

WEAK TYPES OF LIMIT POINTS AND SEPARATION AXIOMS ON SUPRA TOPOLOGICAL SPACES

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ABSTRACT. Supra topology was defined by neglecting an intersection condition of topology which makes it more flexible to describe some real-life problems and to easily construct some examples whom show some relationships between certain topological concepts. In fact, it is one of the most important developments of topology in the recent years. The purpose of this paper is to introduce new kinds of limit points of a set and separation axioms on supra topological spaces using supra α -open sets. We discuss the main properties of supra α limit points of a set and describe their behaviour on the spaces that possess the difference property. We probe some equivalent conditions for each one of supra α regular, supra α normal and $S\alpha T_i$ -spaces ($i = 0, 1, 2, 3, 4$). For comparison, we prove that every $S\alpha T_i$ -space is ST_{i-1} for $i = 1, 2, 3, 4$ and show that $S\alpha T_i$ -spaces are weaker than ST_i -spaces in the cases of $i = 0, 1, 2$. Some examples and counterexamples are given. In the end, we draw attention to that the concepts and results obtained in this work will be a guide to investigate their counterparts in other structures such weak and minimal structures.

1. INTRODUCTION AND PRELIMINARIES

Topology forms a new type of geometry that relies on nearness or neighbourhood of points instead of measuring distance between them. That is, it explicitly defines what sets of objects are considered to be near to one another. Abstractly, topology is defined on a nonempty set X as a subset of its power set $P(X)$

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which is closed under finite intersection and arbitrary union. Actually, the universal and empty sets belong to topology according to the celebrated result of the set theory that reports that the intersection and union of an empty collection of sets are the universal and empty sets, respectively.

However, we need sometimes to relax the topological conditions in order to model some phenomena and practical problems, or even seek to preserve some topological properties under fewer conditions than topology, see, for example [25,27]. Historically, generalized trend emerged first, in 1940, by Alexandroff [2]. He strengthened the intersection condition. Then, in 1983, Mashhour et al. [29] defined a supra topology by dropping only the intersection condition, and in 1996, Maki et al. [28] introduced a minimal structure as a collection contains the empty and universal sets. In 2002, Császár [18] introduced generalized topology as a collection contains the empty set and is closed under a nonempty union. He continued this path by defining a weak structure as a collection contains the empty set. In 2015, Al-Odhari [3] defined infra topological spaces by cancelling the union condition.

After Mashhour et al. [29] initiated the concepts of continuity and separation axioms on supra topological spaces, Al-shami [4] investigated the classical topological notions such as limit points of a set, compactness, and separation axioms on the supra topological spaces. In fact, some results via topology do not still valid via supra topology such as the distribution of the closure operator between the union of two sets and the distribution of the interior operator between the intersection of two sets. Also, the property of a compact subset of a T_2 -space is closed is another example of an invalid topological result on the supra topologies.

In a similar ways of their counterparts in general topology, the concepts of supra α -open [19], supra pre-open [32], supra b -open [34], supra β -open [26], supra R -open [20] and supra semi-open sets [5] have been introduced and their main properties have been discussed. In other words, their definitions were formulated using supra interior and supra closure operators instead of interior and closure operators. These generalizations have been utilized to define new types of compactness, connectedness, limit points and separation axioms, see, for example [8, 9, 12, 24, 30, 33]. The class of supra R -open sets have been studied in [6, 15] under the name of somewhere dense sets. Mustafa and Qoqazeh [31] took advantage of supra D -sets to define separation axioms on supra topological spaces. Recently, Al-shami [11] has studied the concept of paracompactness

on supra topological spaces and he and El-Shafei [14] investigated two types of separation axioms on supra soft topological spaces.

To extended supra topological spaces, Abo-Elhamayel and Al-shami [1] studied supra topological spaces. Then the authors of [7, 10, 16, 17, 21, 23] defined various ordered maps using supra α -open, supra pre-open, supra semi-open, supra b -open, supra β -open and supra R -open sets. El-Shafei et al. [22] investigated novel supra separation axioms with the help of illustrative examples.

This paper is organized as follows: Section 2 defines supra α limit points of a set using supra α -open sets. Section 3 introduces $S\alpha T_i$ -spaces ($i = 0, 1, 2, 3, 4$) and shows the relationships between them with the help of examples. Section 4 concludes the paper with summary and further works.

In the following, we recall some definitions and results of supra topology and supra α -open set that help to investigate results obtained in this work.

Definition 1.1. [29] For $X \neq \emptyset$, a subfamily μ of $P(X)$ is called a supra topology if it is closed under arbitrary union and X is a member of μ .

Then the pair (X, μ) is called a supra topological space. Terminologically, a member of μ is called a supra open set and its complement is called a supra closed set.

Remark 1.1.

- (1) μ is called an associated supra topology with a topology τ if $\tau \subseteq \mu$.
- (2) Through this paper, we consider (X, μ) and (Y, ν) are associated supra topological spaces with the topological spaces (X, τ) and (Y, θ) , respectively.

Definition 1.2. [29] For $A \subseteq (X, \mu)$, $\text{int}^\mu(A)$ is the union of all supra open sets contained in A and $\text{cl}^\mu(A)$ is the intersection of all supra closed sets containing A .

If there is no confusion, we write $\text{int}(A)$ and $\text{cl}(A)$ in the places of $\text{int}^\mu(A)$ and $\text{cl}^\mu(A)$, respectively.

Definition 1.3. [19] A subset A of (X, μ) is said to be supra α -open if $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$.

Definition 1.4. [19] For $A \subseteq (X, \mu)$, $\alpha\text{int}^\mu(A)$ is the union of all supra α -open sets contained in A and $\alpha\text{cl}^\mu(A)$ is the intersection of all supra α -closed sets containing A .

If there is no confusion, we write $\alpha\text{int}(A)$ and $\alpha\text{cl}(A)$ in the places of $\alpha\text{int}^\mu(A)$ and $\alpha\text{cl}^\mu(A)$, respectively.

Definition 1.5. [19] A map $g : (X, \mu) \rightarrow (Y, \nu)$ is said to be:

- (1) *supra α -continuous* if the inverse image of each open subset of Y is a supra α -open subset of X .
- (2) *supra α -open (resp. supra α -closed)* if the image of each open (resp. closed) subset of X is a supra α -open (resp. supra α -closed) subset of Y .

Definition 1.6. [4] Let A be a subset of (X, μ) . The family $\mu_A = \{A \cap G : G \in \mu\}$ is called a supra relative topology on A . A pair (A, μ_A) is called a supra subspace of (X, μ) .

Definition 1.7. [13] β is called a basis for a supra topology (X, μ) if every member of μ can be expressed as a union of elements of β .

Definition 1.8. [13] Let $\{(X_i, \mu_i) : i = 1, 2, \dots, n\}$ be the collection of supra topological spaces. Then $\beta = \prod_{i=1}^n \mu_i = \{\prod_{i=1}^n G_i : G_i \in \mu_i\}$ defines a basis for a supra topology T on $X = \prod_{i=1}^n X_i$. The pair (X, T) is called a finite product supra spaces.

