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MATHEMATICAL AND NUMERICAL ANALYSIS OF A HEAT TRANSFER PROBLEM

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ABSTRACT. In this work we study a problem of heat transfer between two compartments. The heat transfer is modeled by a coupling of heat equation. First, we proved the existence and uniqueness of the solution of the coupled problem using different mathematical analysis tools. Then, we used a numerical scheme based on the finite difference method to approach the solution of the problem.

1. INTRODUCTION

Global warming is currently almost no longer contested by the international scientific community since sensitive effects are now observable on a global scale. Indeed, we already note an increase in temperatures atmospheric and oceanic means, massive melting of snow, and ice and a rise in sea level. So the insulation and the improvement thermal equipment is an important factor in the fight against global warming. We also note that a decrease in needs energy consumption of buildings thanks to the improved insulation of the walls, will have significant economic and environmental benefits [1], [2]. The building sector is not the only one concerned since the optimization of thermal insulation [3] also represents an indisputable interest in all areas where energy consumption is high in volume and cost: storage sites (cold room, distribution centers, etc.),

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transport (road, rail ...) but also where product conservation is essential (food and medical fields ...). These works also present a strong environmental benefit because reducing energy consumption remains an objective priority in the context of sustainable development. Our study is of significant scientific interest because it allows to enrich and to advance research on semi-transparent insulation. Indeed, to increase product insulation capabilities, it is necessary to understand the conduction-radiation coupled heat transfer phenomena within these complex semi-transparent media of fibrous and alveolar type. Still, the modeling and thermal characterization of these environments remain poorly controlled.

It is in order to improve this area that we set out in the work to perform a mathematical and numerical analysis of a heat transfer problem.

For the mathematical analysis of the heat transfer problem, we use the tools of the variable separation method, of the theory of semi-groups and of the spectral theory. Then for the numerical part we use the finite difference method for a better understanding of the problem solution.

This work brings a combination of several mathematical tools for the analysis of the coupled problem and a rigorous numerical study of these types of problem.

2. Presentation of the model

Let the following coupled model between two compartments 1 and 2 of respective temperatures : $T_1 : (l_0, l_1) \times \mathbb{R}_+^* \to \mathbb{R}$ and $T_2 : (l_1, l_2) \times \mathbb{R}_+^* \to \mathbb{R}$ such that:

$$\left\{ \begin{array}{l} \dfrac{\partial T_1(x,t)}{\partial t} - \dfrac{1}{\alpha_1}\dfrac{\partial^2 T_1(x,t)}{\partial x^2} = F_1(x,t) \; \forall t > 0, \forall x \in (l_0, l_1) \\ T_1(l_0, t) = G_1(t) \; \forall t > 0 \\ \dfrac{\partial T_2(x,t)}{\partial t} - \dfrac{1}{\alpha_2}\dfrac{\partial^2 T_2(x,t)}{\partial x^2} = F_2(x,t) \; \forall t > 0, \forall x \in (l_1, l_2) \\ -\mu_2 \dfrac{\partial T_2(l_2,t)}{\partial x} = \sigma_2 G_2(t) \forall t > 0 \\ T_2(l_1, t) = T_1(l_1, t) \; \forall t > 0 \\ \mu_2 \dfrac{\partial T_2(l_1,t)}{\partial x} = \mu_1 \dfrac{\partial T_1(l_1,t)}{\partial x} \; \forall t > 0 \end{array} \right. \tag{2.1}$$

with the initial conditions

(2.2) $$T_k(x,0) = T_{k0}(x), \forall x \in (l_{k-1}, l_k),\ k = 1, 2.$$

T_1 and T_2 are the respective temperatures of the first and second compartments; $(l_1 - l_0)$ the length of the first compartment and $(l_2 - l_1)$ the length of the second compartment; α_1 and α_2 are the heat diffusion coefficients of the two respective compartments; σ_2 is the convective exchange coefficient of the second compartment; μ_1 and μ_2 are the respective thermal conductivity coefficients between the two compartments; F_1 and F_2 are respectively the external sources of heat of the two compartments; G_1 and G_2 are the respective temperatures of the outer ends of the two compartments; the last two relations of (2.1) represent the coupling conditions.

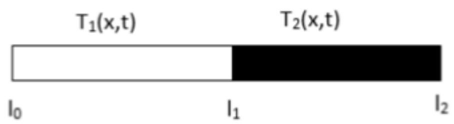

FIGURE 1. The representation of the domain

3. EXISTENCE AND UNIQUENESS OF SOLUTIONS

In this part we are interested in the existence and the unicity of the solutions of the problem (2.1) - (2.2). Thus, we make the following assumptions:

3.1. Hypotheses.

We consider the following Hilbert space:

$$\mathbf{H} := \prod_{k=1}^{2} L^2((l_{k-1}, l_k), \mathbb{R})$$

provided with the dot product

(3.1) $$\left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right\rangle := \sum_{k=1}^{2} \mu_k \int_{l_{k-1}}^{l_k} f_k(x) g_k(x) dx.$$

Consequently the norm on $\mathbf{H}$ will be the norm induced by the scalar product i.e.

$$\left\| \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\| = \left(\sum_{k=1}^{2} \mu_k \int_{l_{k-1}}^{l_k} f_k^2(x)dx \right)^{\frac{1}{2}}, \ \forall \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in L_1^2 \times L_2^2.$$

