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APPLICATION OF SUMUDU TRANSFORM ON FRACTIONAL KINETIC EQUATION PERTAINING TO THE GENERALIZED K-WRIGHT FUNCTION

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ABSTRACT. Fractional kinetic equations (FKEs) that includes a wide range of special functions have been applied to the description and resolution of many important physics and astrophysics problems greatly and successfully.We derive solutions for (FKEs) in this paper with the help of Sumudu transforms, including the product of the generalized k-Wright function. After that, other important special cases have been revealed. The use of the Generalized k-Wright function to obtain the (FKEs) solution is relatively general and can be used effectively to construct many well-known and novel (FKEs).

1. Introduction and Preliminaries

The importance of differential fractional equations in the files not only in math- ematics have applied science gained more attention dynamical systems, direction but also in mathematical physics, control and engineering systems to generate a mathematical model of many physical phenomena see [1,5–9,11,13, 19–21,28,29,33], [18].

The fractional calculus with the Mittag-Leffler law has been widely studied recently due to its significance and applicability in various fields see [16,30,31] (FKEs) of various models have been successfully applied in the last decades to describe and explain numerous physics and astrophysics problems [35–39].

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Thus, we remember the differential fractional equation associated with the rate of reaction change $\mathfrak{M} = \mathfrak{M}(t)$,the destruction rate $d = d(\mathfrak{M})$ and the production rate $p = p(\mathfrak{M})$ that given by Haubold and Mathai see [17] as the follows

$$\frac{d(\mathfrak{M})}{dt} = -d(\mathfrak{M}_{\mathrm{t}}) + p(\mathfrak{M}_{\mathrm{t}})$$

where $\mathfrak{M}_{\mathrm{t}}$ is the function identified by

$$\mathfrak{M}_{\mathrm{t}}(t^*) = \mathfrak{M}(t - t^*), t^* > 0.$$

Neglecting the inhomogeneity in the quantity $\mathfrak{M}(t)$ that is the equation

$$\frac{d\mathfrak{M}}{dt} = -c_i \mathfrak{M}_i(t) \tag{1.1}$$

is part of the initial condition $\mathfrak{M}_{\mathrm{i}}(t = 0) = \mathfrak{M}_0$ is the number of density of index i at time $t = 0$

The equation solution (1.1) is referred as

$$\mathfrak{M}_{\mathrm{i}}(t) = \mathfrak{M}_0 \; e^{-c_i t}$$

On the other hand, we can take

$$\mathfrak{M}(t) - \mathfrak{M}_0 = c_0 \; D_t^{-1} \mathfrak{M}(t) \tag{1.2}$$

where the ${}_0D_t^{-1}$ is the standard fractional integral operator. In addition, the fractional generalization for the standard kinetic equation (1.2) defined by Haubold and Mathai see [17]as the form

$$\mathfrak{M}(t) - \mathfrak{M}_0 = c^\gamma \; {}_0D_t^{-\gamma} \; \mathfrak{M}(t) \tag{1.3}$$

where ${}_0D_t^{-\gamma}$ is the Riemann-Liouville fractional integral operator expressed as

$${}_0D_t^{-\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t - s)^{\gamma - 1} f(s) ds, \; (t > 0, \mathfrak{R}(\gamma) > 0).$$

Haubold and Mathai [17] provide the equation solution (1.3) in the form:

$$\mathfrak{M}(t) = \mathfrak{M}_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\gamma k + 1)} (ct)^{\gamma k}.$$

Further, Saxena and Kalla see [32] expressed the following fractional kinetic equation as the form

$$\mathfrak{M}(t) - \mathfrak{M}_0 f(t) = -c^\gamma \; ({}_0D_t^{-\gamma} \; \mathfrak{M})(t), \; (\mathfrak{R}(\gamma) > 0$$

where $\mathfrak{M}(t)$ refers to density number of a given species at every time t , $\mathfrak{M}_0 = \mathfrak{M}(0)$ is a density number that species at time t = 0, c is a constant and $f \in L(0,\infty)$.