Proposition 1.1. [13] Let A and B be two subsets of (X, μ) and (Y, ν) , respectively. Then:

- (i) $cl(A) \times cl(B) = cl(A \times B)$.
- (ii) $int(A) \times int(B) = int(A \times B)$.

2. LIMIT POINTS OF A SET WITH RESPECT TO SUPRA α -OPEN SETS

In this section, we formulate a concept of supra α limit points of a set and establish basic properties. Some examples and counterexamples are provided.

Definition 2.1. A subset A of (X, μ) is said to be a supra α neighbourhood of $x \in X$ provided that there is a supra α open set F containing x such that $x \in F \subseteq A$.

Definition 2.2. A point $x \in X$ is said to be a supra α limit point of a subset A of (X, μ) provided that every supra α neighborhood of x contains at least one point of A other than x itself.

All supra α limit points of A is said to be a supra α derived set of A and is denoted by $A^{\alpha'}$.

Proposition 2.1. If $A \subseteq B$, then $A^{\alpha'} \subseteq B^{\alpha'}$ for every subsets A and B of (X, μ) .

Proof. Straightforward. □

Corollary 2.1. *We have the following results for any two subsets A and B of (X, μ) .*

- (1) $A^{\alpha'} \cup B^{\alpha'} \subseteq (A \cup B)^{\alpha'}$.
- (2) $(A \cap B)^{\alpha'} \subseteq A^{\alpha'} \cap B^{\alpha'}$.

The following example illustrates that the converse of the above proposition and corollary fails.

Example 1. Let $\mu = \{\emptyset, G \subseteq \mathcal{N} : 1 \in G \text{ or } 2 \in G \text{ such that } G^c \text{ is finite}\}$ be a supra topology on the natural numbers set $\mathcal{N}$. If A is a nonempty supra α -open set, then $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$. This means $\text{int}(A) \neq \emptyset$. Therefore $1 \in \text{int}(A)$ or $2 \in \text{int}(A)$ such that $(\text{int}(A))^c$ is finite. Since $\text{int}(A) \subseteq A$ and $A^c \subseteq (\text{int}(A))^c$, then A is a nonempty supra open set. Thus the collection of all supra α -open sets coincides with the collection of all supra open sets.

Now, let $A = \{1\}$, $B = \{2\}$, C is the odd numbers set and D is the even numbers set. Then $A^{\alpha'} = B^{\alpha'} = \emptyset$, $C^{\alpha'} = D^{\alpha'} = \mathcal{N}$. Note the following cases:

- (i) $A^{\alpha'} \subseteq B^{\alpha'}$ and $B^{\alpha'} \subseteq A^{\alpha'}$, but neither $B \subseteq A$ nor $A \subseteq B$.
- (ii) $A^{\alpha'} \cup B^{\alpha'} = \emptyset$, but $(A \cup B)^{\alpha'} = \mathcal{N} \setminus \{1, 2\}$.
- (iii) $C^{\alpha'} \cap D^{\alpha'} = \mathcal{N}$, but $(C \cap D)^{\alpha'} = \emptyset$.

Proposition 2.2. *Let A be a subset of (X, μ) and $\lambda \in X$. Then $\lambda \in A^{\alpha'}$ if and only if $\lambda \in (A \setminus \{\lambda\})^{\alpha'}$.*

Proof. *Necessity:* Let $\lambda \in A^{\alpha'}$. Then for every supra α open set G containing λ , we have $(G \setminus \{\lambda\}) \cap A \neq \emptyset$. Therefore $(G \setminus \{\lambda\}) \cap (A \setminus \{\lambda\}) \neq \emptyset$. Thus $\lambda \in (A \setminus \{\lambda\})^{\alpha'}$. *Sufficiency:* It follows from Proposition 2.1. $\square$

Theorem 2.1. *Let A be a subset of (X, μ) . Then the following results hold.*

- (i) A is a supra α -closed set iff $A^{\alpha'} \subseteq A$.
- (ii) $A \cup A^{\alpha'}$ is a supra α -closed set.
- (iii) $\alpha\text{cl}(A) = A \cup A^{\alpha'}$.

Proof.

- (i) Suppose that A is a supra α -closed set and $\lambda \notin A$. Then A^c is a supra α -open set containing λ . In this case $A^c \cap A = \emptyset$ leads to $\lambda \notin A^{\alpha'}$. Therefore $A^{\alpha'} \subseteq A$. Conversely, let $\lambda \in A^c$ and let $A^{\alpha'} \subseteq A$. Then $\lambda \notin A^{\alpha'}$. Therefore there is a supra α -open set G_λ such that $G_\lambda \setminus \{\lambda\} \cap A = \emptyset$. Since $\lambda \in A^c$, then $G_\lambda \cap A = \emptyset$. Now, $G_\lambda \subseteq A^c$. Therefore $A^c = \bigcup \{G_\lambda : \lambda \in A^c\}$. Thus A is supra α -closed.

- (ii) Let $\lambda \notin (A \cup A^{\alpha'})$. Then $\lambda \notin A$ and $\lambda \notin A^{\alpha'}$. Therefore there is a supra α -open set G such that

$$(2.1) \quad G \cap A = \emptyset$$

Now, for each $\lambda \in G$, we have $\lambda \notin A^{\alpha'}$. This means that

$$(2.2) \quad G \cap A^{\alpha'} = \emptyset$$

From (2.1) and (2.2), we obtain $G \cap (A \cup A^{\alpha'}) = \emptyset$. This implies that $\lambda \notin (A \cup A^{\alpha'})^{\alpha'}$. Hence $(A \cup A^{\alpha'})^{\alpha'} \subseteq (A \cup A^{\alpha'})$. By (i), $A \cup A^{\alpha'}$ is a supra α -closed set, as required.

- (iii) Since $A \subseteq \alpha cl(A)$ and $A^{\alpha'} \subseteq (\alpha cl(A))^{\alpha'} \subseteq \alpha cl(A)$, then $A \cup A^{\alpha'} \subseteq \alpha cl(A)$. Since $A \cup A^{\alpha'}$ is a supra α -closed set containing A and $\alpha cl(A)$ is the smallest supra α -closed set containing A , then $\alpha cl(A) \subseteq A \cup A^{\alpha'}$. Therefore $\alpha cl(A) = A \cup A^{\alpha'}$.