By analogy with Sobolev spaces [4] we define the subspaces $\mathbf{H}^1$ and $\mathbf{H}^2$ of $\mathbf{H}$ by

$$\mathbf{H}^m := \prod_{k=1}^{2} W^{m,2}((l_{k-1}, l_k), \mathbb{R}), \ m = 1, 2$$

and we pose for everything

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \mathbf{H}$$

$$\nabla f := \begin{pmatrix} f_1' \\ f_2' \end{pmatrix} \ \text{ si } \ f \in \mathbf{H}^1$$

and

$$\Delta f := \begin{pmatrix} f_1'' \\ f_2'' \end{pmatrix} \ \text{ si } \ f \in \mathbf{H}^2.$$

We can thus define a standard on $\mathbf{H}^1$ as follows:

$$\|f\|_{\mathbf{H}^1} := \left(\|f\|_{\mathbf{H}}^2 + \|\nabla f\|_{\mathbf{H}}^2 \right)^{\frac{1}{2}}, \ \forall f \in \mathbf{H}^1.$$

This choice of the space $\mathbf{H}$ may seem arbitrary, but it will be especially justified in the demonstration of the existence of a Hilbertian basis of $\mathbf{H}$.

3.2. **Existence of a Hilbertian basis of H.** The problem (2.1) can be associated with a problem with eigenvalues so we will solve it using a Hilbertian basis of $\mathbf{H}$. Thus demonstrating the existence of a base of $\mathbf{H}$ amounts to verifying the conditions of the theorem [5, Theorem 7.2.8]. Indeed, let's define the following problem:

$$\text{(3.2)}\qquad \begin{cases} \\ T_1''(x) = -\alpha_1 f_1(x),\ \forall x \in (l_0, l_1) \\ T_2''(x) = -\alpha_2 f_2(x),\ \forall x \in (l_1, l_2) \\ T_1(l_0) = 0 \\ T_1(l_1) = T_2(l_1) \\ \mu_1 T_1'(l_1) = \mu_2 T_2'(l_1) \\ T_2'(l_2) = 0 \end{cases}$$

Lemma 3.1. *For all $f \in \mathbf{H}$, the system* (3.2) *admits a single solution with $T \in \mathbf{H}^2$. More precisely, we have for all $k = 1, 2$:*

$$T_k(x) = -\alpha_k \int_{l_{k-1}}^{x} \left(\int_{l_{k-1}}^{s} f_k(l)dl \right) ds + \delta_{k1}(x - l_{k-1}) + \delta_{k2},\ \forall x \in (l_{k-1}, l_k)$$

with

$$\delta_{k1} = \frac{1}{\mu_k} \sum_{n=k}^{2} \alpha_n \mu_n \int_{l_{n-1}}^{l_n} f_n(l)dl,\ k = 1, 2$$

et

$$\delta_{k2} = -\alpha_1 \int_{l_0}^{l_{k-1}} \left(\int_0^s f_1(l)dl \right) ds + l_{k-1}\delta_{11},\ k = 1, 2.$$

Proof. By integrating the equations in T_1 and T_2 without taking into account the boundary conditions, we obtain for $k = 1, 2$:

$$\text{(3.3)}\quad T_k(x) = -\alpha_k \int_{l_{k-1}}^{x} \left(\int_{l_{k-1}}^{s} f_k(l)dl \right) ds + \delta_{k1}(x - l_{k-1}) + \delta_{k2},\ \forall x \in (l_{k-1}, l_k)$$

with δ_{k_j}, $k, j = 1, 2$ of the real constants. We will now determine the constants using the boundary conditions. Indeed we have:

$$T_1(0) = 0 = \delta_{12} \ \text{ et } \ T_2'(l_2) = 0 = -\alpha_2 \int_{l_1}^{l_2} f_2(l)dl + \delta_{21}$$

where,

$$\text{(3.4)}\qquad \delta_{12} = 0 \ \text{ et } \ \delta_{21} = \alpha_2 \int_{l_1}^{l_2} f_2(l)dl.$$

Furthermore using (3.2) we have:

$$\text{(3.5)}\qquad T_1(l_1) = T_2(l_1) \Rightarrow -\alpha_1 \int_0^{l_1} \left(\int_0^s f_1(l)dl \right) ds + \delta_{11} l_1 = \delta_{22}$$

and

$$\mu_1 T_1'(l_1) = \mu_2 T_2'(l_1) \Rightarrow -\mu_1\alpha_1 \int_0^{l_1} f_1(l)dl + \mu_1\delta_{11} = \mu_2\delta_{21}. \tag{3.6}$$

By replacing the expression of δ_{21} in (3.4) in (3.6) we get

$$\delta_{11} = \frac{1}{\mu_1}\left[\mu_2\alpha_2 \int_{l_1}^{l_2} f_2(l)dl + \mu_1\alpha_1 \int_0^{l_1} f_1(l)dl\right] \tag{3.7}$$

and replacing the expression of δ_{11} in (3.7) we deduce that:

$$\delta_{22} = -\alpha_1 \int_0^{l_1}\left(\int_0^{s} f_1(l)dl\right)ds + \frac{l_1}{\mu_1}\left[\mu_2\alpha_2 \int_{l_1}^{l_2} f_2(l)dl + \mu_1\alpha_1 \int_0^{l_1} f_1(l)dl\right]. \tag{3.8}$$

The result is obtained by incorporating the expressions obtained in (3.4), (3.7) and (3.8) in (3.3). □

