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ON COLOURABILITY OF HYPERGRAPHS

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ABSTRACT. This article encompasses around the concept of colourability in hypergraphs. The deletion-contraction attribute frequently observed in 2-graph has also come in connection with hypergraphs leading to a certain kind of structure theorem. The decomposition theorem on alternating polynomials and the inductional way of revealment used in the chromatic polynomials lead us to some elegant formations.

1. INTRODUCTION

A figurative analysis basically motivates us to give theoretical approach of various aspects of chromatic polynomials whether it is weakly chromatic polynomial or strongly chromatic polynomial, its factorisation, quotient structure etc. Structure theory of a mathematical idea reveals some interesting elementary phenomena of the concept in discussion and such phenomena finally becomes a responsible factor in exploring the importance of the idea imposed on. Motivated by this aspect of chromatic polynomials, we would try some types of factorisations of strongly chromatic polynomial and weakly chromatic polynomial.

The topic of graph colouring is one of the most enthralling classical research areas in graph theory. There has been a capacious application of this topic in

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various fields. Large scheduling [7], assignment of radio frequency [5, 13], separating combustible chemical combinations [11] are some of the spheres where the exercise of colourability is extensively perpetrated. The deletion-contraction paradigm shift in hypergraph from graph in this article is a distinguished feature. The concept of hypergraph was first introduced by C. Berge [2] in his book Graphs and Hypergraphs, and developed many properties of vertex colourings of a hypergraph. In contrast to the prevailing notation of vertex colouring, we acquire two types of vertex colourings in a hypergraph viz., weak vertex colouring and strong vertex colouring. Keeping on par with the important fundamental theorem relating to the chromatic polynomials in case of a graph we come upon a general perspective of the said theorem in case of hypergraph too, that also is named as the fundamental theorem. In case of a hypergraph, the derivation of this theorem leads us to an elegant algorithmic figurative description of the theorem. It finally coincides with the graph theoretic fundamental version already mentioned that may be claimed as an elaborate analytical presentation. Some proximate form of decomposition theorems of chromatic polynomials in case of a hypergraph is considered which in turn leads to an intricate quotient structure of such a chromatic polynomial. The permutational application to prove that the chromatic polynomial of a hypergraph is equal to the sum of the chromatic polynomial of the hypergraph after deletion and contraction is an elemental one. A brief tabular representation of the Pre–basic model discussion of strong vertex colouring is shown. In this table, the main emphasis is given on size 0, 1 and 2 of a hypergraph together with the order 0, 1, 2, m etc. and also on disjoint edges of hypergraphs so as to have a look into the changes in the formation of the chromatic polynomials hence calculated. The method of induction is used to prove most of the theorems giving an overall, between the lines understanding of the various notions that are present and used therein.

2. Preliminaries

Definition 2.1. *Hypergraph: [1] A hypergraph H denoted by $H = (V, E = (e_i)_{i \in I})$ on a finite set V is a family $(e_i)_{i \in I}$ (I is a finite set of indexes) of subsets of V called hyperedges. Sometimes V is denoted by $V(H)$ and E by $E(H)$.*

Definition 2.2. *Order of a Hypergraph: [1] The order of a hypergraph $H = (V, E)$ is the cardinality of V i.e. $|V| = n$.*

Definition 2.3. *Size of a Hypergraph: [1] The size of a hypergraph $H = (V, E)$ is the cardinality of E i.e., $|E| = m$.*

Definition 2.4. *λ- colouring of a Hypergraph (or, Proper vertex colouring): [1] For a hypergraph H and $\lambda \geq 2, \lambda \in N$, a λ- colouring of the vertices of H is an allocation of colours to the vertices such that:*

(i) *A vertex has just one colour.*
(ii) *We use λ colours to colour the vertices.*
(iii) *No hyperedge with cardinality more than 1 is monochromatic.*

Definition 2.5. *Chromatic number of a Hypergraph: [1] The chromatic number $\chi(H) = H$ of a hypergraph H is the smallest λ such that H has a λ- colouring.*

Definition 2.6. *Strong vertex colouring of a Hypergraph: [8] A strong vertex colouring of a hypergraph $H = (V, E)$ is a map $\psi : V(H) \to \mathbb{N}$ such that whenever $u, v \in e$ for some $e \in E(H)$, we have $\psi(u) \neq \psi(v)$.*

In other words, the strong vertex colouring of a hypergraph H is such a vertex colouring, so that, no two vertices of an edge possess the same colour.

Definition 2.7. *Weak vertex colouring of a Hypergraph: [3] A weak vertex colouring or a weak λ-colouring of a hypergraph $H = (V, E)$ is a labelling of its vertices V with the colours from the set $\{1, 2, \ldots, \lambda\}$ in such a way that every edge $e_i \in E$ such that if $|e_i| \geq 2$ has atleast two vertices coloured differently.*

Definition 2.8. *Chromatic polynomial of a Graph: [10] For any graph $G = (V, E)$, the chromatic polynomial of G is the function $P(G, \lambda)$ such that for any positive integer λ, $P(G, \lambda)$ denotes the number of proper λ-colourings of G.*

Definition 2.9. *Strongly chromatic polynomial of a Hypergraph: [9] Strongly chromatic polynomial in a hypergraph H is the function $P_s(H, \lambda)$ such that for any positive integer λ, $P_s(H, \lambda)$ is the number of strongly vertex colouring of H.*

Definition 2.10. *Weakly chromatic polynomial of a Hypergraph: [9] Weakly chromatic polynomial in a hypergraph H is the function $P_w(H, \lambda)$ such that for any positive integer λ, $P_w(H, \lambda)$ is the number of weak vertex colouring of H. For example, let a hypergraph $H = (V, E)$ where $V = \{a, b, c, d\}$ and $E = \{E_1 = \{a, b, c\}, E_2 = \{a, d\}\}$ be given as in the following figure 1.*

Then $P_s(H, \lambda) = [\lambda(\lambda - 1)(\lambda - 2)](\lambda - 1) = \lambda(\lambda - 1)^2(\lambda - 2)$ and $P_w(H, \lambda) = (\lambda^3 - \lambda)(\lambda - 1)$.