□

Corollary 2.2. *If A is a supra α -closed subset of (X, μ) , then $A^{\alpha'}$, $(A^{\alpha'})^{\alpha'}$, $((A^{\alpha'})^{\alpha'})^{\alpha'}$, ... are supra α -closed sets.*

Definition 2.3. *A map $g : (X, \mu) \rightarrow (Y, \nu)$ is said to be:*

- (1) *supra α^* -continuous if $g^{-1}(H)$ is a supra α -open set in X for every supra α -open set in Y .*
- (2) *supra α^* -open (resp. supra α^* -closed) if $g(H)$ is a supra α -open (resp. supra α -closed) set in Y for every supra α -open (resp. supra α -closed) set in X .*
- (3) *supra α^* -homeomorphism if it is bijective, supra α^* -continuous and supra α^* -open.*

Theorem 2.2. *If $g : (X, \mu) \rightarrow (Y, \nu)$ is a supra α^* -homeomorphism map, then $g(A^{\alpha'}) = (g(A))^{\alpha'}$ for each $A \subseteq X$.*

Proof. Let $\lambda \notin (g(A))^{\alpha'}$. Then there is a supra α -open set H containing λ such that $(H \setminus \{\lambda\}) \cap g(A) = \emptyset$. So $g^{-1}[(H \setminus \{\lambda\}) \cap g(A)] = g^{-1}(\emptyset)$. This implies that $(g^{-1}(H) \setminus g^{-1}(\lambda)) \cap A = \emptyset$. Thus $g^{-1}(\lambda) \notin A^{\alpha'}$. Since g is bijective, then $\lambda \notin g(A^{\alpha'})$. Therefore $g(A^{\alpha'}) \subseteq (g(A))^{\alpha'}$. By reversing the preceding steps, we find that $(g(A))^{\alpha'} \subseteq g(A^{\alpha'})$. Hence, the proof is complete. □

Definition 2.4. *For a nonempty set X , a sub collection Λ of 2^X is said to have the difference property provided that $G \in \Lambda$ implies that $G \setminus \{\lambda\} \in \Lambda$.*

The following two examples illustrate the existence and uniqueness of the difference property.

Example 2. Let $\mu = \{\emptyset, G \subseteq \mathcal{N} : G \text{ is infinite}\}$ be a supra topology on the set of natural numbers $\mathcal{N}$. It is clear that the infinity of G implies the infinity of $G \setminus \{\lambda\}$. That is, $G \in \mu$ implies $G \setminus \{\lambda\} \in \mu$. Then $(\mathcal{N}, \mu)$ has the difference property. Also, it can be seen that the collection of supra α -open subsets of $(\mathcal{N}, \mu)$ coincides with the collection of supra open sets. Hence, $(\mathcal{N}, \mu)$ has the difference property for the collection of supra α -open sets.

Example 3. Let $\mu = \{\emptyset, G \subseteq \mathcal{N} : G \text{ such that } \{1, 2\} \subseteq G \text{ or } \{2, 3\} \subseteq G\}$ be a supra topology on the set of natural numbers $\mathcal{N}$. Then $\{1, 2\} \in \mu$, but $\{1, 2\} \setminus \{2\} = \{1\} \notin \mu$. Therefore (X, μ) does not have the difference property.

Theorem 2.3. If (X, μ) has the difference property for the collection of supra α -open sets, then the following properties hold for $A \subseteq X$.

- (1) $(A^{\alpha'})^{\alpha'} \subseteq A^{\alpha'}$.
- (2) $\alpha cl(A^{\alpha'}) = A^{\alpha'} = (\alpha cl(A))^{\alpha'}$.
- (3) $A^{\alpha'} = \emptyset$ if A is finite.

Proof.

- (1) Let $\lambda \notin A^{\alpha'}$. Then there is a supra α -open set G containing λ such that $G \setminus \{\lambda\} \cap A = \emptyset$. Since (X, μ) has the difference property for the collection of supra α -open sets, then $G \setminus \{\lambda\}$ is a supra α -open set. Therefore $G \setminus \{\lambda\} \cap A^{\alpha'} = \emptyset$. Since $\lambda \notin A^{\alpha'}$, then $G \cap (A^{\alpha'})^{\alpha'} = \emptyset$. Thus $\lambda \notin (A^{\alpha'})^{\alpha'}$. Hence, $(A^{\alpha'})^{\alpha'} \subseteq A^{\alpha'}$.
- (2) Since $(A^{\alpha'})^{\alpha'} \subseteq A^{\alpha'}$, then it follows from Theorem 2.1 that $A^{\alpha'}$ is a supra α -closed set. Therefore

$$(2.3) \quad \alpha cl(A^{\alpha'}) = A^{\alpha'}$$

Also, $(A)^{\alpha'} \subseteq (\alpha cl(A))^{\alpha'}$ because $A \subseteq \alpha cl(A)$. On the other hand, let $\lambda \notin (A)^{\alpha'}$. Then it follows from 1 above that $G \setminus \{\lambda\} \cap A = \emptyset$ and $G \setminus \{\lambda\} \cap A^{\alpha'} = \emptyset$. This means that $G \setminus \{\lambda\} \cap \alpha cl(A) = \emptyset$. Therefore $\lambda \notin (\alpha cl(A))^{\alpha'}$. Thus $(\alpha cl(A))^{\alpha'} \subseteq (A)^{\alpha'}$. Hence

$$(2.4) \quad (\alpha cl(A))^{\alpha'} = (A)^{\alpha'}$$

From (2.3) and (2.4), the desired result is proved.

- (3) Let A be a finite subset of X . Suppose that there exists an element $\lambda \in X$ such that $\lambda \in A^{\alpha'}$. Then for every supra α -open set G containing λ , we

have $G \setminus \{\lambda\} \cap A \neq \emptyset$. Therefore for every $\zeta \in A$ such that $\zeta \neq \lambda$, we have $G \setminus \{\lambda, \zeta\}$ is a supra α -open set. Thus $G \setminus [A \cup \{\lambda\}]$ is a supra α -open set such that $G \setminus [A \cup \{\lambda\}] \cap A = \emptyset$. This implies that $\lambda \notin A^{\alpha'}$. But this is a contradiction. Hence, it must be that $A^{\alpha'} = \emptyset$.

□

We explain that the three properties mentioned in the above theorem need not be true if (X, μ) does not have the difference property for the collection of supra α -open sets. Let $A = \{1, 3\}$ be a subset of supra topological space given in Example 3. Then $A^{\alpha'} = \mathcal{N} \setminus \{1, 3\}$, $(A^{\alpha'})^{\alpha'} = \mathcal{N}$ and $\alpha cl(A^{\alpha'}) = \mathcal{N}$. This leads to the following three properties.

- (1) $(A^{\alpha'})^{\alpha'} \not\subseteq A^{\alpha'}$.
- (2) $\alpha cl(A^{\alpha'}) \neq A^{\alpha'}$.
- (3) $A^{\alpha'} \neq \emptyset$ in spite of A is finite.

3. SEPARATION AXIOMS WITH RESPECT TO SUPRA α -OPEN SETS

In this section, we exploit supra α -open sets to define new types of separation axioms, namely supra α regular, supra α normal and $S\alpha T_i$ -spaces ($i = 0, 1, 2, 3, 4$). We study some equivalent conditions for each one of them and provide some examples to elaborate the relationships between them.