So we can define our operator as follows:

$$\begin{array}{rccc} \mathcal{L}: & \mathbf{H} & \to & \mathbf{H^1} \\ & f & \mapsto & \mathcal{L}f \end{array} \tag{3.9}$$

with $\mathcal{L}f$ the solution of the equation (3.2) defined in Lemma 3.1. More precisely

$$T = \mathcal{L}f \Leftrightarrow \begin{cases} T_1''(x) = -\alpha_1 f_1(x), \ \forall x \in (l_0, l_1) \\ T_2''(x) = -\alpha_2 f_2(x), \ \forall x \in (l_1, l_2) \\ T_1(0) = 0 \\ T_1(l_1) = T_2(l_1) \\ \mu_1 T_1'(l_1) = \mu_2 T_2'(l_1) \\ T_2'(l_2) = 0 \end{cases}$$

The objective now is to demonstrate that the operator $\mathcal{L}$ thus defined is continuous linear, self-adjoint and compact.

Lemma 3.2. *The operator* $\mathcal{L}$ *defined in* (3.9) *is linear and satisfies the following properties:*

i) $\mathcal{L}f \in \mathbf{H}^2$, $\forall f \in \mathbf{H}$,
ii) $\langle \Delta\mathcal{L}f, \mathcal{L}g\rangle = -\langle \nabla\mathcal{L}f, \nabla\mathcal{L}g\rangle$, $\forall f, g \in \mathbf{H}$,
iii) $\langle \Delta\mathcal{L}f, \mathcal{L}g\rangle = \langle \mathcal{L}f, \Delta\mathcal{L}g\rangle$, $\forall f, g \in \mathbf{H}$.

Proof.

(i) The linearity of $\mathcal{L}$ and the property *i)* are direct consequences of the definition of $\mathcal{L}$ and of the Lemma 3.1.

(ii) Let $f, g \in \mathbf{H}$. We pose:

$$\mathcal{L}f = T = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} \in \mathbf{H}^2 \text{ et } \mathcal{L}g = K = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \in \mathbf{H}^2.$$

We then have:

$$\begin{aligned} \langle \Delta T, K \rangle &= \sum_{k=1}^{2} \mu_k \int_{l_{k-1}}^{l_k} T_k''(x) K_k(x) dx \\ &= \sum_{k=1}^{2} \mu_k [T_k'(l_k) K_k(l_k) - T_k'(l_{k-1}) K_k(l_{k-1})] - \sum_{k=1}^{2} \mu_k \int_{l_{k-1}}^{l_k} T_k'(x) K_k'(x) dx \\ &= [\mu_1 T_1'(l_1) K_1(l_1) - \mu_2 T_2'(l_1) K_2(l_1)] \\ &\quad + [-\mu_1 T_1'(l_0) K_1(l_0) + \mu_2 T_2'(l_2) K_2(l_2)] - \langle \nabla T, \nabla K \rangle \end{aligned}$$

from where by using the conditions with the edges and the conditions of coupling ie $K_1(l_0) = T_2'(l_2) = 0$, $\mu_1 T_1'(l_1) = \mu_2 T_2'(l_1)$ and $K_1(l_1) = K_2(l_1)$ we deduce that:

$$\langle \Delta T, K \rangle = -\langle \nabla T, \nabla K \rangle$$

and property ii) ensues.

(iii) Similar to the proof of *ii)* we get:

$$\langle \nabla T, \nabla K \rangle = -\left\langle T, \nabla^2 K \right\rangle = -\langle T, \Delta K \rangle$$

where,

$$\langle \Delta T, K \rangle = \langle T, \Delta K \rangle$$

which proves property *iii)*.

□

Before demonstrating the continuity of the operator $\mathcal{L}$ we will first prove the following lemma:

Lemma 3.3. *The product norm is equivalent to the norm* $\|\cdot\|$ *in* $\mathbf{H}$.

Proof. We suppose that $\mu_1, \mu_2 \geq 1$. We have:

$$\begin{array}{rcl} \langle f, f\rangle & = & \\ \left\langle \begin{pmatrix} \mu_1 f_1 \\ \mu_2 f_2 \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\rangle & = & \mu_1 \left\langle \begin{pmatrix} f_1 \\ 0 \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\rangle + \mu_2 \left\langle \begin{pmatrix} 0 \\ f_2 \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\rangle \\ & \leq & (\mu_1+\mu_2) \left(\left\langle \begin{pmatrix} f_1 \\ 0 \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} 0 \\ f_2 \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\rangle \right) \\ & \leq & (\mu_1+\mu_2) \left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\rangle \end{array}$$

where

$$\langle f, f\rangle \leq (\mu_1+\mu_2) \left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\rangle. \tag{3.10}$$

Likewise:

$$\begin{array}{rcl} \langle f, f\rangle & \geq & \min(\mu_1^{-1}, \mu_2^{-1}) \left\langle \begin{pmatrix} f_1 \\ 0 \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\rangle + \min(\mu_1^{-1}, \mu_2^{-1}) \left\langle \begin{pmatrix} 0 \\ f_2 \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\rangle \\ & \geq & \min(\mu_1^{-1}, \mu_2^{-1}) \left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\rangle, \end{array}$$

where,

$$\langle f, f\rangle \geq \min(\mu_1^{-1}, \mu_2^{-1}) \left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\rangle. \tag{3.11}$$