The transform of Sumudu is widely used to solve different types of science and engineering problems, and was presented by Watugala see [42, 43]. The interested readers can use see [2–4, 22, 41] references for the all information of Sumudu transforms, properties, and their applications. The Sumudu transform over the set function:
$A = f(t)\big|\exists\mathfrak{M}, \eta_1, \eta_2 > 0, \big|f(t)\big| < \mathfrak{M}e^{\frac{|t|}{\tau_j}}, t \in (-1)^j \times [0,\infty)$.
Is defined by the

$$G(\tau) = S[f(t);\tau] = \int_0^\infty e^{-t} f(\tau t)dt\ ;\ \tau \in (-\eta_1, \eta_2).$$

In the following investigation, we establish generalized fractional kinetic equations' solution. The definitions of various Wright functions are presented in order to establish the fractional kinetic equation's solutions in the following investigation.

2. Generalized K-Wright Function

In 2007, Diaz and Pariguan see [10] provided the symbol of k-Pochhemmer and the function of k- gamma which are defined as follows:

$$(y)_{n,k} := \begin{cases} \frac{\Gamma_k(y+nk)}{\Gamma_k(y)} & (k \in \mathbb{R}; y \in \mathbb{C} \setminus \{0\}) \\ y(y+k)...(y+(n-1)k) & (n \in \mathbb{N}; y \in \mathbb{C}) \end{cases} \tag{2.1}$$

also the relation with the classical Euler's gamma function as the form

$$\Gamma_k(y) = \int_0^\infty t^{y-1} e^{-\frac{t^k}{k}} dt$$

that is

$$\Gamma_k(y) = k^{\frac{y}{k}-1}(\frac{y}{k}).,\quad (\mathbb{R}(y) > 0).$$

where $y \in \mathbb{C}, k \in \mathbb{R}\ and\ n \in \mathbb{N}$.

Now, when k = 1 the equation (2.1)reduces to the classical Pochhammer symbol and Euler's gamma function respectively. For more details see [26, 27].

Srivastava and Karisson, see [40] defined The Fox-Wright function as

$$ {}_n\psi_m[z] = {}_n\psi_m\left[\begin{matrix}(p_1,\alpha_1),...,(p_n,\alpha_n);\\ (q_1,\beta_1),...,(q_m,\beta_m);\end{matrix} z\right] = \sum_{r=0}^{\infty} \frac{\prod_{i=1}^{n} \Gamma(p_i+\alpha_i r)}{\prod_{j=1}^{m} \Gamma(q_j+\beta_j r)} \frac{z^r}{r!}, $$

where the coefficients $\alpha_1,...,\alpha_n,\beta_1,...,\beta_m \in \mathbb{R}^+$ such that

$$ 1+\sum_{j=1}^{m}\beta_j-\sum_{i=1}^{n}\alpha_i \geq 0 $$

presented the generalized $k-$Wright function concept, that is shown in the following definition.

Definition 2.1. *[14]*

For $k \in \mathbb{R}^+$; $z \in \mathbb{C}$; $p_i, q_j \in \mathbb{C}$, $\alpha_i, \beta_j \in \mathbb{R}$ $(\alpha_i, \beta_j \neq 0;\ i = 1,2,...,n;\ j = 1,2,...,m)$ *and* $(p_i+\alpha_i r),\ (q_j+\beta_j r) \in \mathbb{C}\setminus k\mathbb{Z}^-$, *the generalized* $k-$*Wright function* ${}_n\psi_m^k$ *is defined by*

$$ {}_n\psi_m^k(z) = {}_n\psi_m^k\left[\begin{matrix}(p_i,\alpha_i)_{1,n}\\ (q_j,\beta_j)_{1,m}\end{matrix}\middle| z\right] = \sum_{r=0}^{\infty} \frac{\prod_{i=1}^{n} \Gamma_k(p_i+\alpha_i r)}{\prod_{j=1}^{m} \Gamma_k(q_j+\beta_j r)} \frac{z^r}{r!}. \tag{2.2} $$