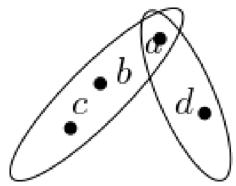


FIGURE 1

Definition 2.11. *r-uniform hypergraph: A hypergraph $H = (V, E)$ is r-uniform if each hyperedge contains precisely r-vertices.*

Definition 2.12. *r-uniform q-edge hypergraph: A hypergraph $H = (V, E)$ is r-uniform q-edge hypergraph if the size of the hypergraph H is q and each of its edges has r-vertices.*

Definition 2.13. *Edge-tree hypergraph: [6] A hypergraph $H = (X, E)$ is called an edge-tree hypergraph if there exists a tree T whose set of edges is E, such that every $e \in E$ is a path in T.*

Definition 2.14. *Connected component of a hypergraph: [4] A connected component of a hypergraph is a maximal set of vertices which are pair wise connected by a non-trivial path. Thus if $H_1, H_2, \ldots, H_k$ are k connected components of a hypergraph H then $H_i \cap H_j = \phi$, for $i \neq j$ and $H = H_1 \cup H_2 \cup \ldots \cup H_k$.*

Definition 2.15. *Monic polynomial: [12] A monic polynomial is a single-variable polynomial in which the leading coefficient (the non-zero coefficient of highest degree) is equal to 1.*

$$P(x) = x^n + a_{n-1}x^{n-1} + ... + a_1x + a_0.$$

Definition 2.16. *Alternating monic polynomial: A monic polynomial is alternating if its terms are alternatively positive and negative beginning with the first term as positive.*

$$P(t) = t^n + a_{n-1}t^{n-1} + ..., [a_i \in \mathbb{Z}^+].$$

Definition 2.17. *Edge deleted graph: In case of a graph $G = (V, E)$ where $e \in E$, the deletion of e from G, denoted $G-e$, is defined as the graph $G-e = (V, E-\{e\})$.*

Definition 2.18. *Edge contracted graph: In case of a graph $G = (V, E)$ where $e \in E$, the contraction of $e(= u, v)$ in G, denoted G/e, is defined as the graph $G/e = (V(u$ and v are merged together), $E - \{e\})$.*

Definition 2.19. *Point hyperpath: A point hyperpath in a hypergraph $H = (V, E)$ between two distinct vertices v_1 and v_2 is a sequence $v_1E_1v_2E_2 \ldots v_{n-1}E_{n-1}v_n$, where*

(i) *n is a positive integer.*
(ii) *$v_1, \ldots, v_n$ are distinct integers.*
(iii) *$E_i \cap E_{i+1} \neq \phi$.*

Definition 2.20. *Deletion-Contraction property: [10] The two definitions of edge deleted graph and edge contracted graph together are often referred as the Deletion-Contraction property.*

In the following three figure 2, (i) is the graph G containing the edge $e = (u, v)$, (ii) is the deleted graph $G - e$ with deletion of edge e, and (iii) is the contracted graph G/e with the edge e deleted and the vertices u and v merged together.

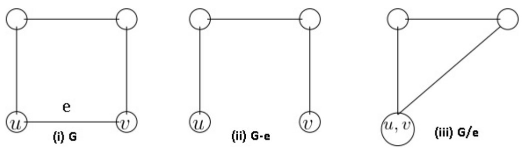


FIGURE 2

2.1. **Numeric illustrative presentation of graph colouring.** As an illustration of a graph with 4 vertices a, b, c, d with 4 distinct colors Red, Purple, Orange and Blue each of them corresponding to the numbers $1, 2, 3, 4$ respectively where each of the vertices takes distinct colours appears as in Figure 3,

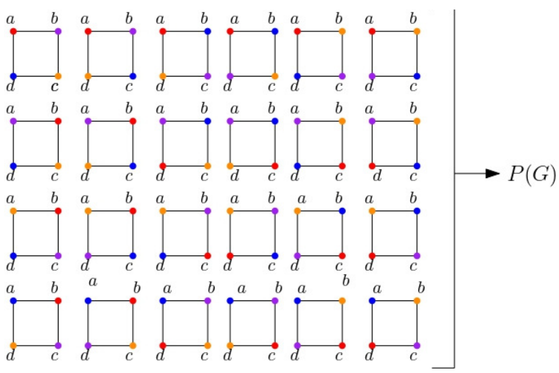


FIGURE 3

Here, $P(G)$ stands for number of colourings of G with the condition that no two adjacent vertices have the same colour.

Its corresponding numeric representation is given in Figure 4,

1 2 3 4	2 1 3 4	3 1 2 4	4 1 2 3
1 2 4 3	2 1 4 3	3 1 4 2	4 1 3 2
1 3 2 4	2 4 3 1	3 2 1 4	4 2 3 1
1 3 4 2	2 4 1 3	3 2 4 1	4 2 1 3
1 4 2 3	2 3 1 4	3 4 1 2	4 3 1 2
1 4 3 2	2 3 4 1	3 4 2 1	4 3 2 1

→ (1)

FIGURE 4

When one edge is deleted its possible colourings with the condition that no adjacent vertices takes the same colour appears as in Figure 5,

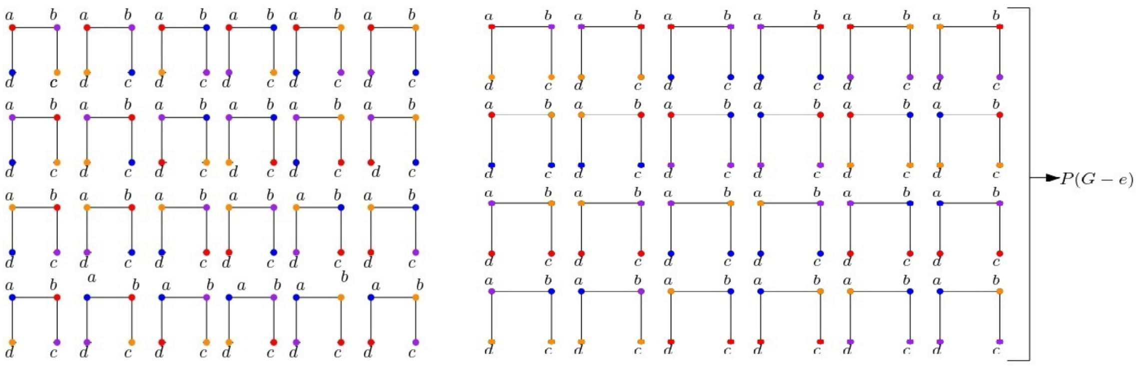


FIGURE 5

Its numeric representation together with (1) will be as in Figure 6.