(3.10) and (3.11) show that the two norms are equivalent. ◻

Lemma 3.4. *Let $\mathcal{L}$ be the linear operator defined in* (3.9) *then there is a constant $\gamma > 0$ such that:*

$$\|\mathcal{L}f\|_{\mathbf{H}} \leq \gamma \|\nabla \mathcal{L} f\|_{\mathbf{H}}. \tag{3.12}$$

Moreover for all $f \in \mathbf{H}$, $\mathcal{L}f$ satisfies the Poincare inequality:

$$\|\mathcal{L}f\|_{\mathbf{H}^1} \leq (1+\gamma^2) \|\nabla \mathcal{L} f\|_{\mathbf{H}}. \tag{3.13}$$

Proof. Let $f \in \mathbf{H}$. We pose

$$T = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \mathcal{L}f.$$

Since the product norm is equivalent to the norm $\|\cdot\|$ in $\mathbf{H}$ (see Lemma 3.3), it suffices to show that the inequality (3.12) is true for the product standard. We note that the conditions $T_1(l_1) = T_2(l_1)$ and $T_1(0) = 0$ result in:

$$|T_2(x)| = \left|T_1(l_1) + \int_{l_1}^{x} T_2'(s)ds\right| \leq |T_1(l_1)| + \int_{l_1}^{l_2} |T_2'(s)|ds, \ \forall x \in [l_1, l_2]$$

and

$$|T_1(x)| = \left|\int_{l_0}^{x} T_1'(s)ds\right| \leq \int_{l_0}^{l_1} |T_1'(s)|ds, \ \forall x \in [l_0, l_1]$$

where

$$\begin{cases} \|T_1\|_{L^\infty((l_0,l_1),\mathbb{R})} \leq \|T_1'\|_{L^1((l_0,l_1),\mathbb{R})} \\ \\ \|T_2\|_{L^\infty((l_1,l_2),\mathbb{R})} \leq \|T_1\|_{L^\infty((l_0,l_1),\mathbb{R})} + \|T_2'\|_{L^1((l_1,l_2),\mathbb{R})} \end{cases}$$

Therefore we have:

$$\begin{cases} \|T_1\|_{L^\infty((l_0,l_1),\mathbb{R})} \leq \|T_1'\|_{L^1((l_0,l_1),\mathbb{R})} \\ \\ \|T_2\|_{L^\infty((l_1,l_2),\mathbb{R})} \leq \|T_1'\|_{L^1((l_0,l_1),\mathbb{R})} + \|T_2'\|_{L^1((l_1,l_2),\mathbb{R})}. \end{cases} \tag{3.14}$$

As in the case of bounded intervals there is a continuous injection of L^∞ in L^2 and of L^2 in L^1 we deduce that there are constants $C_1, C_2 > 0$ (dependent only on the intervals (l_0, l_1) and (l_1, l_2)) such that:

$$\begin{cases} \|T_1\|_{L^2((l_0,l_1),\mathbb{R})} \leq C_1\|T_1\|_{L^\infty((l_0,l_1),\mathbb{R})} \\ \\ \|T_1'\|_{L^1((l_0,l_1),\mathbb{R})} \leq C_2\|T_1'\|_{L^2((l_0,l_1),\mathbb{R})} \\ \\ \|T_2\|_{L^2((l_1,l_2),\mathbb{R})} \leq C_1\|T_2\|_{L^\infty((l_1,l_2),\mathbb{R})} \\ \\ \|T_2'\|_{L^1((l_1,l_2),\mathbb{R})} \leq C_2\|T_2'\|_{L^2((l_1,l_2),\mathbb{R})}. \end{cases} \tag{3.15}$$

By combining the inequalities (3.14) and (3.15) we obtain:

$$\begin{cases} \|T_1\|_{L^2((l_0,l_1),\mathbb{R})} \leq C_1C_2\|T_1'\|_{L^2((l_0,l_1),\mathbb{R})} \\ \\ \|T_2\|_{L^2((l_1,l_2),\mathbb{R})} \leq C_1C_2\left[\|T_1'\|_{L^2((l_0,l_1),\mathbb{R})} + \|T_2'\|_{L^2((l_1,l_2),\mathbb{R})}\right]. \end{cases}$$

where,

$$\|T_1\|_{L^2((l_0,l_1),\mathbb{R})} + \|T_2\|_{L^2((l_1,l_2),\mathbb{R})} \leq 2C_1C_2\left[\|T_1'\|_{L^2((l_0,l_1),\mathbb{R})} + \|T_2'\|_{L^2((l_1,l_2),\mathbb{R})}\right]$$

which proves the inequality (3.12). The Poincare inequality (3.13) is obtained by using (3.12). More precisely we have:

$$\|\mathcal{L}f\|^2_{\mathbf{H}^1} = \|\mathcal{L}f\|^2_{\mathbf{H}} + \|\nabla\mathcal{L}f\|^2_{\mathbf{H}} \leq (1+\gamma^2)\|\nabla\mathcal{L}f\|^2_{\mathbf{H}}.$$

□

Proposition 3.1. ***[continuity]*** *The operator $\mathcal{L}$ defined in* (3.9) *checks the following properties:*

i) $\mathcal{L} : \mathbf{H} \to \mathbf{H}^1$ *is linear and continuous from* $\mathbf{H}$ *to values in* $\mathbf{H}^1$*,*
ii) $\mathcal{L} : \mathbf{H} \to \mathbf{H}$ *is compact.*

Proof.