We use the following notes to explain the state of convergence

$$ \Delta = \sum_{j=1}^{m}(\frac{\beta_j}{k}) - \sum_{i=1}^{n}(\frac{\alpha_i}{k});\ \delta = \Pi_{i=1}^{n}\left|\frac{\alpha_i}{k}\right|^{\frac{-\alpha_i}{k}}\ \Pi_{j=1}^{m}\left|\frac{\beta_j}{k}\right|^{\frac{-\beta_j}{k}} $$

$$ \mu = \sum_{j=1}^{m}(\frac{q_j}{k}) - \sum_{i=1}^{n}(\frac{p_i}{k}) + \frac{n-m}{2} $$

(a). if $\Delta > -1$ the series (2.2) is completely convergent for all $z \in \mathbb{C}$ and generalized k-Wright function ${}_n\psi_m^k(z)$ is an entire function of z

(b). if $\Delta = -1$ then the series (2.2) is completely convergent for all $|z| < \delta$ and of $|z| = \delta,\ \Re(\mu) > \frac{1}{2}$.

By giving the proper parameter values the follwoing relation of the generalized k-Wright function ${}_n\psi_m(z)$ in terms of family of Mittag-Leffler function see [12, 15, 34, 44, 45], [23, 25] defined as follows

$$ {}_1\psi_2^k(z) = {}_1\psi_2^k\left[\begin{matrix}(y,k)\\ (\beta,\alpha),(y,0)\end{matrix}\middle| z\right] = \sum_{r=0}^{\infty} \frac{(y)_{r,k} z^r}{\Gamma_k(r\alpha+\beta)r!} = E_{k,\alpha,\beta}^{y}(z) \tag{2.3} $$

$$ {}_1\psi_2^k(z) = {}_1\psi_2^k\left[\begin{matrix}(y\tau,k)\\ (\beta,\alpha),(y,0)\end{matrix}\middle| z\right] = \sum_{r=0}^{\infty} \frac{(y)_{r\ \tau,k} z^r}{\Gamma_k(r\alpha+\beta)r!} = E_{k,\alpha,\beta}^{y,\tau}(z) \tag{2.4} $$

$$(2.5)\qquad {}_1\psi_2^1(z) = {}_1\psi_2^1\left[\begin{matrix}(1,1)\\(1,\alpha),(y,0)\end{matrix}\middle| z\right] = \sum_{r=0}^{\infty}\frac{z^r}{\Gamma(r\alpha+1)} = E_\alpha(z)$$

$$(2.6)\qquad {}_1\psi_2^1(z) = {}_1\psi_2^k\left[\begin{matrix}(y,1)\\(\beta,\alpha),(y,0)\end{matrix}\middle| z\right] = \sum_{r=0}^{\infty}\frac{(y)_r z^r}{\Gamma(r\alpha+\beta)r!} = E_{\alpha,\beta}^{y}(z)$$

$$(2.7)\qquad {}_1\psi_2^1(z) = {}_1\psi_2^1\left[\begin{matrix}(y,\tau)\\(\beta,\alpha),(y,0)\end{matrix}\middle| z\right] = \sum_{r=0}^{\infty}\frac{(y)_{r\ \tau} z^r}{\Gamma(r\alpha+\beta)r!} = E_{\alpha,\beta}^{y,\tau}(z)$$

$$(2.8)\qquad {}_1\psi_2^1(z) = {}_1\psi_2^1\left[\begin{matrix}(1,1)\\(\beta,\alpha),(y,0)\end{matrix}\middle| z\right] = \sum_{r=0}^{\infty}\frac{z^r}{\Gamma(r\alpha+\beta)} = E_{\alpha,\beta}(z)$$

$$ {}_n\psi_m^1(z) = {}_n\psi_m^1\left[\begin{matrix}(p_i,\alpha_i)_{1,n}\\(q_j,\beta_j)_{1,m}\end{matrix}\middle| z\right] = {}_n\psi_m(z).$$

3. Solution of generalized fractional kinetic equations by using Sumudu transform

In this part, we investigated the solutions of the generalized fractional kinetic equations by considering generalized $k-$Wright function using the method of Sumudu transforms.