1 2 3 3	1 3 4 4	2 3 1 1	2 4 3 3
2 1 3 3	3 1 4 4	3 2 1 1	4 2 3 3
1 2 4 4	1 4 2 2	2 3 4 4	3 4 1 1
2 1 4 4	4 1 2 2	3 2 4 4	4 3 1 1
1 3 2 2	1 4 3 3	2 4 1 1	3 4 2 2
3 1 2 2	4 1 3 3	4 2 1 1	4 3 2 2

FIGURE 6

When the two end vertices after deletion of any one of the edges, say (here), (c, d) are glued then it appears in Figure 7.

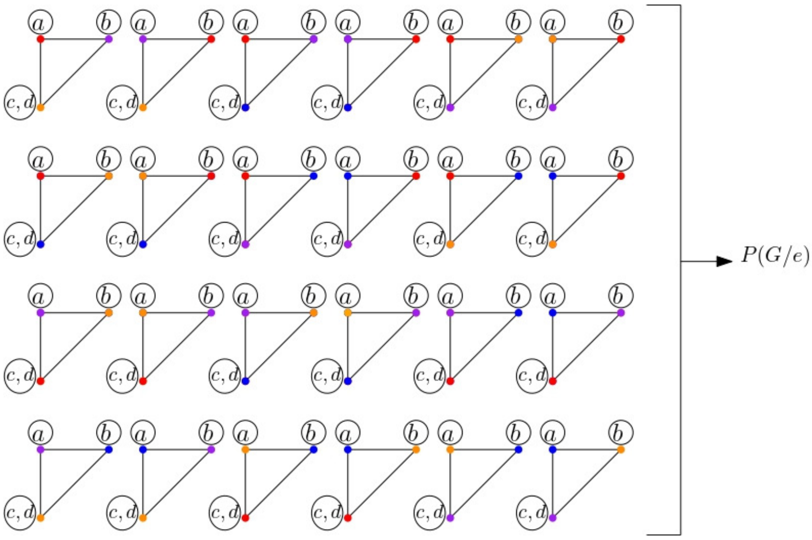


FIGURE 7

And its corresponding numeric presentation is as in Figure 8.

1	2	3	1	3	4	2	3	1	2	4	3
2	1	3	3	1	4	3	2	1	4	2	3
1	2	4	1	4	2	2	3	4	3	4	1
2	1	4	4	1	2	3	2	4	4	3	1
1	3	2	1	4	3	2	4	1	3	4	2
3	1	2	4	1	3	4	2	1	4	3	2

FIGURE 8

And clearly it appears that,

$$P_s(G-e) = P_s(G) + P_s(G/e),$$

or,

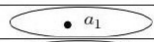
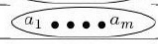
$$P_s(G) = P_s(G-e) - P_s(G/e).$$

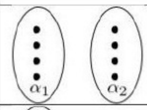
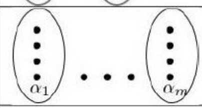
If the number of vertices is n and number of colours is k. $P_s(G,k)$ stands for number of colourings of n vertices with k colours. $P_s(G-e,k)$ stands for number of colourings with the edge e deleted. $P_s(G/e,k)$ stands for number of colourings when the two vertices of the deleted edge is glued.

2.2. **Pre-basic model discussion of strong vertex colouring [Figure 9, 10].**

Size	Order	$P_s(H,\lambda)$
0	Any	1

FIGURE 9

Size	Order	$P(H,\lambda)$	
1	1	$\lambda = [\lambda]_1$	
1	2	$\lambda(\lambda-1) = [\lambda]_2$	
1	m	$\lambda(\lambda-1)\ldots(\lambda-m) = [\lambda]_m$	

Size	Order	$P(H,\lambda)$	
2(disjoint)	α_1, α_2	$\lambda = [\lambda]_{\alpha_1}[\lambda]_{\alpha_2}$	
m(disjoint)	$\alpha_1, \ldots, \alpha_m$	$[\lambda]_{\alpha_1} \cdots [\lambda]_{\alpha_m}$	

$\rightarrow (*)$

FIGURE 10

2.3. **Hypergraph theoretic vertex colouring analytic presentation.** For basic model size is always 2 with varying order and the same process repeats for any size. The basic model for the purpose, we consider the hypergraph, $H = (V, E)$ where $V = \{\alpha, a, \beta, b, \gamma\}$ And $E = \{E_1, E_2\}$ with $E_1 = \{\alpha, a, \beta\}$, $E_2 = \{\alpha, b, \gamma\}$ and its representation is shown in Figure 11.

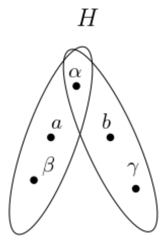
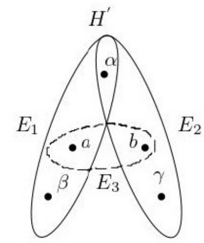


FIGURE 11

We concentrate on a particular pair (a, b) in two different edges E_1 and E_2. The strong vertex colouring of H in λ colours are of two types,

(i) vertices a and b are given different colours which is equivalent to a colouring of the hypergraph $H' = (V, \mathcal{E}')$ where $\mathcal{E}' = \{E_1, E_2, E_3\}$ obtained from H by adding the edge $E_3 = \{a, b\}$.



(ii) a strong vertex colouring of H of this type is the vertex colouring of the hypergraph H'' obtained from H' by coinciding the vertices a and b. Thus, we get the hypergraph, $H'' = (V'', \mathcal{E}'')$ where $V'' = V$ (with a, b coincident) and $\mathcal{E}'' = \{\mathcal{E}_1'' = \{\alpha, a(b), \beta\}, \mathcal{E}_2'' = \{\alpha, a(b), \gamma\}\}$, [Note that $a(b)$ means a glued with b or a and b are with same colour [Figure 12].