(i) Let $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \mathbf{H}$. First note that thanks to the property *ii)* of the Lemma 3.2 with $f = g$ and to the Cauchy-Schwartz inequality we have:

$$\begin{aligned} |\langle\nabla\mathcal{L}f, \nabla\mathcal{L}f\rangle| = |\langle\Delta\mathcal{L}f, \mathcal{L}f\rangle| \quad &\Rightarrow \quad |\langle\nabla\mathcal{L}f, \nabla\mathcal{L}f\rangle| \leq \|\mathcal{L}f\|_{\mathbf{H}}\|\Delta\mathcal{L}f\|_{\mathbf{H}} \\ &\Rightarrow \qquad \|\nabla\mathcal{L}f\|^2_{\mathbf{H}} \leq \|\mathcal{L}f\|_{\mathbf{H}}\|\Delta\mathcal{L}f\|_{\mathbf{H}} \end{aligned}$$

and since

$$\Delta\mathcal{L}f = \begin{pmatrix} -\alpha_1 f_1 \\ -\alpha_2 f_2 \end{pmatrix} \Rightarrow \|\Delta\mathcal{L}f\|_{\mathbf{H}} \leq (\alpha_1+\alpha_2)\|f\|_{\mathbf{H}}$$

we deduce that

$$\|\nabla\mathcal{L}f\|^2_{\mathbf{H}} \leq (\alpha_1+\alpha_2)\|\mathcal{L}f\|_{\mathbf{H}}\|f\|_{\mathbf{H}}.$$

According to the inequality (3.12) we have:

$$\|\mathcal{L}f\|_{\mathbf{H}} \leq \gamma\|\nabla\mathcal{L}f\|_{\mathbf{H}} \Rightarrow \|\nabla\mathcal{L}f\|^2_{\mathbf{H}} \leq (\alpha_1+\alpha_2)\gamma\|\nabla\mathcal{L}f\|_{\mathbf{H}}\|f\|_{\mathbf{H}}$$

where,

$$\|\nabla\mathcal{L}f\|_{\mathbf{H}} \leq (\alpha_1+\alpha_2)\gamma\|f\|_{\mathbf{H}}$$

and recalling the inequality (3.13)

$$\|\mathcal{L}f\|_{\mathbf{H}^1} \leq (1+\gamma^2)\|\nabla\mathcal{L}f\|_{\mathbf{H}}$$

we deduce that

$$\|\mathcal{L}f\|_{\mathbf{H}^1} \leq \gamma(1+\gamma^2)(\alpha_1+\alpha_2)\|f\|_{\mathbf{H}}.$$

(ii) Let $\mathcal{I} : \mathbf{H}^1 \to \mathbf{H}$, $f \mapsto \mathcal{I}f = f$ the injection operator and $\mathcal{L}f$ the operator defines in (3.9). So we have: $\mathcal{I} \circ \mathcal{L}$ defined from $\mathbf{H}$ to value in $\mathbf{H}$.

$$\mathcal{L}f \in \mathbf{H}^1, \ \forall f \in \mathbf{H} \Rightarrow \mathcal{L}f = (\mathcal{I} \circ \mathcal{L})f, \ \forall f \in \mathbf{H}$$

and since $\mathcal{I}$ is compact then $\mathcal{L}$ is compact as composed of compact and continuous operator.

$\square$

Theorem 3.1. *Let $\mathcal{L}$ be the linear operator defined in* (3.1). *So we have the following properties:*

i) *The space* $\mathbf{H}$ *provided with the scalar product*

$$\langle\langle f, g\rangle\rangle := \sum_{k=1}^{2} \alpha_k \mu_k \int_{l_{k-1}}^{l_k} f_k(x) g_k(x) dx, \ \forall f, g \in \mathbf{H}$$

admits a Hilbertian basis formed by eigenvectors associated with the eigenvalues of $\mathcal{L}$,

ii) *$\mathcal{L}$ is defined positive for the dot product $\langle\langle \cdot, \cdot \rangle\rangle$. In particular all the eigenvalues of $\mathcal{L}$ form a sequence $(\nu_n)_{n\geq 1}$ such that:*

$$\nu_n > 0, \ \forall n \geq 1 \ \textit{ et } \lim_{n\to+\infty} \nu_n = 0.$$

Remark 3.1. *We note that the product $\langle\langle \cdot, \cdot \rangle\rangle$ is linked by the scalar product $\langle \cdot, \cdot \rangle$ by the following relation:*

$$\langle\langle f, g\rangle\rangle = \left\langle \begin{pmatrix} \alpha_1 f_1 \\ \alpha_2 f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} \alpha_1 g_1 \\ \alpha_2 g_2 \end{pmatrix} \right\rangle, \ \forall f, g \in \mathbf{H}.$$

We will use this equality in the proof of Theorem 3.1.