Watugala [42, 43], described and studied Sumudu transform in order to simplify the steps of solving integral and differential equations in the time domain. Sumudu transform has very unique and beneficial properties in solving the engineering and science problems that govern kinetic equation.

Remark 3.1. *In this part, solutions for the fractional kinetic equations as for the generalized Mittag-Leffler are acquired $E_{\alpha,\beta}(z)$see [24] , Which is defined as the form:*

$$E_{\alpha,\beta}(z) = \sum_{r=0}^{\infty}\frac{z^r}{\Gamma(\alpha r+\beta)},\ `\Re(\alpha) > 0,\ \Re(\beta) > 0.$$

Theorem 3.1. *Let* $\Re(\gamma) > 0, \delta > 0, c > 0, k \in \mathbb{R}^+;\ c, z \in \mathbb{C};\ p_i, q_j \in \mathbb{C},\ \alpha_i, \beta_j \in \mathbb{R}\ (\alpha_i, \beta_j \neq 0;\ i = 1, 2, ..., n;\ j = 1, 2, ..., m)$ *and* $(p_i + \alpha_i r),\ (q_j + \beta_j r) \in \mathbb{C} \setminus k\mathbb{Z}^-$, *then the following equation*

$$\mathfrak{N}(t) - \mathfrak{N}_0\left(\prod_{\lambda=1}^{l} {}_{n_\lambda}\psi_{m_\lambda}^{k_\lambda}\left[\begin{matrix}(p_{\lambda i}, \alpha_{\lambda i})_{1,n_\lambda}\\(q_{\lambda j}, \beta_{\lambda j})_{1,m_\lambda}\end{matrix}\Big| c^\gamma t^\gamma\right]\right) = -\delta^\gamma\ {}_0D_t^{-\gamma}\mathfrak{N}(t) \tag{3.1}$$

has a solution given by

$$\mathfrak{N}(t) = \mathfrak{N}_0\left(\prod_{\lambda=1}^{l}\sum_{r=0}^{\infty}\frac{\prod_{i=1}^{n_\lambda}\ \Gamma_{k_\lambda}(p_{\lambda i} + \alpha_{\lambda i} r)}{\prod_{j=1}^{m_\lambda}\Gamma_{k_\lambda}(q_{\lambda j} + \beta_{\lambda j} r)}\right)\left(\frac{c^{r\gamma}t^{r\gamma}}{r!}\right)^\lambda \times \frac{1}{t}$$

$$\times\Gamma(\gamma r\lambda + 1)\ E_{\gamma,\gamma r\lambda}\ (-\delta^\gamma t^\gamma).$$

Proof. The Sumudu transform of Riemann- Liouville fractional integral operator is presented as

$$\mathcal{S}\ \{{}_0D_t^{-\gamma} f(t); \tau)\} = (\tau)^\gamma G(\tau) \tag{3.2}$$

where $G(\tau) = \int_0^\infty e^{-t} f(\tau t)dt$. Now, applying the Sumudu transform to both sides of equation (3.1) and using (3.2) we have

$$\mathcal{S}\Big(\mathfrak{N}(t); \tau\Big) = \mathfrak{N}_0\ \mathcal{S}\left(\prod_{\lambda=1}^{l} {}_{n_\lambda}\psi_{m_\lambda}^{k_\lambda}\left[\begin{matrix}(p_{\lambda i}, \alpha_{\lambda i})_{1,n_\lambda}\\(q_{\lambda j}, \beta_{\lambda j})_{1,m_\lambda}\end{matrix}\Big| c^\gamma t^\gamma; \tau\right]\right) - \delta^\gamma\ \mathcal{S}\Big(\ {}_0D_t^{-\gamma}\mathfrak{N}(t); \tau\Big)$$

