powers of natural numbers and other kinds of summation methods as well. This paper explores the relationship that exists between summation of various powers of natural numbers in detail. Such relationships are important to understand their behaviour. The derivation of such relationships has been presented in this paper in detail. The figures were provided at the concluding section to understand the geometric significance of the functional relations obtained.

1. INTRODUCTION

The formula for computing sums of powers of natural numbers is well known (see [1,2]). The German mathematician Johann Faulhaber provided an exclusive method for expressing odd powers of sum of natural numbers in terms of the sum of first power of natural numbers. Several mathematicians had done great research in this area. This paper attempts to provide five expressions, which gives clear picture about inter-relationship that exists between various powers of sum of natural numbers. We begin with some basic definitions and concepts.

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Definition 1.1. *Let n and k be natural numbers. We denote the sum of kth powers of first n natural numbers by the expression:*

$$S_k(n) = 1^k + 2^k + ... + n^k. \tag{1.1}$$

2. Finding Basic Relationships

We begin our exploration by comparing the sums of first and third powers of natural numbers. For future discussion let us assume that $S_1(n) = s$. Then from the formulas described above, we have:

$$s = \frac{n^2}{2} + \frac{n}{2} = \frac{n(n+1)}{2}, \tag{2.1}$$

$$S_3(n) = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} = \left[\frac{n(n+1)}{2}\right]^2 = s^2. \tag{2.2}$$

Thus the sum of third powers of first n natural numbers is the square of the sum of first n natural numbers as shown in equation (3.2). If we try to write the functional equation (an equation which expresses the connection) $F_1(x, y) = 0$ between $x = S_1(n), y = S_3(n)$ then from equations (3.1), (3.2), we have:

$$F_1(x, y) = x^2 - y = 0 \tag{2.3}$$

Equation (3.3) represents a parabola open upwards with vertex at origin.

We now determine relationship between sum of first and second powers of natural numbers. From set of equations described in section 2, we have:

$$S_2(n) = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} = \frac{n(n+1)(2n+1)}{6}. \tag{2.4}$$

If we consider $F_2(x, y) = 0$ to be the functional equation where $x = S_1(n), y = S_2(n)$ then from equations (3.1) and (3.4), we have:

$$y = \frac{n(n+1)(2n+1)}{6} = \frac{n(n+1)}{2} \times \frac{2n+1}{3} = x(\frac{2n+1}{3}).$$

Finding n from this, we get $n = \frac{3y-x}{2x}$. Substituting this in (3.1) to eliminate n we get $2x = \frac{3y-x}{2x} \times \frac{3y+x}{2x}$. This gives $8x^3 + x^2 - 9y^2 = 0$. Hence the functional equation between sum of first and second powers of natural numbers is:

$$F_2(x, y) = 8x^3 + x^2 - 9y^2 = 0. \tag{2.5}$$

We now determine relationship between sum of second and third powers of natural numbers. For this we consider equations (3.2) and (3.4) and try to eliminate n as above. If we consider $F_3(x, y)$ to be the functional equation where $x = S_2(n), y = S_3(n)$ then from equations (3.2) and (3.4), we have:

$$x^2 = \left[\frac{n(n+1)}{2}\right]^2 \times \left(\frac{2n+1}{3}\right)^2 = y\left(\frac{2n+1}{3}\right)^2,$$

$$\frac{2n+1}{3} = \frac{x}{\sqrt{y}}, n = \frac{3x-\sqrt{y}}{2\sqrt{y}}, n+1 = \frac{3x+\sqrt{y}}{2\sqrt{y}}.$$

Substituting these values in equation (3.4), we get:

$$x = \frac{1}{2} \times \left(\frac{3x-\sqrt{y}}{2\sqrt{y}}\right) \times \left(\frac{3x+\sqrt{y}}{2\sqrt{y}}\right) \times \frac{x}{\sqrt{y}}.$$

From this, we get: $64y^3 = \left(3x^2 - y\right)^2 \Rightarrow 81x^4 - 18x^2y + y^2 - 64y^3 = 0.$

Hence the functional equation between sum of second and third powers of natural numbers is:

$$F_3(x, y) = 81x^4 - 18x^2y + y^2 - 64y^3 = 0. \tag{2.6}$$

We now determine relationship between sum of third and fifth powers of natural numbers. From set of equations described in section 2, we have:

$$S_3(n) = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} = \left[\frac{n(n+1)}{2}\right]^2,$$

$$S_5(n) = \frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} - \frac{n^2}{12} = \frac{[n(n+1)]^2\left(2n^2+2n-1\right)}{12}.$$

If we consider F4(x, y) = 0 to be the functional equation where $x = S_3(n), y = S_5(n)$ then we have:

$$x = \frac{[n(n+1)]^2}{4} \Rightarrow n(n+1) = 2\sqrt{x},$$
$$y = \frac{[n(n+1)]^2\left(2n^2+2n-1\right)}{12} = \frac{[n(n+1)]^2}{4} \times \frac{\left(2n^2+2n-1\right)}{3} = \frac{x}{3} \times [2n(n+1)-1],$$
$$3y = x[4\sqrt{x}-1] \Rightarrow (x+3y)^2 = 16x^3 \Rightarrow 16x^3 - x^2 - 6xy - 9y^2 = 0.$$

Hence the functional equation between sum of third and fifth powers of natural numbers is:

$$F_4(x, y) = 16x^3 - x^2 - 6xy - 9y^2 = 0. \tag{2.7}$$

We now determine relationship between sum of second and fourth powers of natural numbers. From set of equations described in section 2, we have:

$$S_2(n) = \frac{n(n+1)(2n+1)}{6}, \quad S_4(n) = \frac{n(n+1)(2n+1)\left(3n^2+3n-1\right)}{30}.$$