Lemma 3.5. *The norm induced by the scalar products $\langle \cdot, \cdot \rangle$ and $\langle\langle \cdot, \cdot \rangle\rangle$ are equivalent.*

Proof. We suppose that $\alpha_1, \alpha_2 \geq 1$. We have:

$$\begin{aligned}
\langle\langle f, f\rangle\rangle &= \\
\left\langle \begin{pmatrix} \alpha_1 f_1 \\ \alpha_2 f_2 \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\rangle &= \alpha_1 \left\langle \begin{pmatrix} f_1 \\ 0 \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\rangle + \alpha_2 \left\langle \begin{pmatrix} 0 \\ f_2 \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\rangle \\
&\leq (\alpha_1 + \alpha_2) \left(\left\langle \begin{pmatrix} f_1 \\ 0 \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} 0 \\ f_2 \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\rangle \right) \\
&\leq (\alpha_1 + \alpha_2) \left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\rangle = (\alpha_1 + \alpha_2) \langle f, f\rangle
\end{aligned}$$

from where,

$$\langle\langle f, f\rangle\rangle = (\alpha_1 + \alpha_2) \langle f, f\rangle \tag{3.16}$$

similarly

$$\begin{aligned}
\langle\langle f, f\rangle\rangle &\geq \min(\alpha_1^{-1}, \alpha_2^{-1}) \left\langle \begin{pmatrix} f_1 \\ 0 \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\rangle + \min(\alpha_1^{-1}, \alpha_2^{-1}) \left\langle \begin{pmatrix} 0 \\ f_2 \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\rangle \\
&\geq \min(\alpha_1^{-1}, \alpha_2^{-1}) \left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\rangle = \min(\alpha_1^{-1}, \alpha_2^{-1}) \langle f, f\rangle
\end{aligned}$$

from where

$$\langle\langle f, f\rangle\rangle \geq \min(\alpha_1^{-1}, \alpha_2^{-1}) \langle f, f\rangle . \tag{3.17}$$

(3.16) and (3.17) show that the two norms are equivalent. □

We now establish the proof of the previous Theorem 3.1:

Proof.

(i) Since the norm induced by the dot products $\langle\cdot,\cdot\rangle$ and $\langle\langle\cdot,\cdot\rangle\rangle$ are equivalent, it is clear that the Proposition 3.1 remains true, that is to say the continuity and the compactness, when $\mathbf{H}$ is provided with the standard induced by $\langle\langle\cdot,\cdot\rangle\rangle$. It suffices to show that $\mathcal{L}$ is self-adjoint for the dot product $\langle\langle\cdot,\cdot\rangle\rangle$. Let $f, g \in \mathbf{H}$. According to Lemma 3.2 we have

$$\langle \Delta\mathcal{L}f, \mathcal{L}g\rangle = \langle \mathcal{L}f, \Delta\mathcal{L}g\rangle \Leftrightarrow \langle \Delta\mathcal{L}f, \mathcal{L}g\rangle = -\langle \mathcal{L}f, -\Delta\mathcal{L}g\rangle .$$

as

$$-\Delta\mathcal{L}f = \begin{pmatrix} \alpha_1 f_1 \\ \alpha_2 f_2 \end{pmatrix} \text{ et } -\Delta\mathcal{L}g = \begin{pmatrix} \alpha_1 g_1 \\ \alpha_2 g_2 \end{pmatrix}$$

we deduce that

$$\langle\langle f, \mathcal{L}g\rangle\rangle = \langle -\Delta\mathcal{L}f, \mathcal{L}g\rangle = \langle \mathcal{L}f, -\Delta\mathcal{L}g\rangle = \langle\langle \mathcal{L}f, g\rangle\rangle .$$

So $\mathcal{L}$ is self-adjoint for the scalar product $\langle\langle\cdot,\cdot\rangle\rangle$.

(ii) We have:

$$\langle\langle \mathcal{L}f, f\rangle\rangle = \left\langle \mathcal{L}f, \begin{pmatrix} \alpha_1 f_1 \\ \alpha_2 f_2 \end{pmatrix} \right\rangle = \langle \mathcal{L}f, -\Delta\mathcal{L}f\rangle = -\langle \mathcal{L}f, \Delta\mathcal{L}f\rangle = \langle \nabla\mathcal{L}f, \nabla\mathcal{L}f\rangle \geq 0.$$

Here we use the Lemma 3.2.

□

3.2.1. *Determination of eigenvectors.* We know from Theorem 3.1 that there is a Hilbertian basis of $\mathbf{H}$ formed by eigenvectors associated with the eigenvalues of $\mathcal{L}$. The objective of this subsection is to determine these eigenvectors. Let $\nu \in \sigma(\mathcal{L})$ (the spectrum of $\mathcal{L}$). An eigenvector V associated with the eigenvalue ν is a non-zero solution of the equation:

$$\nu V = \mathcal{L}V \Leftrightarrow \begin{cases} \nu V_1''(x) = -\alpha_1 V_1(x),\ \forall x \in (l_0, l_1) \\ \nu V_2''(x) = -\alpha_2 V_2(x),\ \forall x \in (l_1, l_2) \\ \nu V_1(0) = 0 \\ \nu V_1(l_1) = \nu V_2(l_1) \\ \nu\mu_1 V_1'(l_1) = \nu\mu_2 V_2'(l_1) \\ \nu V_2'(l_2) = 0. \end{cases} \tag{3.18}$$

Recall that according to Theorem [5, Theorem 7.2.8], the eigenvalues of $\mathcal{L}$ are strictly positive. Consequently we can rewrite the system (3.18) in the form:

$$\nu V = \mathcal{L}V \Leftrightarrow \begin{cases} V_1''(x) = -\frac{\alpha_1}{\nu} V_1(x),\ \forall x \in (l_0, l_1) \\ V_2''(x) = -\frac{\alpha_2}{\nu} V_2(x),\ \forall x \in (l_1, l_2) \\ V_1(0) = 0 \\ V_1(l_1) = V_2(l_1) \\ \mu_1 V_1'(l_1) = \mu_2 V_2'(l_1) \\ V_2'(l_2) = 0. \end{cases}$$