that is

$$\mathfrak{N}(\tau) = \mathfrak{N}_0\int_0^\infty e^{-t}\left(\prod_{\lambda=1}^{l}\sum_{r=0}^{\infty}\frac{\prod_{i=1}^{n_\lambda}\ \Gamma_{k_\lambda}(p_{\lambda i} + \alpha_{\lambda i} r)}{\prod_{j=1}^{m_\lambda}\Gamma_{k_\lambda}(q_{\lambda j} + \beta_{\lambda j} r)}\right)\left(\frac{c^{r\gamma}(\tau t)^{r\gamma}}{r!}\right)^\lambda dt - \delta^\gamma(\tau)^\gamma\mathfrak{N}(\tau), \tag{3.3}$$

by interchanging the order of integration and summation in the equation (3.3), we obtain

$$\begin{aligned}\mathfrak{N}(\tau)\big[1 + \delta^\gamma(\tau)^\gamma\big] &= \mathfrak{N}_0\left(\prod_{\lambda=1}^{l}\sum_{r=0}^{\infty}\frac{\prod_{i=1}^{n_\lambda}\ \Gamma_{k_\lambda}(p_{\lambda i} + \alpha_{\lambda i} r)}{\prod_{j=1}^{m_\lambda}\Gamma_{k_\lambda}(q_{\lambda j} + \beta_{\lambda j} r)}\right)\int_0^\infty e^{-t}\left(\frac{c^{r\gamma}(\tau t)^{r\gamma}}{r!}\right)^\lambda dt\\
&= \mathfrak{N}_0\left(\prod_{\lambda=1}^{l}\sum_{r=0}^{\infty}\frac{\prod_{i=1}^{n_\lambda}\ \Gamma_{k_\lambda}(p_{\lambda i} + \alpha_{\lambda i} r)}{\prod_{j=1}^{m_\lambda}\Gamma_{k_\lambda}(q_{\lambda j} + \beta_{\lambda j} r)}\right)\left(\frac{c^{r\gamma}(\tau)^{r\gamma}}{r!}\right)^\lambda\int_0^\infty e^{-t}t^{\gamma r\lambda}dt\\
&= \mathfrak{N}_0\left(\prod_{\lambda=1}^{l}\sum_{r=0}^{\infty}\frac{\prod_{i=1}^{n_\lambda}\ \Gamma_{k_\lambda}(p_{\lambda i} + \alpha_{\lambda i} r)}{\prod_{j=1}^{m_\lambda}\Gamma_{k_\lambda}(q_{\lambda j} + \beta_{\lambda j} r)}\right)\left(\frac{c^{r\gamma}(\tau)^{r\gamma}}{r!}\right)^\lambda\Gamma(\gamma r\lambda + 1). \qquad (3.4)\end{aligned}$$

Equation (3.4) leads to

(3.5)
$$\mathfrak{N}(\tau)=\mathfrak{N}_0\left(\prod_{\lambda=1}^{l}\sum_{r=0}^{\infty}\frac{\prod_{i=1}^{n_\lambda}\Gamma_{k_\lambda}(p_{\lambda i}+\alpha_{\lambda i}r)}{\prod_{j=1}^{m_\lambda}\Gamma_{k_\lambda}(q_{\lambda j}+\beta_{\lambda j}r)}\right)\left(\frac{c^{r\gamma}(\tau)^{r\gamma}}{r!}\right)^{\lambda}\Gamma(\gamma r\lambda+1)\sum_{\mu=0}^{\infty}(-1)^{\mu}(\delta\tau)^{\gamma\mu}$$