If we consider $F_5(x,y) = 0$ to be the functional equation where $x = S_2(n), y = S_4(n)$ then we have:

$$y = \tfrac{n(n+1)(2n+1)}{6} \cdot \tfrac{[3n(n+1)-1]}{5} = \tfrac{x}{5} \cdot [3n(n+1)-1],$$
$$n(n+1) = \tfrac{x+5y}{3x} \quad \Rightarrow \quad 2n+1 = \tfrac{6x}{n(n+1)} = \tfrac{18x^2}{x+5y},$$
$$n = \tfrac{1}{2}\left[\tfrac{18x^2-x-5y}{x+5y}\right] \quad \Rightarrow \quad n+1 = \tfrac{1}{2}\left[\tfrac{18x^2+x+5y}{x+5y}\right].$$

Substituting the values of n, $n+1$ and $2n+1$ in $S_2(n)$ we get:

$$6x = \frac{1}{2}\left[\frac{18x^2-x-5y}{x+5y}\right] \cdot \frac{1}{2}\left[\frac{18x^2+x+5y}{x+5y}\right] \cdot \frac{18x^2}{x+5y}$$
$$4(x+5y)^3 = 3x\left[324x^4-(x+5y)^2\right]$$
$$\Rightarrow \quad 972x^5 - 7x^3 - 90x^2y - 375xy^2 - 500y^3 = 0.$$

Hence the functional equation between sum of second and fourth powers of natural numbers is:

$$F_5(x,y) = 972x^5 - 7x^3 - 90x^2y - 375xy^2 - 500y^3 = 0. \tag{2.8}$$

Equations (3.3), (3.5), (3.6), (3.7), (3.8) provide the inter-relationship between corresponding sums of powers of natural numbers. By following similar procedures it will certainly be possible for us to obtain relationships between various powers of sums of natural numbers. We now present some figures to display graphically the relationships that we had obtained.

CONCLUSION

In this paper, the inter-relationships between sums of first-third, first-second, second-third, third-fifth, second-fourth powers of natural numbers have been derived in equations (3.3), (3.5), (3.6), (3.7), (3.8) respectively. These relationships exhibits functional equations which when viewed as surface representations, provides wonderful pictures as displayed in Figures 1 to 5 in section 4. From these five figures, we notice that Figure 1, provides the trivial relationship

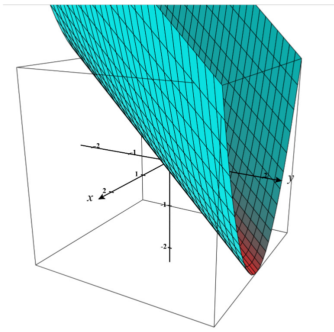

FIGURE 1. **Graph of** $F_1(x, y) = x^2 - y$

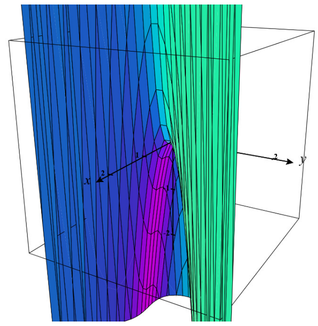

FIGURE 2. **Graph of** $F_2(x, y) = 8x^3 + x^2 - 9y^2$

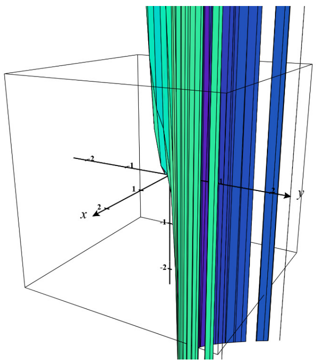

FIGURE 3. **Graph of** $F_3(x, y) = 81x^4 - 18x^2y + y^2 - 64y^3 = 0$

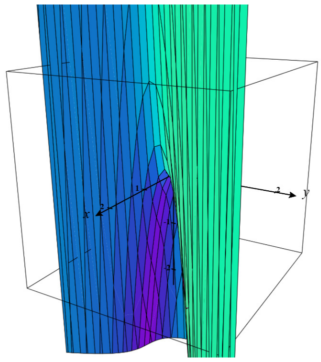

FIGURE 4. **Graph of** $F_4(x, y) = 16x^3 - x^2 - 6xy - 9y^2 = 0$

between sums of first and third powers of first n natural numbers through a paraboloid. The relationships between first-second and third-fifth sums of powers of natural numbers displayed as Figures 2 and 4 in section 4, provide similar surface characterizations. Similarly the relationships between second-third and second-fourth sums of powers of natural numbers displayed as Figures 3 and 5 in section 4, provide surface characterizations which are of even and odd degrees respectively. If we try to determine additional functional equations between sums of two various higher powers of natural numbers, then we can see similar behaviour of surface characterizations. These characterizations will provide an additional insight as to how the sums will behave for given two powers. This also would provide us with a natural extension of real plane to complex

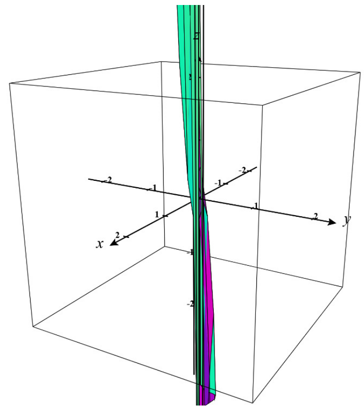

FIGURE 5. **Graph of** $F_5(x, y) = 972x^5 - 7x^3 - 90x^2y - 375xy^2 - 500y^3 = 0$

analytic continuation which supplies many treasures including the most famous Riemann Zeta Function.

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