By solving we get:

$$\begin{cases} V_1(x) = C_{11}\cos\left(\sqrt{\frac{\alpha_1}{\nu}}(x-l_0)\right) + C_{12}\sin\left(\sqrt{\frac{\alpha_1}{\nu}}(x-l_0)\right),\ x\in(l_0,l_1) \\ V_2(x) = C_{21}\cos\left(\sqrt{\frac{\alpha_2}{\nu}}(l_2-x)\right) + C_{22}\sin\left(\sqrt{\frac{\alpha_2}{\nu}}(l_2-x)\right),\ x\in(l_1,l_2). \end{cases}$$

Condition $V_1(l_0) = V_2'(l_2) = 0$ results:

$$C_{11} = C_{22} = 0 \Rightarrow \begin{cases} V_1(x) = C_{12}\sin\left(\sqrt{\frac{\alpha_1}{\nu}}(x-l_0)\right),\ x\in(l_0,l_1) \\ V_2(x) = C_{21}\cos\left(\sqrt{\frac{\alpha_1}{\nu}}(l_2-x)\right),\ x\in(l_1,l_2). \end{cases}$$

Conditions $V_1(l_1) = V_2(l_1)$ and $\mu_1 V_1'(l_1) = \mu_2 V_2'(l_1)$ give us the following system:

$$\begin{cases} C_{12}\sin\left(\sqrt{\frac{\alpha_1}{\nu}}(l_1-l_0)\right) = C_{21}\cos\left(\sqrt{\frac{\alpha_2}{\nu}}(l_2-l_1)\right) \\ \mu_1 C_{12}\sqrt{\frac{\alpha_1}{\nu}}\cos\left(\sqrt{\frac{\alpha_1}{\nu}}(l_1-l_0)\right) = \mu_2 C_{21}\sqrt{\frac{\alpha_2}{\nu}}\sin\left(\sqrt{\frac{\alpha_2}{\nu}}(l_2-l_1)\right) \end{cases} \tag{3.19}$$

hence by multiplying the first line of (3.19) by μ_1 we have:

$$\begin{cases} \mu_1 C_{12}\sqrt{\frac{\alpha_1}{\nu}}\sin\left(\sqrt{\frac{\alpha_1}{\nu}}(l_1-l_0)\right) = \mu_1 C_{21}\sqrt{\frac{\alpha_1}{\nu}}\cos\left(\sqrt{\frac{\alpha_2}{\nu}}(l_2-l_1)\right) \\ \mu_1 C_{12}\sqrt{\frac{\alpha_1}{\nu}}\cos\left(\sqrt{\frac{\alpha_1}{\nu}}(l_1-l_0)\right) = \mu_2 C_{21}\sqrt{\frac{\alpha_2}{\nu}}\sin\left(\sqrt{\frac{\alpha_2}{\nu}}(l_2-l_1)\right). \end{cases}$$

By raising both lines of (3.19) squared then by adding them we have:

$$C_{12}^2\mu_1^2\frac{\alpha_1}{\nu} = C_{21}^2\left(\mu_1^2\frac{\alpha_1}{\nu}\cos^2\left(\sqrt{\frac{\alpha_2}{\nu}}(l_2-l_1)\right) + \mu_2^2\frac{\alpha_2}{\nu}\sin^2\left(\sqrt{\frac{\alpha_2}{\nu}}(l_2-l_1)\right)\right). \tag{3.20}$$

Remark 3.2. *C_{12} and C_{21} are different from 0. Indeed if $C_{12}=0$ or $C_{21}=0$ then using the equality* (3.20) *we get $C_{12}=0 \Leftrightarrow C_{21}=0$ by Consequently $C_{12}\neq 0$ and $C_{21}\neq 0$ because otherwise $\mu_1\equiv 0$ and $\mu_2\equiv 0$.*

Without loss of generality we can therefore take C_{21} as a multiple of C_{12}. Consequently we can set $C_{12}=C$ and $C_{21}=1$. Indeed according to eqref B05 we have:

$$\begin{pmatrix} V_1(x) \\ V_2(x) \end{pmatrix} = \begin{pmatrix} C_{12}\sin\left(\sqrt{\frac{\alpha_1}{\nu}}(x-l_0)\right) \\ C_{21}\cos\left(\sqrt{\frac{\alpha_2}{\nu}}(l_2-x)\right) \end{pmatrix}.$$

By dividing the equality by C_{12} it results:

$$\frac{1}{C_{12}}\begin{pmatrix} V_1(x) \\ V_2(x) \end{pmatrix} = \begin{pmatrix} \sin\left(\sqrt{\frac{\alpha_1}{\nu}}(x-l_0)\right) \\ \frac{C_{21}}{C_{12}}\cos\left(\sqrt{\frac{\alpha_2}{\nu}}(l_2-x)\right) \end{pmatrix}.$$

From where we can take $C_{12} = 1$ and $C_{21} = C$, we get that C must satisfy the following system

$$C^2\mu_1^2\frac{\alpha_1}{\nu} = \left(\mu_1^2\frac{\alpha_1}{\nu}\cos^2\left(\sqrt{\frac{\alpha_2}{\nu}}(l_2-l_1)\right) + \mu_2^2\frac{\alpha_2}{\nu}\sin^2\left(\sqrt{\frac{\alpha_2}{\nu}}(l_2-l_1)\right)\right)$$

with $\nu > 0$.