Now, taking inverse Sumudu transform on both sides of the equation (3.5), and using
$$\mathcal{S}^{-1}\{\tau^{\gamma};t\}=\frac{t^{\gamma-1}}{\Gamma(\gamma)},\ (\mathcal{R}(\gamma)>0)$$
we have
$$\mathcal{S}^{-1}\{\mathfrak{N}(\tau)\}=\mathfrak{N}_0\left(\prod_{\lambda=1}^{l}\sum_{r=0}^{\infty}\frac{\prod_{i=1}^{n_\lambda}\Gamma_{k_\lambda}(p_{\lambda i}+\alpha_{\lambda i}r)}{\prod_{j=1}^{m_\lambda}\Gamma_{k_\lambda}(q_{\lambda j}+\beta_{\lambda j}r)}\right)\left(\frac{c^{r\gamma}}{r!}\right)^{\lambda}$$
$$\times\Gamma(\gamma r\lambda+1)\ \mathcal{S}^{-1}\left(\sum_{\mu=0}^{\infty}(-1)^{\mu}(\delta)^{\gamma\mu}(\tau)^{\gamma r\lambda+\gamma\mu}\right).$$
That is
$$\mathfrak{N}(t)=\mathfrak{N}_0\left(\prod_{\lambda=1}^{l}\sum_{r=0}^{\infty}\frac{\prod_{i=1}^{n_\lambda}\Gamma_{k_\lambda}(p_{\lambda i}+\alpha_{\lambda i}r)}{\prod_{j=1}^{m_\lambda}\Gamma_{k_\lambda}(q_{\lambda j}+\beta_{\lambda j}r)}\right)\left(\frac{c^{r\gamma}}{r!}\right)^{\lambda}$$
$$\times\Gamma(\gamma r\lambda+1)\left(\sum_{\mu=0}^{\infty}(-1)^{\mu}(\delta)^{\gamma\mu}\frac{(t)^{\gamma r\lambda+\gamma\mu-1}}{\Gamma(\gamma r\lambda+\gamma\mu)}\right)$$
$$\mathfrak{N}(t)=\mathfrak{N}_0\left(\prod_{\lambda=1}^{l}\sum_{r=0}^{\infty}\frac{\prod_{i=1}^{n_\lambda}\Gamma_{k_\lambda}(p_{\lambda i}+\alpha_{\lambda i}r)}{\prod_{j=1}^{m_\lambda}\Gamma_{k_\lambda}(q_{\lambda j}+\beta_{\lambda j}r)}\right)\left(\frac{c^{r\gamma}t^{r\gamma}}{r!}\right)^{\lambda}\times\frac{1}{t}$$

(3.6)
$$\times\Gamma(\gamma r\lambda+1)\left(\sum_{\mu=0}^{\infty}(-1)^{\mu}\frac{(t^{\gamma}\delta^{\gamma})^{\mu}}{\Gamma(\gamma r\lambda+\gamma\mu)}\right).$$

Now, we can write equation (3.6) as
$$\mathfrak{N}(t)=\mathfrak{N}_0\left(\prod_{\lambda=1}^{l}\sum_{r=0}^{\infty}\frac{\prod_{i=1}^{n_\lambda}\Gamma_{k_\lambda}(p_{\lambda i}+\alpha_{\lambda i}r)}{\prod_{j=1}^{m_\lambda}\Gamma_{k_\lambda}(q_{\lambda j}+\beta_{\lambda j}r)}\right)\left(\frac{c^{r\gamma}t^{r\gamma}}{r!}\right)^{\lambda}\times\frac{1}{t}$$
$$\times\Gamma(\gamma r\lambda+1)\ E_{\gamma,\gamma r\lambda}\left(-\delta^{\gamma}t^{\gamma}\right).$$

□

Theorem 3.2. *Let* $\Re(\gamma) > 0, c > 0, k \in \mathbb{R}^+;\ c, z \in \mathbb{C};\ p_i, q_j \in \mathbb{C},\ \alpha_i, \beta_j \in \mathbb{R}\ (\alpha_i, \beta_j \neq 0;\ i = 1, 2, ..., n;\ j = 1, 2, ..., m)$ *and* $(p_i + \alpha_i r),\ (q_j + \beta_j r) \in \mathbb{C} \setminus k\mathbb{Z}^-$, *then the following equation*

$$\mathfrak{N}(t) - \mathfrak{N}_0\left(\prod_{\lambda=1}^{l} {}_{n_\lambda}\psi_{m_\lambda}^{k_\lambda}\left[\begin{matrix}(p_{\lambda i}, \alpha_{\lambda i})_{1,n_\lambda}\\(q_{\lambda j}, \beta_{\lambda j})_{1,m_\lambda}\end{matrix}\middle| c^\gamma t^\gamma\right]\right) = -c^\gamma\ {}_0D_t^{-\gamma}\mathfrak{N}(t)$$