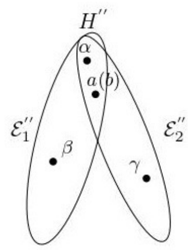


FIGURE 12

And hence we get the chromatic polynomial of the basic model as,

$$P_s(H, \lambda) = P_s(H', \lambda) + P_s(H'', \lambda).$$

3. MAIN RESULTS

Theorem 3.1. $P_s(G, k) = P_s(G - e, k) + P_s(G/e, k)$.

Proof. Clearly, $P_s(G, k) = n_{Pk}$, where n_{Pk} is the number of possible permutations of k objects from a set of n. $P_s(G - e, k)$ gives rise to two cases:

Case (i) The two vertices of the deleted edge contain different colours.

Case (ii) The two vertices of the deleted edge contain same colour.

It is not difficult to see that,

Number in case of (i) coincides with $P_s(G, k)$.

In case of (ii) also, since there is no edge between these two end vertices, colouring these two end vertices with the same colour does not change the numbers with that of different colours. Hence, in this case also, we get the same number as in case (i) . It is not difficult to see that the glued case is again like the case (ii) [Since two same colour vertices are glued].

Thus,

$$P_s(G,k) = n_{Pk}$$
$$P_s(G-e,k) = n_{Pk} + n_{Pk}$$
$$P_s(G/e,k) = n_{Pk}$$

Hence, clearly, $P_s(G,k) = P_s(G-e,k) - P_s(G/e,k)$. □

We now attempt to prove the fundamental theorem of strong vertex colouring with minimal order 3 and minimal size 2 leaving the elementary cases described in (*) depending upon which we tend to develop what has been presented below.

If $H = (V,E)$, then $H^{'} = (V,\mathcal{E}^{'})$, where $\mathcal{E}^{'}$, a new edge set obtained on introducing a new edge $E^{'} = \{a,b\}$ containing two vertices each one from two distinct edges in $\mathcal{E}$. Also we get another hypergraph $H^{''} = (V^{''},\mathcal{E}^{''})$ where $V^{''} = V\backslash\{b\}$ where b is the vertex left after gluing b with a and $\mathcal{E}^{''}$ is the edge set obtained from $\mathcal{E}$ on gluing b with a.

Then, the fundamental theorem states, $P_s(H,\lambda) = P_s(H^{'},\lambda) + P_s(H^{''},\lambda)$.

Now, we prove the pivotal model of the above mentioned fundamental theorem as follows:

Here, in Figure 13, $H = (V,E)$, where $V = \{\alpha,a,b\}$ and $\mathcal{E} = \{E_1 = \{\alpha,a\}, E_2 = \{\alpha,b\}\}$ and $E_1 \cap E_2 = \{\alpha\}$

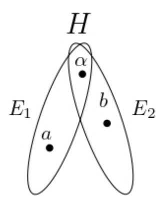


FIGURE 13

$H^{'} = (V,\mathcal{E}^{'})$, in Figure 14, where $\mathcal{E}^{'} = \{E_1, E_2, E_3 = \{a,b\}\}$

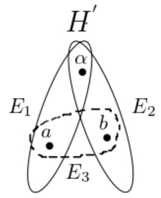


FIGURE 14

$H^{''} = (V^{'}, \mathcal{E}^{''})$, in Figure 15, where $V^{'} = \{\alpha, a(b)\}$ and $\mathcal{E}^{''} = \{E^{''}\}$ where $E^{''} = \{\alpha, a(b)\}$

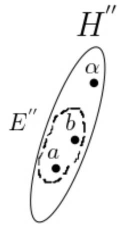


FIGURE 15

Then,

$$P_s(H, \lambda) = [\lambda(\lambda - 1)](\lambda - 1) = (\lambda - 1)^2.$$
$$P_s(H^{'}, \lambda) = \lambda(\lambda - 1)(\lambda - 2).$$
$$P_s(H^{''}, \lambda) = \lambda(\lambda - 1).$$

Now,

$$\begin{aligned} P_s(H^{'}, \lambda) + P_s(H^{''}, \lambda) &= \lambda(\lambda - 1)(\lambda - 2) + \lambda(\lambda - 1) = \lambda(\lambda - 1)(\lambda - 1) \\ &= \lambda(\lambda - 1)^2 \\ &= P_s(H, \lambda). \end{aligned}$$

Hence, the result.

Now, we attempt the necessary endeavour with minimal size 2 and with order one- more than the minimal order 3.

Now, the fundamental theorem with a hypergraph having 4 vertices and 2 edges takes the following form [Figure 16], $H = (V, E)$ where $V = \{\alpha, a, b, c\}$ and $\mathcal{E} = \{E_1 = \{\alpha, a, b\}, E_2 = \{\alpha, c\}\}$ and $E_1 \cap E_2 = \{a\}$

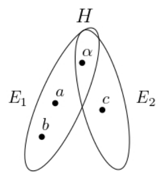


FIGURE 16

$H^{'} = (V, \mathcal{E}^{'})$ where $\mathcal{E}^{'} = \{E_1, E_2, E_3 = \{a, c\}\}$ and $H^{''} = (V^{''}, \mathcal{E}^{''})$ where $V^{''} = \{\alpha, a(c), b\}$ and $\mathcal{E}^{''} = \{\alpha, a(c), b\}$ [here, $a(c)$ means a is glued with c or a and c are with same colour [Figure 17].

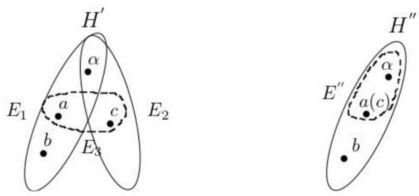


FIGURE 17

Now,

$$P_s(H, \lambda) = [\lambda(\lambda-1)(\lambda-2)](\lambda-1) = \lambda(\lambda-1)^2(\lambda-2)$$
$$P_s(H^{'}, \lambda) = [\lambda(\lambda-1)(\lambda-2)](\lambda-2) = \lambda(\lambda-1)(\lambda-2)^2$$
$$P_s(H^{''}, \lambda) = \lambda(\lambda-1)(\lambda-2)$$

Further,

$$\begin{aligned} P_s(H^{'}, \lambda) + P_s(H^{''}, \lambda) &= \lambda(\lambda-1)(\lambda-2)^2 + \lambda(\lambda-1)(\lambda-2) \\ &= \lambda(\lambda-1)^2(\lambda-2) \\ &= P_s(H, \lambda) \end{aligned}$$

Hence,we get the required result.