Definition 3.1. A supra topological space (X, μ) is said to be:

- (1) $S\alpha T_0$ if for every $\lambda \neq \zeta \in X$, there exists a supra α -open set containing only one of them.
- (2) $S\alpha T_1$ if for every $\lambda \neq \zeta \in X$, there exist two supra α -open sets one of them contains λ but not ζ and the other contains ζ but not λ .
- (3) supra α Hausdorff (or $S\alpha T_2$) if for every $\lambda \neq \zeta \in X$, there exist two disjoint supra α -open sets U and V containing λ and ζ , respectively.
- (4) supra α regular if for every supra α -closed set F and each $\lambda \notin F$, there exist disjoint supra α -open sets U and V containing F and λ , respectively.
- (5) supra α normal if for every disjoint supra α -closed sets F and H , there exist disjoint supra α -open sets U and V containing F and H , respectively.
- (6) $S\alpha T_3$ (resp. $S\alpha T_4$) if it is both supra α regular (resp. supra α normal) and $S\alpha T_1$.

Theorem 3.1. The following three statements are equivalent:

- (1) (X, μ) is an $S\alpha T_0$ -space;

- (2) $\alpha cl(\{\lambda\}) \neq \alpha cl(\{\zeta\})$ for each $\lambda \neq \zeta \in X$;
 (3) For each $\lambda \in X$, we have $\{\lambda\}^{\alpha'}$ is a union of supra α -closed sets.

Proof. $1 \rightarrow 2$: For each $\lambda \neq \zeta \in X$, there exists a supra α -open set G containing λ but not ζ , or containing ζ but not λ . Say $\lambda \in G$ and $\zeta \notin G$. Then $\lambda \notin \alpha cl(\{\zeta\})$ because G is a supra α -open set containing λ such that $G \cap \{\zeta\} = \emptyset$. Since $\lambda \in \alpha cl(\{\lambda\})$, then $\alpha cl(\{\lambda\}) \neq \alpha cl(\{\zeta\})$.

$2 \rightarrow 3$: Let $\zeta \in \{\lambda\}^{\alpha'}$. Then $\zeta \neq \lambda$ and $\zeta \in \{\lambda\} \cup \{\lambda\}^{\alpha'} = \alpha cl(\{\lambda\})$. Therefore $\alpha cl(\zeta) \subseteq \alpha cl(\{\lambda\})$. Thus $\zeta \in \alpha cl(\zeta) \subseteq \{\lambda\}^{\alpha'}$. Hence, $\{\lambda\}^{\alpha'} = \bigcup \{\alpha cl(\zeta) : \text{for each } \zeta \in \{\lambda\}^{\alpha'}\}$.

$3 \rightarrow 1$: Let $\lambda \neq \zeta$. Then we have two cases:

- (i) Either $\zeta \in \{\lambda\}^{\alpha'}$. Then there is a supra α -closed set F such that $\zeta \in F \subseteq \{\lambda\}^{\alpha'}$. Since $\lambda \notin \{\lambda\}^{\alpha'}$, then $\lambda \notin F$. Therefore F^c is a supra α -open set containing λ such that $\zeta \notin F^c$.
 (ii) Or $\zeta \notin \{\lambda\}^{\alpha'}$. Then there is a supra α -open set G containing ζ such that $\lambda \notin G$.

In the both cases above, we infer that (X, μ) is an $S\alpha T_0$ -space. $\square$

Corollary 3.1. *An $S\alpha T_0$ -space (X, μ) contains at most a supra α dense singleton set ($\{\lambda\}$ is a supra α dense set if $\alpha cl\{\lambda\} = X$).*

Proof. Let (X, μ) be an $S\alpha T_0$ -space. Suppose that there are two distinct singleton set $\{\lambda\}$ and $\{\zeta\}$ such that $\alpha cl(\{\lambda\}) = \alpha cl(\{\zeta\}) = X$. Then (X, μ) is not an $S\alpha T_0$ -space, a contradiction. Hence, (X, μ) contains at most a supra α dense singleton set. $\square$

Theorem 3.2. *The following four statements are equivalent:*

- (1) (X, μ) is an $S\alpha T_1$ -space;
 (2) Every singleton subset of (X, μ) is supra α -closed;
 (3) The intersection of all supra α -open sets containing a set A is exactly A ;
 (4) $\{\lambda\}^{\alpha'} = \emptyset$ for each $\lambda \in X$.

Proof. $1 \rightarrow 2$: Consider (X, μ) is an $S\alpha T_1$ -space and let $\{\lambda\} \subseteq X$. For all $\zeta \in X$ such that $\lambda \neq \zeta$, there exists a supra α -open set G containing ζ such that $G \cap \{\lambda\} = \emptyset$. Then $\zeta \notin \alpha cl(\{\lambda\})$. Therefore, $\alpha cl(\{\lambda\}) = \{\lambda\}$. Thus, $\{\lambda\}$ is a supra α -closed set.

$2 \rightarrow 3$: Let A be a subset of (X, μ) . Then for each $a \in A^c$, we have $\{\lambda\}^c$ is a supra α -open set containing A . Now, $A \subseteq \{G : G \text{ is a supra } \alpha\text{-open set}\}$

containing $A\} \subseteq \{\{\lambda\}^c : \lambda \in A^c\} \subseteq A$. Thus, $A = \{G : G \text{ is a supra } \alpha\text{-open set containing } A\}$, as required.

$3 \rightarrow 4$: Suppose that there exists $\lambda \in X$ such that $\{\lambda\}^{\alpha'} \neq \emptyset$. Then there exists $\zeta \neq \lambda$ such that $\zeta \in \{\lambda\}^{\alpha'}$. Therefore $G \setminus \{\zeta\} \cap \{\lambda\} \neq \emptyset$ for every supra α -open set G containing ζ . This implies that any supra α -open set containing ζ contains λ as well. Thus the intersection of all supra α -open sets containing ζ is not equal $\{\zeta\}$. But this contradicts 3. Hence, it must be $\{\lambda\}^{\alpha'} = \emptyset$.

$4 \rightarrow 1$: Let $\lambda \neq \zeta$. Since $\{\lambda\}^{\alpha'} = \emptyset$ and $\{\zeta\}^{\alpha'} = \emptyset$, then $\{\lambda\}$ and $\{\zeta\}$ are supra α -closed sets. Therefore $\{\lambda\}^c$ and $\{\zeta\}^c$ are supra α -open sets containing $\{\zeta\}$ and $\{\lambda\}$, respectively. Thus, (X, μ) is an $S\alpha T_1$ -space. $\square$

Proposition 3.1. *Every (X, μ) satisfying the difference property for the collection of supra α -open sets is an $S\alpha T_1$ -space.*

Proof. Let $\lambda \neq \zeta \in X$. Since X is a supra α -open set and (X, μ) satisfies the difference property for the collection of supra α -open sets, then $X \setminus \{\lambda\}$ and $X \setminus \{\zeta\}$ are supra α -open sets containing ζ and λ , respectively, such that $a \notin X \setminus \{\lambda\}$ and $b \notin X \setminus \{\zeta\}$. Hence, (X, μ) is an $S\alpha T_1$ -space. $\square$

We show by the following example that the converse of the above proposition is not always true.