has a solution given by

$$\mathfrak{N}(t) = \mathfrak{N}_0\left(\prod_{\lambda=1}^{l}\sum_{r=0}^{\infty}\frac{\prod_{i=1}^{n_\lambda}\Gamma_{k_\lambda}(p_{\lambda i} + \alpha_{\lambda i} r)}{\prod_{j=1}^{m_\lambda}\Gamma_{k_\lambda}(q_{\lambda j} + \beta_{\lambda j} r)}\right)\left(\frac{c^{r\gamma}t^{r\gamma}}{r!}\right)^\lambda \times \frac{1}{t}$$
$$\times\Gamma(\gamma r\lambda + 1)\ E_{\gamma,\gamma r\lambda}\left(-c^\gamma t^\gamma\right).$$

Theorem 3.3. *Let* $\Re(\gamma) > 0, c > 0, k \in \mathbb{R}^+;\ c, z \in \mathbb{C};\ p_i, q_j \in \mathbb{C},\ \alpha_i, \beta_j \in \mathbb{R}\ (\alpha_i, \beta_j \neq 0;\ i = 1, 2, ..., n;\ j = 1, 2, ..., m)$ *and* $(p_i + \alpha_i r),\ (q_j + \beta_j r) \in \mathbb{C} \setminus k\mathbb{Z}^-$, *then the following equation*

$$\mathfrak{N}(t) - \mathfrak{N}_0\left(\prod_{\lambda=1}^{l} {}_{n_\lambda}\psi_{m_\lambda}^{k_\lambda}\left[\begin{matrix}(p_{\lambda i}, \alpha_{\lambda i})_{1,n_\lambda}\\(q_{\lambda j}, \beta_{\lambda j})_{1,m_\lambda}\end{matrix}\middle| t\right]\right) = -c^\gamma\ {}_0D_t^{-\gamma}\mathfrak{N}(t)$$

has a solution given by

$$\mathfrak{N}(t) = \mathfrak{N}_0\left(\prod_{\lambda=1}^{l}\sum_{r=0}^{\infty}\frac{\prod_{i=1}^{n_\lambda}\Gamma_{k_\lambda}(p_{\lambda i} + \alpha_{\lambda i} r)}{\prod_{j=1}^{m_\lambda}\Gamma_{k_\lambda}(q_{\lambda j} + \beta_{\lambda j} r)}\right)\left(\frac{t^{r}}{r!}\right)^\lambda \times \frac{1}{t}$$
$$\times\Gamma(r\lambda + 1)\ E_{\gamma,r\lambda}\left(-c^\gamma t^\gamma\right).$$

Proof. The proofs of Theorem 3.2 and Theorem 3.3 are similar with the proof of Theorem 3.1 . So it is omitted here. □

4. Special Cases

(i) From previous results if we choose $\lambda = 1$, in Theorem 3.1, Theorem 3.2 and Theorem 3.3 then we have the following corollaries:

Corollary 4.1. *Let* $\Re(\gamma) > 0, \delta > 0, c > 0, k \in \mathbb{R}^+;\ c, z \in \mathbb{C};\ p_i, q_j \in \mathbb{C},\ \alpha_i, \beta_j \in \mathbb{R}\ (\alpha_i, \beta_j \neq 0;\ i = 1, 2, ..., n;\ j = 1, 2, ..., m)$ *and* $(p_i + \alpha_i r),\ (q_j + \beta_j r) \in \mathbb{C} \setminus k\mathbb{Z}^-$, *then the following equation*

$$\mathfrak{N}(t) - \mathfrak{N}_0\left({}_{n}\psi_{m}^{k}\left[\begin{matrix}(p_i, \alpha_i)_{1,n}\\(q_j, \beta_j)_{1,m}\end{matrix}\middle| c^\gamma t^\gamma\right]\right) = -\delta^\gamma\ {}_0D_t^{-\gamma}\mathfrak{N}(t)$$

has a solution given by

$$\mathfrak{N}(t) = \mathfrak{N}_0 \left(\sum_{r=0}^{\infty} \frac{\prod_{i=1}^{n} \Gamma_k(p_i + \alpha_i r)}{\prod_{j=1}^{m} \Gamma_k(q_j + \beta_j r)} \right) \left(\frac{c^{r\gamma} t^{r\gamma}}{r!} \right) \times \frac{1}{t}$$

$$\times \Gamma(\gamma r + 1)\ E_{\gamma,\gamma r}\ (-\delta^{\gamma} t^{\gamma}).$$