Our next endeavour would be with size 2, one common vertex and order being two-more than the minimal order.

Now, the third basic form [figure 18] of the fundamental theorem with $H = (V, E)$ where $V = \{a, b, c, \alpha, \beta\}$ and $E = \{E_1 = \{a, \alpha, \beta\}, E_2 = \{a, b, c\}\}$.

After renaming the hyperedges E_1 and E_2 as shown the corresponding hypergraph $H^{'}$ takes the following pattern on introducing all possible new edges as

$E_1 = \{a, \alpha, \beta\}$, $E_2 = \{a, b, c\}$, $S_1 = \{\alpha, b\}$, $S_2 = \{b, \beta\}$, $S_3 = \{\alpha, c\}$, $S_4 = \{\beta, c\}$. Using the characteristics of strong vertex colouring with the above mentioned edges of the hypergraph $H^{'} = (V, \mathcal{E}^{'})$ where $\mathcal{E}^{'} = \{E_1, E_2, S_1, S_2, S_3, S_4\}$ as shown the chromatic polynomial appears as $P_s(H^{'}, \lambda)$.

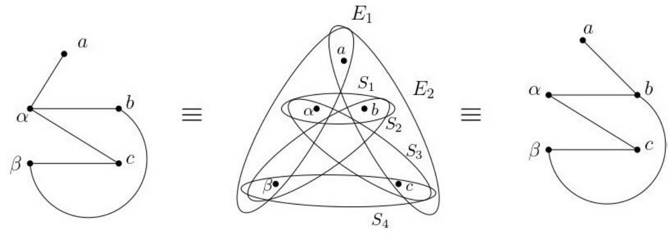


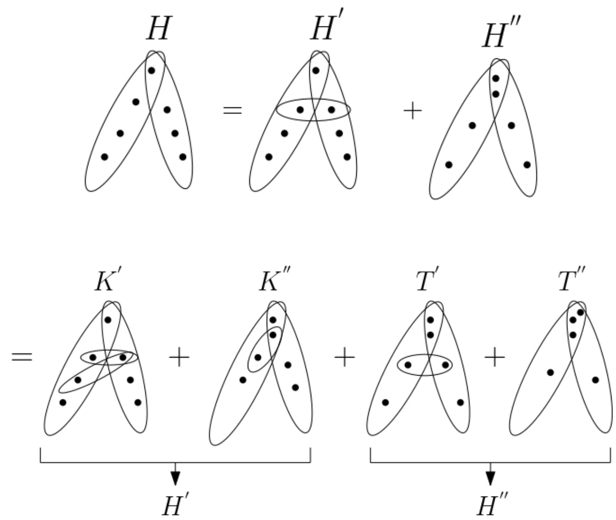
FIGURE 18

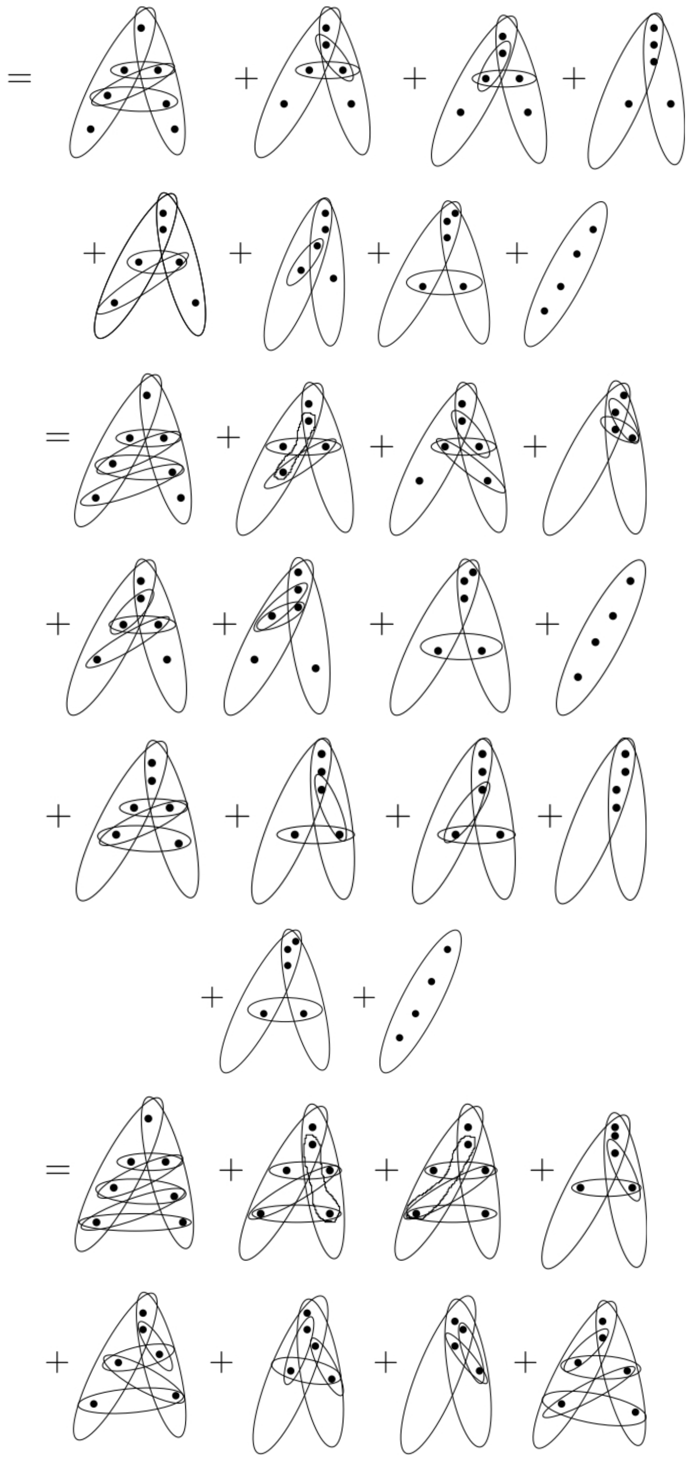
Then $P_s(H^{'}, \lambda)$ is $[\lambda]_5 = \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)$ and thus whatever be the hypergraph the first decomposed term with n vertices appear as $[\lambda]_n$ which will be clear from the total decomposition model given below.