Example 4. Let $\mu = \{\emptyset, X, \{\lambda_1, \lambda_2\}, \{\lambda_1, \lambda_3\}, \{\lambda_2, \lambda_3\}, \{\lambda_1, \lambda_2, \lambda_3\}\}$ be a supra topology on $X = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$. Then the collection of all supra α -open subsets of (X, μ) is $\{\emptyset, X, \{\lambda_1, \lambda_2\}, \{\lambda_1, \lambda_3\}, \{\lambda_2, \lambda_3\}, \{\lambda_1, \lambda_2, \lambda_3\}, \{\lambda_1, \lambda_2, \lambda_4\}, \{\lambda_1, \lambda_3, \lambda_4\}, \{\lambda_2, \lambda_3, \lambda_4\}\}$. Therefore (X, μ) is $S\alpha T_1$, because every singleton set is a supra α -closed set. On the other hand, (X, μ) does not satisfy the difference property for the collection of supra α -open sets because $\{\lambda_2, \lambda_3, \lambda_4\}$ is a supra α -open set, but $\{\lambda_2, \lambda_3, \lambda_4\} \setminus \{\lambda_2\}$ is not supra α -open.

We need the following definition to obtain the equivalence between $S\alpha T_0$ and $S\alpha T_1$.

Definition 3.2. (X, μ) is called a supra α symmetric space if $\lambda \in \alpha cl\{\zeta\}$ implies that $\zeta \in \alpha cl\{\lambda\}$ for $\lambda \neq \zeta \in X$.

Theorem 3.3. Let (X, μ) be a supra α symmetric space. Then it is $S\alpha T_1$ iff it is $S\alpha T_0$.

Proof. The necessary condition is obvious.

To prove the sufficient condition, let $\lambda \neq \zeta$. Then there exist a supra α -open set G containing only one of them. Say, $\lambda \in G$ and $\zeta \notin G$. Therefore $\lambda \notin \alpha cl\{\zeta\}$. By the supra α symmetry of (X, μ) , we have $\zeta \notin \alpha cl\{\lambda\}$. Thus $(\alpha cl\{\lambda\})^c$ is a supra α -open set containing ζ . Hence, (X, μ) is $S\alpha T_1$. $\square$

Theorem 3.4. *The following three statements are equivalent:*

- (1) (X, μ) is an $S\alpha T_2$ -space;
- (2) $\{\lambda\} = \bigcap \{F_i : F_i \text{ is a supra } \alpha\text{-closed neighborhood of } \lambda\}$ for each $\lambda \in X$;
- (3) The diagonal $\Delta = \{(\lambda, \lambda) : \lambda \in X\}$ is supra α -closed in the product supra space $X \times X$.

Proof. $1 \rightarrow 2$: Consider (X, μ) is an $S\alpha T_2$ -space. Then for $\lambda \neq \zeta$, there exist two disjoint supra α -open sets G_i and H_i such that $\lambda \in G_i$ and $\zeta \in H_i$. Obviously, $G_i \subseteq H_i^c$. Therefore $\lambda \in \alpha cl(G_i) \subseteq H_i^c = F_i$. Thus, F_i is a supra α -closed neighborhood of λ such that $\zeta \notin F_i$. Hence, $\{\lambda\} = \bigcap \{F_i : F_i \text{ is a supra } \alpha\text{-closed neighborhood of } \lambda\}$.

$2 \rightarrow 1$: To prove that (X, μ) is an $S\alpha T_2$ -space, let $\lambda \neq \zeta$. Since $\{\lambda\} = \bigcap \{F_i : F_i \text{ is a supra } \alpha\text{-closed neighborhood of } \lambda\}$, then there exists a supra α -closed neighborhood F_{i_0} of λ such that $\zeta \notin F_{i_0}$. Therefore there exists a supra α -open set G containing λ such that $\lambda \in \alpha cl(G) \subseteq F_{i_0}$. It is clear that $(\alpha cl(G))^c$ is a supra α -open set containing ζ and $G \cap (\alpha cl(G))^c = \emptyset$. Hence, (X, μ) is an $S\alpha T_2$ -space.

$1 \rightarrow 3$: Suppose that (X, μ) is $S\alpha T_2$ and let $(\lambda, \zeta) \in X \times X - \Delta$. Then $\lambda \neq \zeta$. Therefore there exist two disjoint supra α -open sets G and H containing λ and ζ , respectively. Thus, $(\lambda, \zeta) \in G \times H \subseteq X \times X - \Delta$, proving that $X \times X - \Delta$ is a supra α neighbourhood of any of its points. Hence, Δ is supra α -closed.

$3 \rightarrow 1$: Suppose that Δ is a supra α -closed subset of $X \times X$ and let $\lambda \neq \zeta \in X$. Then $X \times X - \Delta$ is a supra α -open set containing (λ, ζ) . Therefore there exist two supra α -open subsets G and H of (X, μ) such that $(\lambda, \zeta) \in G \times H \subseteq X \times X - \Delta$. This implies that G and H are two disjoint supra α -open sets containing λ and ζ , respectively. Hence, (X, μ) is $S\alpha T_2$. $\square$

Theorem 3.5. *The the following three statements are equivalent:*

- (1) (X, μ) is supra α regular;
- (2) For each supra α -open subset U of (X, μ) containing λ , there exists a supra α -open subset V of (X, μ) such that $\lambda \in V \subseteq \alpha cl(V) \subseteq U$;
- (3) Every supra α -open subset U of (X, μ) can be represented as follows: $U = \bigcup \{H : H \text{ is a supra } \alpha\text{-open subset of } (X, \mu) \text{ and } \alpha cl(H) \subseteq U\}$.

Proof. $1 \rightarrow 2$: Let (X, μ) be a supra α regular space and U be a supra α -open set such that $\lambda \in U$. Then there exist disjoint supra α -open sets V and W containing λ and U^c , respectively. Therefore $\lambda \in V \subseteq W^c \subseteq U$. Thus $\lambda \in V \subseteq \alpha cl(V) \subseteq U$.

$2 \rightarrow 3$: Suppose that U is a supra α -open set. By hypothesis, for each $\lambda \in U$, there exists a supra α -open set H such that $\lambda \in H \subseteq \alpha cl(H) \subseteq U$. Then $U = \bigcup \{H : H \text{ is supra } \alpha\text{-open and } \alpha cl(H) \subseteq U\}$.