Corollary 4.2. *Let* $\Re(\gamma) > 0, \delta > 0, c > 0, k \in \mathbb{R}^+;\ c, z \in \mathbb{C};\ p_i, q_j \in \mathbb{C},\ \alpha_i, \beta_j \in \mathbb{R}\ (\alpha_i, \beta_j \neq 0;\ i = 1, 2, ..., n;\ j = 1, 2, ..., m)$ *and* $(p_i + \alpha_i r),\ (q_j + \beta_j r) \in \mathbb{C} \setminus k\mathbb{Z}^-$, *then the following equation*

$$\mathfrak{N}(t) - \mathfrak{N}_0 \left({}_n\psi_m^k \left[\begin{matrix} (p_i, \alpha_i)_{1,n} \\ (q_j, \beta_j)_{1,m} \end{matrix} \middle| c^{\gamma} t^{\gamma} \right] \right) = -c^{\gamma}\ {}_0D_t^{-\gamma} \mathfrak{N}(t)$$

has a solution given by

$$\mathfrak{N}(t) = \mathfrak{N}_0 \left(\sum_{r=0}^{\infty} \frac{\prod_{i=1}^{n} \Gamma_k(p_i + \alpha_i r)}{\prod_{j=1}^{m} \Gamma_k(q_j + \beta_j r)} \right) \left(\frac{c^{r\gamma} t^{r\gamma}}{r!} \right) \times \frac{1}{t}$$

$$\times \Gamma(\gamma r + 1)\ E_{\gamma,\gamma r}\ (-c^{\gamma} t^{\gamma}).$$

Corollary 4.3. *Let* $\Re(\gamma) > 0, \delta > 0, c > 0, k \in \mathbb{R}^+;\ c, z \in \mathbb{C};\ p_i, q_j \in \mathbb{C},\ \alpha_i, \beta_j \in \mathbb{R}\ (\alpha_i, \beta_j \neq 0;\ i = 1, 2, ..., n;\ j = 1, 2, ..., m)$ *and* $(p_i + \alpha_i r),\ (q_j + \beta_j r) \in \mathbb{C} \setminus k\mathbb{Z}^-$, *then the following equation*

$$\mathfrak{N}(t) - \mathfrak{N}_0 \left({}_n\psi_m^k \left[\begin{matrix} (p_i, \alpha_i)_{1,n} \\ (q_j, \beta_j)_{1,m} \end{matrix} \middle| t \right] \right) = -c^{\gamma}\ {}_0D_t^{-\gamma} \mathfrak{N}(t)$$

has a solution given by

$$\mathfrak{N}(t) = \mathfrak{N}_0 \left(\sum_{r=0}^{\infty} \frac{\prod_{i=1}^{n} \Gamma_k(p_i + \alpha_i r)}{\prod_{j=1}^{m} \Gamma_k(q_j + \beta_j r)} \right) \left(\frac{t^{r}}{r!} \right) \times \frac{1}{t}$$

$$\times \Gamma(r + 1)\ E_{\gamma,r}\ (-c^{\gamma} t^{\gamma}).$$

(ii) When we choose parameter values according as in equations (2.3), (2.4), (2.5), (2.6), (2.7) and (2.8). The new results for the product of the Mittag-Leffler functions can then be derived from the results in Theorem 3.1, Theorem 3.2 and Theorem 3.3.

5. Conclusion

In this study, a new and generalized solution of the fractional kinetic equation using the Sumudu transform technique, which involves generalized k-Wright function, is developed. In terms of the solution of the kinetic fractional equation described above, the manifold generality of the generalized function k-Wright is defined. In fact, the findings obtained here are very capable of generating a quite large number of established and presumably new results.

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