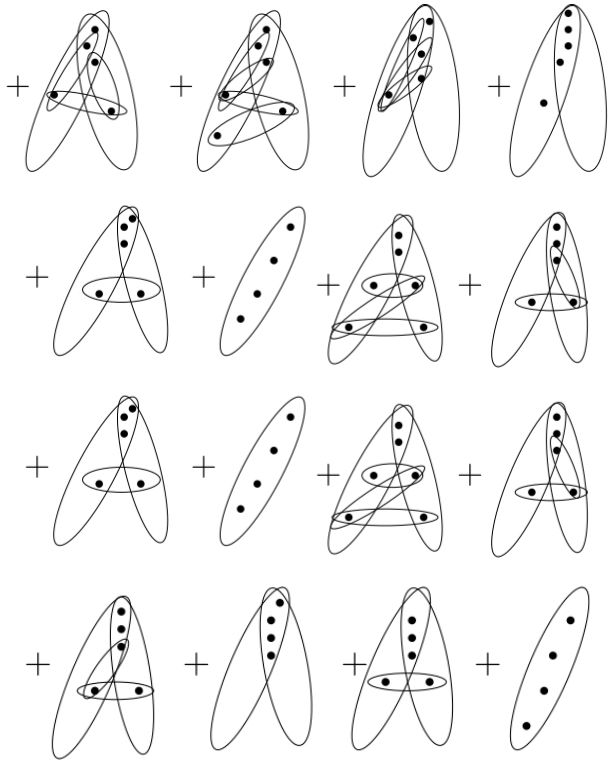
The total decomposition model of the fundamental theorem:

Analytic view:

Keeping in note of the above form we further illustrate the fundamental theorem by decomposing a given hypergraph H with size 2 and order 7 to $H^{'}$, $H^{''}$ and further $H^{'}$ to $K^{'}$, $K^{''}$ and $H^{''}$ to $T^{'}$, $T^{''}$ and so on. Continuing the process in this way we henceforth arrive at the strong chromatic polynomial of H.







$$= [\lambda]_7 + 6[\lambda]_6 + 11[\lambda]_5 + 2[\lambda]_4$$

FIGURE 19

[In general [Figure 19] in case of a hypergraph with order n it leads us to the form $[\lambda]_n + a_1[\lambda]_{n-1} + ... + a_t[\lambda]_{n-t}$].

Note:

(1) The first term of the decomposition takes the form of a hypergraph consisting of 2-graph edges.
(2) Each of the decomposed component of the above takes the third basic form with some integral coefficient.

Thus the decomposition theorem finally gives the structure of $P(H, \lambda)$ as $P_s(H, \lambda) = [\lambda]_n + a_1[\lambda]_{n-1} + ... + a_t[\lambda]_{n-t}$. Following the discussion in (*) above we would like to present the more general result of hypergraph with connected components as below:

Theorem 3.2. *Suppose H is a hypergraph with connected components $H_1, H_2, \ldots, H_k$ then $P_s(H, \lambda) = P_s(H_1, \lambda)P_s(H_2, \lambda)\ldots P_s(H_k, \lambda)$.*

Proof. The proof of the theorem suffices with two components H_1 and H_2. In other words, if H has connected components H_1, H_2 with $V(H_1) \cap V(H_2) = \phi$. Then, $P_s(H_1 \cup H_2, \lambda) = P_s(H_1, \lambda)P_s(H_2, \lambda)$. Since, $V(H_1) \cap V(H_2) = \phi$ clearly the strong vertex colouring of H_1 is independent with that of H_2. Hence, the number of ways of strong vertex colouring of $H_1 \cup H_2$ is clearly the product of the number of strong vertex colourings of H_1 and H_2.

Thus, $P_s(H_1 \cup H_2, \lambda) = P_s(H_1, \lambda)P_s(H_2, \lambda)$. □

As a partial endeavour of our search for chromatic polynomial of a point hyperpath leads us to a sectional result in connection with 3-uniform such point hyperpaths which appear in a special make up that follows from the following discussion:

We have already discussed about chromatic polynomial of an arbitrary hypergraph specifically of two edges with its elegant successive figurative development. Now we would like to present a model of a 3-uniform point hyperpath of length n leading to its corresponding chromatic polynomial. First we take a point hyperpath of length 3, then 4 and finally coining the chromatic polynomial of a 3-uniform point hyperpath of length n. Our 3-uniform point hyperpath $H = (V, E)$ where $V = \{a, b, c, d, e, f, g\}$ and $E = \{E_1 = \{a, b, c\}, E_2 = \{c, d, e\}, E_3 = \{e, f, g\}\}$ of length 3 is shown in Figure 20.

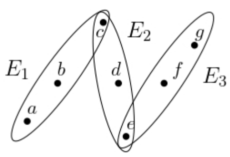


FIGURE 20

Now, we break it into two connected components H_1 and H_2 as shown in Figure 21.

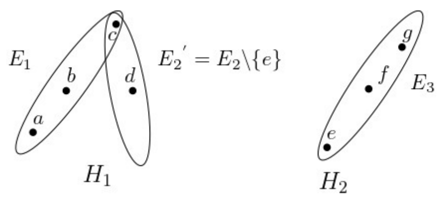


FIGURE 21

Following the method described in the Basic Model we have the following figurative sequence [Figures 22, 23] for component H_1.