$3 \rightarrow 1$: Let F be a supra α -closed set such that $\lambda \notin F$. Then $F^c = \bigcup \{H : H \text{ is supra } \alpha\text{-open and } \alpha cl(H) \subseteq F^c\}$. Since $\lambda \in F^c$, then there exists a supra α -open set H_λ containing λ such that $\alpha cl(H_\lambda) \subseteq F^c$. Take $V = (\alpha cl(H_\lambda))^c$. Then V is a supra α -open set containing F and $V \cap H_\lambda = \emptyset$. This completes the proof. $\square$

Theorem 3.6. *Consider (X, μ) is a supra α regular space. Then the following concepts are equivalent:*

- (1) (X, μ) is an $S\alpha T_2$ -space;
- (2) (X, μ) is an $S\alpha T_1$ -space;
- (3) (X, μ) is an $S\alpha T_0$ -space.

Proof. The implications $1 \rightarrow 2 \rightarrow 3$ are obvious.

$3 \rightarrow 1$: Let $\lambda, \zeta \in X$ such that $\lambda \neq \zeta$. Since (X, μ) is an $S\alpha T_0$ -space, then from Theorem (3.1), we get $\alpha cl\{\lambda\} \neq \alpha cl\{\zeta\}$. Therefore $\lambda \notin \alpha cl\{\zeta\}$ or $\zeta \notin \alpha cl\{\lambda\}$. Say, $\lambda \notin \alpha cl\{\zeta\}$. Since (X, μ) is supra α regular, then there exist disjoint supra α -open sets G and H containing λ and $\alpha cl\{\zeta\}$, respectively. Thus (X, μ) is an $S\alpha T_2$ -space. $\square$

Theorem 3.7. *The following statements are equivalent:*

- (1) (X, μ) is supra α normal;
- (2) For each supra α -closed set F and each supra α -open set U containing F , there exists a supra α -open set V such that $F \subseteq V \subseteq \alpha cl(V) \subseteq U$;
- (3) For every supra α -open sets U and V such that $U \cup V = X$, there are two supra α -closed sets F and H contained in U and V , respectively, such that $F \cup H = X$.

Proof. $1 \rightarrow 2$: Consider (X, μ) is supra α normal and F is a supra α -closed subset of a supra α -open set U . Then U^c and F are disjoint supra α -closed sets. Therefore there exist two disjoint supra α -open sets W and V containing U^c and F , respectively. Thus $F \subseteq V \subseteq W^c = \alpha cl(W^c) \subseteq U$. Hence, $F \subseteq V \subseteq \alpha cl(V) \subseteq U$.

$2 \rightarrow 3$: Consider U and V are supra α -open sets such that $U \cup V = X$. Then U^c is a supra α -closed sets such that $U^c \subseteq V$. By 2, there is a supra α -open set

G such that $U^c \subseteq G \subseteq \alpha cl(G) \subseteq V$. Thus $G^c \subseteq U$ and $\alpha cl(G) \subseteq V$ are supra α -closed sets such that $G^c \cup \alpha cl(G) = X$.

$3 \rightarrow 1$: Consider F and H are disjoint supra α -closed sets. Since F^c and H^c are supra open sets such that $F^c \cup H^c = X$, then there are two supra α -closed sets M and N such that $M \subseteq F^c$, $N \subseteq H^c$ and $M \cup N = X$. Thus M^c and N^c are two disjoint supra α -open sets containing F and H , respectively. Hence, (X, μ) is supra α normal. $\square$

Now, we show the implications of these separation axioms among themselves as well as with ST_i -space.

It should be noted that the concepts of ST_i -space which were defined by replacing 'supra α -open' by 'supra open' in Definition 3.1, see, [4, 29].

Theorem 3.8. *Every $S\alpha T_i$ -space is $S\alpha T_{i-1}$ for $i = 1, 2, 3, 4$.*

Converse of this theorem is not necessary true as it is seen in the following examples.

Example 5. Let $\mu = \{\emptyset, X, \{\lambda_1\}\}$ be a supra topology on $X = \{\lambda_1, \lambda_2, \lambda_3\}$. Then the collection of all supra α -open subsets of (X, μ) is $\{\emptyset, X, \{\lambda_1\}, \{\lambda_1, \lambda_2\}, \{\lambda_1, \lambda_3\}\}$. Therefore (X, μ) is not an $S\alpha T_1$ -space, because $\lambda_1 \neq \lambda_2$ and every supra α -open set containing λ_2 contains λ_1 as well. On the other hand, it can be checked that (X, μ) is $S\alpha T_0$.

Example 6. Assume that (X, μ) is the same as in Example 4. Then (X, μ) is $S\alpha T_1$. On the other hand, (X, μ) is not an $S\alpha T_2$ -space, because $\lambda_3 \neq \lambda_4$ and there do not exist disjoint supra α -open sets such that one of them contains λ_3 and the other contains λ_4 .

Example 7. Let $\mu = \{\emptyset, X, \{\lambda_1, \lambda_2\}, \{\lambda_3, \lambda_4\}, \{\lambda_1, \lambda_3\}, \{\lambda_2, \lambda_4\}, \{\lambda_2, \lambda_3\}, \{\lambda_1, \lambda_2, \lambda_3\}, \{\lambda_1, \lambda_2, \lambda_4\}, \{\lambda_1, \lambda_3, \lambda_4\}, \{\lambda_2, \lambda_3, \lambda_4\}\}$ be a supra topology on $X = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$. In (X, μ) , every set is supra open iff it is supra α -open. Now, $\{\lambda_1, \lambda_4\}$ is a supra α -closed set and $\lambda_2 \notin \{\lambda_1, \lambda_4\}$. Since there do not exist two disjoint supra α -open sets such that one of them contains λ_2 and the other contains $\{\lambda_1, \lambda_4\}$, then (X, μ) is not $S\alpha T_3$. On the other hand, it can be checked that (X, μ) is $S\alpha T_2$.

Example 8. Let $\mu = \{\emptyset, X, \{\lambda_2\}, \{\lambda_4\}, \{\lambda_2, \lambda_4\}, \{\lambda_1, \lambda_3\}, \{\lambda_1, \lambda_4\}, \{\lambda_1, \lambda_5\}, \{\lambda_2, \lambda_3\}, \{\lambda_2, \lambda_5\}, \{\lambda_3, \lambda_5\}, \{\lambda_4, \lambda_5\}, \{\lambda_1, \lambda_2, \lambda_3\}, \{\lambda_1, \lambda_2, \lambda_4\}, \{\lambda_1, \lambda_2, \lambda_5\}, \{\lambda_1, \lambda_3, \lambda_4\}, \{\lambda_1, \lambda_3, \lambda_5\}, \{\lambda_1, \lambda_4, \lambda_5\}, \{\lambda_2, \lambda_3, \lambda_4\}, \{\lambda_2, \lambda_3, \lambda_5\}, \{\lambda_2, \lambda_4, \lambda_5\}, \{\lambda_3, \lambda_4, \lambda_5\}, \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}, \{\lambda_1, \lambda_2, \lambda_3, \lambda_5\}, \{\lambda_1, \lambda_2, \lambda_4, \lambda_5\}, \{\lambda_1, \lambda_3, \lambda_4, \lambda_5\}, \{\lambda_2, \lambda_3, \lambda_4, \lambda_5\}\}$ be a supra topology on $X = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$. In (X, μ) , every set is supra open iff it is supra

α -open. Now, $\{\lambda_1, \lambda_2\}$ and $\{\lambda_3, \lambda_4\}$ are disjoint supra α -closed subsets of (X, μ) . Since there do not exist two disjoint supra α -open sets such that one of them contains $\{\lambda_1, \lambda_2\}$ and the other contains $\{\lambda_3, \lambda_4\}$, then (X, μ) is not supra α normal. Therefore it is not $S\alpha T_4$. On the other hand, one can check that (X, μ) is $S\alpha T_3$.