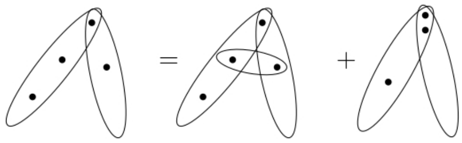


FIGURE 22

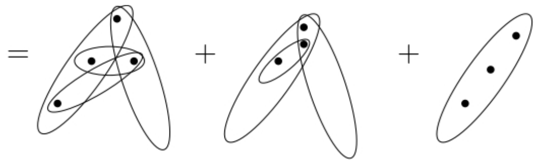


FIGURE 23

Now, we take a look into the second component H_2 where the vertex e can be coloured in $\lambda - 2$ ways and the remaining two vertices of this component can be coloured in $\lambda - 1$ and $\lambda - 2$ ways. So, the chromatic polynomial for this component H_2 would be, $(\lambda - 2)(\lambda - 1)(\lambda - 2) = (\lambda - 1)(\lambda - 2)^2$.

Together, the chromatic polynomial for the 3-uniform point hyperpath H of length 3 (Theorem 3.2) would be, $P(H, \lambda) = P(H_1, \lambda).P(H_2, \lambda) = [\lambda(\lambda-1)^2(\lambda-2)][(\lambda - 1)(\lambda - 2)^2] = \lambda(\lambda - 1)^3(\lambda - 2)^3$ [verified base of induction] Similarly, for a 3-uniform point hyperpath $H = (V, E)$ where $V = \{a, b, c, d, e, f, g, h, i\}$ and $E = \{E_1 = \{a, b, c\}, E_2 = \{c, d, e\}, E_3 = \{e, f, g\}, E_4 = \{g, h, i\}\}$ of length 4 is shown below [Figure 24].

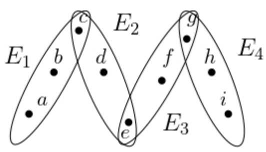


FIGURE 24

We take the following connected components H_1, H_2 and H_3 of [Figure 25] the above point hyperpath,

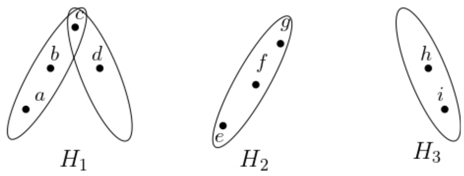


FIGURE 25

And hence we get the chromatic polynomial of 3-uniform point hyperpath of length 4 as,

$$\begin{aligned}P(H,\lambda) &= P(H_1,\lambda).P(H_2,\lambda).P(H_3,\lambda)\\ &= [\lambda(\lambda-1)^2(\lambda-2)][(\lambda-2)(\lambda-1)(\lambda-2)][(\lambda-1)(\lambda-2)]\\ &= \lambda(\lambda-1)^4(\lambda-2)^4\end{aligned}$$

[verified for $n=4$]

Induction hypothesis:

Suppose the result is true for 3-uniform point hyperpath [Figure 26] of length $n-1$.

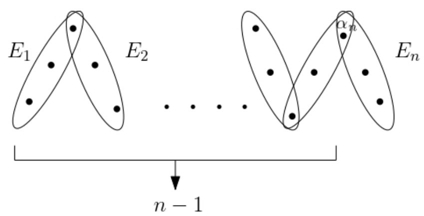


FIGURE 26

Now, the 3-uniform point hyperpath of length n consists of two components H_1 of length $n-1$ and component H_2 consisting of edge $E_n^{'} = E_n \backslash \{\alpha_n\}$. So, by the above theorem,

$$\begin{aligned}P(H,\lambda) &= P(H_1,\lambda).P(H_2,\lambda)\\ &= [\lambda(\lambda-1)^{n-1}(\lambda-2)^{n-1}][(\lambda-1)(\lambda-2)] \;\; [Induction\;\; hypothesis]\\ &= \lambda(\lambda-1)^n(\lambda-2)^n\end{aligned}$$

Hence, the result is true for 3-uniform point hyperpath of any length n.

Thus we get by induction,

Theorem 3.3. *The Chromatic polynomial of a 3-uniform point hyperpath H of length n, (for all n) is $P(H,\lambda) = \lambda(\lambda-1)^n(\lambda-2)^n$.*

The next theorem is another interesting and important theorem which may be termed as the quotient fundamental theorem.

Theorem 3.4. *If H_1 and H_2 are two hypergraphs and $H = H_1 \cup H_2$, then $P_s(H,\lambda) = P_s(H_1,\lambda)P_s(H_2,\lambda)/P_s(H_1 \cap H_2,\lambda)$.*

Proof. Let $V(H_1) = \{v_1, ..., v_\alpha, v_{\alpha+1}, ..., v_n\}$ and $V(H_2) = \{v_1, ..., v_\alpha, w_{\alpha+1}, ..., w_k\}$ So, $V(H_1) \cap V(H_2) = \{v_1, ..., v_\alpha\}$.

Again, suppose $V(H_1^{'}) = \{v_{\alpha+1}, ..., v_n\} = \{a_1, ..., a_{n-\alpha}\}$ and $V(H_2^{'}) = \{w_{\alpha+1}, ..., w_k\} = \{c_1, ..., c_{k-\alpha}\}$.

Now, respective strong vertex colourings are as follows:

$P_s(H_1 \cap H_2, \lambda) = \lambda(\lambda-1)\dots(\lambda-\alpha+1) = \lambda(\lambda-1)\dots(\lambda-\overline{\alpha-1})$.

Again, colouring $H_1 \cap H_2$ we are left with $\lambda - \alpha = \beta$ colours for colouring of $H_1^{'}$.

Thus, $P_s(H_1^{'}) = \beta(\beta-1)\dots(\beta-\overline{n-\alpha}+1)$.

Similarly, $P_s(H_2^{'}) = \beta(\beta-1)\dots(\beta-\overline{k-\alpha}+1)$.

Moreover, $(V(H_1) \cap V(H_2)) \cap V(H_2^{'}) = \phi$ and $V(H_1^{'}) \cap V(H_2^{'})$.

Now, the number of ways of colouring of common part is $P_s(H_1 \cap H_2, \lambda)$. First, we fix a vertex colouring of the common part $H_1 \cap H_2$. Then we consider the vertex colouring of the rest $H_1^{'}$ of H_1.

The number is then,

$$\begin{aligned}
P_s(H_1^{'}, \beta) &= \beta(\beta-1)...(\beta-(n-\alpha)+1) \\
&= \frac{\beta(\beta-1)...(\beta-(n-\alpha)+1)\lambda(\lambda-1)...(\lambda-(n-\alpha)+1)}{\lambda(\lambda-1)...(\lambda-(n-\alpha)+1)} \\
&= \frac{\lambda(\lambda-1)...(\lambda-(n-\alpha)+1)(\lambda-\alpha)(\lambda-\alpha-1)...(\lambda-n+1)}{\lambda(\lambda-1)...(\lambda-(n-\alpha)+1)} \\
&= \frac{P_s(H_1,\lambda)}{P_s(H_1 \cap H_2,\lambda)}
\end{aligned}$$

Similarly, for the part $H_2^{'}$,

$$P_s(H_1^{'}, \beta) = \frac{P_s(H_1,\lambda)}{P_s(H_1 \cap H_2,\lambda)}$$

Thus we get,

$$\begin{aligned} P_s(H,\lambda) &= P_s(H_1\cap H_2,\lambda)P_s(H_1^{'},\beta)P_s(H_2^{'},\beta) \\ &= P_s(H_1\cap H_2,\lambda)[\frac{P_s(H_1,\lambda)}{P_s(H_1\cap H_2,\lambda)}][\frac{P_s(H_2,\lambda)}{P_s(H_1\cap H_2,\lambda)}] \\ &= P_s(H_1,\lambda)P_s(H_2,\lambda)/P_s(H_1\cap H_2,\lambda) \end{aligned}$$

□

The above theorem leads us to the following corollary in regards to the degree and monic character of chromatic polynomial.

Corollary 3.1. *The degree of the chromatic polynomial $P_s(H,\lambda)$ of a hypergraph with order n is n itself.*

Corollary 3.2. *The chromatic polynomial $P_s(H,\lambda)$ is a monic one being independent of the order of the hypergraph.*

The constant free character together with alternately sign changing nature of a chromatic polynomial follows from the two small theorems.

Theorem 3.5. *$P_s(H,\lambda)$ has no constant term.*

Proof. Suppose we have,

$P_s(H,\lambda) = [\lambda]_n + a_1[\lambda]_{n-1} + ... + a_t[\lambda]_{n-t} + ... + a_0[\lambda]_0$,($[\lambda]_0 = 0$)

Hence, $P_s(H,\lambda)$ is of the form,

$P_s(H,\lambda) = [\lambda]_n + a_1[\lambda]_{n-1} + .. + a_t[\lambda]_{n-t}$, (for some $t < n$). Thus, it has no constant term.

□

Theorem 3.6. *$P_s(H,\lambda)$ is always an alternating monic polynomial. [$\mathbf{P_s(H,n,k)}$ is used to denote the strong chromatic polynomial of the hypergraph H with order n and size k.]*

Proof. In case of a hypergraph with order n and size k the fundamental theorem gives rise to other two hypergraphs $H^{'}$ (order n and size $k+1$) and hypergraph $H^{''}$(order $n-1$ and size k) so that,

$P_s(H,n,k) = P_s(H^{'},n,k+1) + P_s(H^{''},n-1,k)$ that gives

$$P_s(H^{'},n,k+1) = P_s(H,n,k) - P_s(H^{''},n-1,k) \tag{3.1}$$

We prove our result for fixed order and varying size.

In particular we note, for $n = 1, 2, 3$ the result is obvious.

Now we fix our order as n with

Hypothesis: the result is true for order n or less and size k or less.

Claim: The result is true for size $k + 1$.

In (3.1) gives $P_s(H', n, k+1) = P_s(H, n, k) - P_s(H'', n-1, k)$ and by hypothesis each of the terms in the right hand side is an alternating monic polynomial of degree n and $n - 1$ respectively. And hence clearly their difference is also so.

Thus, the result is true for fixed n and size $k+1$. In other words, by induction, the result is true for fixed n and varying k.

Similarly, in turn same is true for fixed k and varying n.

Hence, $P_s(H, \lambda)$ is always an alternating monic polynomial.

The following theorem gives some information of chromatic character in case of the algoritheorem for constructing edge-trees from hypergraphs as mentioned in Gavril and Tamari [6].

□

Theorem 3.7. *$H(X, E)$ is a r-uniform q-edge tree hypergraph with $|E_i \cap E_j| \leq 1$ for all $E_i, E_j \in E$. Then $P_s(H, \lambda) = \lambda(\lambda-1)^q(\lambda-2)^q...(\lambda-r+1)^q$ and $P_w(H, \lambda) = (\lambda^r - \lambda)(\lambda^{r-1} - 1)^{q-1}$.*

Proof. Suppose $E_1 = \{a_{11}, a_{12}, \ldots, a_{1r}\}, E_2 = \{a_{21} = a_{1r}, a_{22}, \ldots, a_{2r}\}, ..., E_q = \{a_{q1} = a_{(q-1)}, a_{q2}, \ldots, a_{qr}\}, E_2' = \{a_{22}, \ldots, a_{2r}\}, E_q' = \{a_{q2}, \ldots, a_{qr}\}$.

Note, $E_i' \cap E_j' = \phi, E_1 \cap E_i' \neq \phi$.

And, $P_s(E_1) = \lambda(\lambda - 1)...(\lambda - r + 1)$, $P_s(E_2') = (\lambda - 1)(\lambda - 2)...(\lambda - r + 1)$, $P_s(E_q') = (\lambda - 1)(\lambda - 2)...(\lambda - r + 1)$

So, $P_s(H, \lambda) = P_s(E_1)P_s(E_2')\ldots P_s(E_q') = \lambda(\lambda - 1)^q(\lambda - 2)^q...(\lambda - r + 1)^q$

And, $P_w(H, \lambda) = (\lambda^r - \lambda)(\lambda^{r-1} - 1)^{q-1}$, $P_w(E_1, \lambda) = \lambda^{r-1}(\lambda - 1), P_w(E_2', \lambda) = \lambda^{r-1} - 1, ...P_w(E_q', \lambda) = \lambda^{r-1} - 1$.

Therefore, $P_w(H, \lambda) = P_w(E_1, \lambda).P_w(E_2', \lambda)...P_w(E_q', \lambda) = (\lambda^r - \lambda)(\lambda^{r-1} - 1)... (\lambda^{r-1} - 1) = (\lambda^r - \lambda)(\lambda^{r-1} - 1)^{q-1}$.

□

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