Theorem 3.9. Every ST_i -space (X, μ) is $S\alpha T_i$ for $i = 0, 1, 2$.

Proof. It follows from the fact that every supra open set is supra α -open. $\square$

Converse of this theorem is not necessary true as it is seen in the following examples.

Example 9. Assume that (X, μ) is the same as in Example 5. Then (X, μ) is not an ST_0 -space. On the other hand, the collection of all supra α -open subsets of (X, μ) is $\{\emptyset, X, \{\lambda_1\}, \{\lambda_1, \lambda_2\}, \{\lambda_1, \lambda_3\}\}$. Hence, (X, μ) is $S\alpha T_0$.

Example 10. Let $\mu = \{\emptyset, X, \{\lambda_1\}, \{\lambda_2\}, \{\lambda_1, \lambda_2\}\}$ be a supra topology on $X = \{\lambda_1, \lambda_2, \lambda_3\}$. Then (X, μ) is not an ST_1 -space. On the other hand, the collection of all supra α -open subsets of (X, μ) is $\{\emptyset, X, \{\lambda_1\}, \{\lambda_2\}, \{\lambda_1, \lambda_2\}, \{\lambda_1, \lambda_3\}, \{\lambda_2, \lambda_3\}\}$. Now, it can be checked that (X, μ) is $S\alpha T_4$.

We complete this section by discussing these separation axioms in terms of hereditary and topological properties and finite product space.

Definition 3.3. For a nonempty subset A of (X, μ) , the family $\mu_A = \{A \cap G : G \text{ is a supra } \alpha\text{-open subset of } (X, \mu)\}$ is called a relative α -topology on A . A pair (A, μ_A) is called an α -subspace of (X, μ) .

One can easily prove that an α -subspace (A, μ_A) of (X, μ) is a supra topological space.

Proposition 3.2. Let (Y, μ_Y) be an α -subspace of (X, μ) . A subset H of Y is supra α -closed in (Y, μ_Y) iff there exists a supra α -closed subset F of (X, μ) such that $H = Y \cap F$.

Proof. *Necessity:* Let H be a supra α -closed subset of (Y, μ_Y) . Then there exists a supra α -open subset W of (Y, μ_Y) such that $H = Y \setminus W$. Now, there exists a supra α -open subset V of (X, μ) such that $W = Y \cap V$. Therefore $H = Y \setminus (Y \cap V) = Y \cap V^c$. By taking $F = V^c$, the proof of the *necessary* part is complete.

Sufficiency: Let $H = Y \cap F$ such that F is a supra α -closed subset of (X, μ) . Then $Y \setminus H = Y \setminus (Y \cap F) = (Y \cap X) \setminus (Y \cap F) = Y \cap (X \setminus F)$. Since $X \setminus F$ is a supra α -open subset of (X, μ) , then $Y \setminus H$ is a supra α -open subset of (Y, μ_Y) . Thus H is a supra α -closed subset of (Y, μ_Y) . $\square$

Definition 3.4. A property is said to be a relative α -hereditary property if the property passes from a supra topological space to every relative α -subspace.

Theorem 3.10. A property of being an $S\alpha T_i$ -space is a relative α -hereditary for $i = 0, 1, 2, 3$.

Proof. We prove the theorem in the case of $i = 3$ and the other cases follow similar lines.

Suppose that (A, μ_A) is a relative α -subspace of an $S\alpha T_3$ -space (X, μ) . We first show that (A, μ_A) is an $S\alpha T_1$ -space. Let $\lambda \neq \zeta \in A \subseteq X$. Then there are two supra α -open subsets U and V of (X, μ) containing λ and ζ , respectively, such that $\lambda \notin V$ and $\zeta \notin U$. Now, $G = A \cap U$ and $H = A \cap V$ are two supra α -open subsets of (A, μ_A) containing λ and ζ , respectively, such that $\lambda \notin H$ and $\zeta \notin G$. Thus, (A, μ_A) is $S\alpha T_1$. Second, we show that (A, μ_A) is supra α regular. Let H be a supra α -closed subset of (A, μ_A) and $\lambda \in A$ such that $\lambda \notin H$. It follows from Proposition 3.2 that there is a supra α -closed subset F of (X, μ) such that $H = F \cap A$. Since $\lambda \notin F$, then there exist disjoint supra α -open subsets U and V of (X, μ) containing F and λ , respectively. Now, $M = U \cap A$ and $N = V \cap A$ are disjoint supra α -open subsets of (A, μ_A) containing H and λ , respectively. Thus (A, μ_A) is supra α regular. Hence, the proof is complete. $\square$

Proposition 3.3. Let $g : (X, \mu) \rightarrow (Y, \theta)$ be an injective supra α -continuous map. If (Y, θ) is T_i , then (X, μ) is $S\alpha T_i$ for $i = 0, 1, 2$.

Proof. We only prove the proposition in the case of $i = 2$ and the other cases can be made similarly.

Let $\lambda \neq \zeta \in X$. Then, it follows from the injectivity of g , that there are $x \neq y \in Y$ such that $x = f(\lambda)$ and $y = f(\zeta)$. Since (Y, θ) is T_2 , then there are two disjoint open subsets U and V of (Y, θ) containing x and y , respectively. Now, $g^{-1}(U)$ and $g^{-1}(V)$ are disjoint supra α -open subsets of (X, μ) containing λ and ζ , respectively. Hence, (X, μ) is $S\alpha T_2$, as required. $\square$

In a similar way, one can prove the following results.

Proposition 3.4. Let $g : (X, \tau) \rightarrow (Y, \nu)$ be a bijective supra α -open map. If (X, τ) is T_i , then (Y, ν) is $S\alpha T_i$ for $i = 0, 1, 2$.

Proposition 3.5. Let $g : (X, \tau) \rightarrow (Y, \nu)$ be an injective supra α^* -continuous map. If (X, τ) is $S\alpha T_i$, then (Y, ν) is $S\alpha T_i$ for $i = 0, 1, 2$.

Proposition 3.6. Let $g : (X, \tau) \rightarrow (Y, \nu)$ be a bijective supra α^* -open map. If (X, τ) is $S\alpha T_i$, then (Y, ν) is $S\alpha T_i$ for $i = 0, 1, 2$.

Proposition 3.7. Let $g : (X, \tau) \rightarrow (Y, \nu)$ be a supra α^* -homeomorphism map. Then (X, τ) is $S\alpha T_i$ iff (Y, ν) is $S\alpha T_i$ for $i = 0, 1, 2, 3, 4$.

Theorem 3.11. A and B are supra α -open sets iff the product of them is supra α -open.

Proof. Let A and B be two supra α -open sets. Then $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$ and $B \subseteq \text{int}(\text{cl}(\text{int}(B)))$. So $A \times B \subseteq \text{int}(\text{cl}(\text{int}(A))) \times \text{int}(\text{cl}(\text{int}(B))) = \text{int}(\text{cl}(\text{int}(A \times B)))$. Thus $A \times B$ is a supra α -open set. Conversely, let $A \times B$ be a supra α -open set. Then $A \times B \subseteq \text{int}(\text{cl}(\text{int}(A \times B))) = \text{int}(\text{cl}(\text{int}(A))) \times \text{int}(\text{cl}(\text{int}(B)))$. So $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$ and $B \subseteq \text{int}(\text{cl}(\text{int}(B)))$. Hence, the proof is complete. $\square$

Theorem 3.12. The finite product of $S\alpha T_i$ -spaces is $S\alpha T_i$ for $i = 0, 1, 2$.

Proof. We prove the theorem for two supra topological spaces (X, μ) and (Y, ν) in the case of $i = 2$. One can prove the other cases similarly.

Let $(X \times Y, T)$ be the product supra space of (X, μ) and (Y, ν) . Suppose that $(\lambda_1, \zeta_1) \neq (\lambda_2, \zeta_2)$. Then either $\lambda_1 \neq \lambda_2$ or $\zeta_1 \neq \zeta_2$. Without loss of generality, suppose that $\lambda_1 \neq \lambda_2$. Therefore there exist two disjoint supra α -open subsets U and V of (X, μ) containing λ_1 and λ_2 , respectively. It follows from Theorem (3.11) that $U \times Y$ and $V \times Y$ are two supra α -open subsets of $(X \times Y, T)$ containing (λ_1, ζ_1) and (λ_2, ζ_2) , respectively, such that $(U \times Y) \widetilde{\cap} (V \times Y) = \emptyset$. Hence, $(X \times Y, T)$ is $S\alpha T_2$. $\square$

Definition 3.5. Let (X, μ) and (Y, ν) be two supra topological spaces and $(X \times Y, T)$ be their product supra space such that $\mathcal{C}_1$ and $\mathcal{C}_2$ are the collections of all supra α -open subsets of (X, μ) and (Y, ν) , respectively. Then $\beta = \{G \times H : G \in \mathcal{C}_1 \text{ and } H \in \mathcal{C}_2\}$ defines a basis for a supra topology $\mathcal{C}$ on $X \times Y$. We called $(X \times Y, \mathcal{C})$ an α -finite product supra space.

Lemma 3.1. Let (X, μ) and (Y, ν) be two supra topological spaces and $(X \times Y, \mathcal{C})$ be their α -product supra space. If E is a supra closed subset of $(X \times Y, \mathcal{C})$, then $E = \bigcap_{i \in I} [(F_i \times Y) \cup (X \times H_i)]$, where F_i and H_i are supra α -closed subsets of (X, μ) and (Y, ν) , respectively.

Theorem 3.13. The α -finite product of $S\alpha T_i$ -spaces is ST_i for $i = 0, 1, 2, 3$.

Proof. We prove the theorem for two supra topological spaces (X, μ) and (Y, ν) in the case of $i = 3$. One can prove the other cases similarly.

Let $(X \times Y, \mathcal{C})$ be the α -product supra space of (X, μ) and (Y, ν) . We first prove that $(X \times Y, \mathcal{C})$ is ST_1 . Suppose that $(\lambda_1, \zeta_1) \neq (\lambda_2, \zeta_2)$. Then either $\lambda_1 \neq \lambda_2$ or

$\zeta_1 \neq \zeta_2$. Without loss of generality, suppose that $\lambda_1 \neq \lambda_2$. Therefore there exist two supra α -open subsets U and V of (X, μ) containing λ_1 and λ_2 , respectively. According to Definition(3.5), $U \times Y$ and $V \times Y$ are two supra open subsets of $(X \times Y, \mathcal{C})$ containing (λ_1, ζ_1) and (λ_2, ζ_2) such that $(\lambda_1, \zeta_1) \notin V \times Y$ and $(\lambda_2, \zeta_2) \notin U \times Y$. Hence, $(X \times Y, \mathcal{C})$ is ST_1 . Second, we prove that $(X \times Y, \mathcal{C})$ is supra regular. Suppose that $(\lambda, \zeta) \in X \times Y$ and E is a supra closed subset of $(X \times Y, \mathcal{C})$ such that $(\lambda, \zeta) \notin E = \bigcap_{i \in I} [(F_i \times Y) \cup (X \times H_i)]$, where F_i and H_i are supra α -closed subsets of (X, μ) and (Y, ν) , respectively. Then there exists $j \in I$ such that $(\lambda, \zeta) \notin [(F_j \times Y) \cup (X \times H_j)]$. This means that $\lambda \notin F_j$ and $\zeta \notin H_j$. Since (X, μ) and (Y, ν) are supra α regular, then there exist disjoint supra α -open subsets U and V of (X, μ) containing λ and F_j , respectively, and there exist disjoint supra α -open subsets M and N of (Y, ν) containing ζ and H_j , respectively. Therefore $U \times M$ and $[(V \times Y) \cup (X \times N)]$ are two supra open subsets of $(X \times Y, \mathcal{C})$ containing (λ, ζ) and $[(F_j \times Y) \cup (X \times H_j)]$, respectively. Obviously, $E \subseteq [(F_j \times Y) \cup (X \times H_j)]$ and $(U \times M) \cap [(V \times Y) \cup (X \times N)] = \emptyset$. Thus, $(X \times Y, \mathcal{C})$ is supra regular. Hence, the proof is complete. $\square$

4. CONCLUSION

There are many generalizations of topological spaces which help us to picture and satisfy many topological properties under fewer conditions such as supra topology, minimal structure and generalized topology. This work is devoted to introducing and discussing the concepts of limits points of a set and separation axioms with respect to α -open sets. We have established their main properties and provided some examples to show the obtained results. From the concrete thoughts given in this work, it can be done more investigations on the theoretical parts of these generalized ideas which is valuable by studying the following themes:

- (1) Define weak types of supra α regular and supra α normal spaces.
- (2) Study $S\alpha T_i$ -spaces for $(i = \frac{1}{2}, 2\frac{1}{2}, 3\frac{1}{2}, 5)$
- (3) Explore the concepts introduced herein using the classes of supra b -open and supra β -open sets.
- (4) Investigate of the possibility of applying these concepts on information system, especially, separation axioms.

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