So our system admits a solution $(C,1) \neq (0,1)$ if and only if

$$0 = \begin{vmatrix} \sin\left(\sqrt{\frac{\alpha_1}{\nu}}(l_1-l_0)\right) & -\cos\left(\sqrt{\frac{\alpha_2}{\nu}}(l_2-l_1)\right) \\ \mu_1\sqrt{\frac{\alpha_1}{\nu}}\cos\left(\sqrt{\frac{\alpha_1}{\nu}}(l_1-l_0)\right) & -\mu_2\sqrt{\frac{\alpha_2}{\nu}}\sin\left(\sqrt{\frac{\alpha_2}{\nu}}(l_2-l_1)\right) \end{vmatrix}$$

or equivalent

$$\begin{aligned} 0 = &-\mu_2\sqrt{\tfrac{\alpha_2}{\nu}}\sin\left(\sqrt{\tfrac{\alpha_1}{\nu}}(l_1-l_0)\right)\sin\left(\sqrt{\tfrac{\alpha_2}{\nu}}(l_2-l_1)\right) \\ &+\mu_1\cos\left(\sqrt{\tfrac{\alpha_2}{\nu}}(l_2-l_1)\right)\sqrt{\tfrac{\alpha_1}{\nu}}\cos\left(\sqrt{\tfrac{\alpha_1}{\nu}}(l_1-l_0)\right). \end{aligned} \tag{3.21}$$

Which gives the characteristic equation:

$$\begin{aligned} \chi(\nu) = &-\mu_2\sqrt{\tfrac{\alpha_2}{\nu}}\sin\left(\sqrt{\tfrac{\alpha_1}{\nu}}(l_1-l_0)\right)\sin\left(\sqrt{\tfrac{\alpha_2}{\nu}}(l_2-l_1)\right) \\ &+\mu_1\cos\left(\sqrt{\tfrac{\alpha_2}{\nu}}(l_2-l_1)\right)\sqrt{\tfrac{\alpha_1}{\nu}}\cos\left(\sqrt{\tfrac{\alpha_1}{\nu}}(l_1-l_0).\right) \end{aligned}$$

So, we have the following proposition:

Proposition 3.2. *the eigenvector of $\mathcal{L}$ is given by:*

- *If* $\sin\left(\sqrt{\frac{\alpha_1}{\nu}}(l_1-l_0)\right) = 0$ *then the eigenvector of $\mathcal{L}$ is of the form:*

$$V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \text{ with } \begin{cases} V_1(x) = \sin\left(\sqrt{\frac{\alpha_1}{\nu}}(x-l_0)\right), \ x \in (l_0,l_1) \\ V_2(x) = \dfrac{\mu_1}{\mu_2}\dfrac{\cos\left(\sqrt{\frac{\alpha_1}{\nu}}(l_1-l_0)\right)}{\sin\left(\sqrt{\frac{\alpha_2}{\nu}}(l_2-l_1)\right)}\cos\left(\sqrt{\frac{\alpha_2}{\nu}}(l_2-x)\right), \ x \in (l_1,l_2). \end{cases}$$

  with $\nu > 0$ the solution of the characteristic equation (3.21) *of $\mathcal{L}$.*

- *If* $\sin\left(\sqrt{\frac{\alpha_1}{\nu}}(l_1-l_0)\right) \neq 0$ *then the eigenvector of $\mathcal{L}$ is of the form:*

$$V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \text{ with } \begin{cases} V_1(x) = \sin\left(\sqrt{\frac{\alpha_1}{\nu}}(x-l_0)\right), \ x \in (l_0,l_1) \\ V_2(x) = \frac{\sin(\sqrt{\frac{\alpha_1}{\nu}}(l_1-l_0))}{\cos(\sqrt{\frac{\alpha_2}{\nu}}(l_2-l_1))}\cos\left(\sqrt{\frac{\alpha_2}{\nu}}(l_2-x)\right), \ x \in (l_1,l_2). \end{cases}$$

  with $\nu > 0$ the solution of the characteristic equation (3.21) *of $\mathcal{L}$.*

Proof.

- if $\cos\left(\sqrt{\frac{\alpha_2}{\nu}}(l_2-l_1)\right) = 0$.
  According to (3.19) if $\sin\left(\sqrt{\frac{\alpha_1}{\nu}}(l_1-l_0)\right) = 0$ then $\cos\left(\sqrt{\frac{\alpha_2}{\nu}}(l_1-l_0)\right) = 0$ because $C_{21} \neq 0$.

Consequently the equality (3.20) is checked for all $C_{21} \neq 0$ and $C_{12} \neq 0$.
We know that :
$\left\{ \begin{array}{l} \sin\left(\sqrt{\frac{\alpha_1}{\nu}}(l_1 - l_0)\right) = 0 \\ \cos\left(\sqrt{\frac{\alpha_2}{\nu}}(l_1 - l_0)\right) = 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} \cos\left(\sqrt{\frac{\alpha_1}{\nu}}(l_1 - l_0)\right) \neq 0 \\ \sin\left(\sqrt{\frac{\alpha_2}{\nu}}(l_1 - l_0)\right) \neq 0 \end{array} \right.$
So we assume that:

$$C_{21} = \frac{\mu_1}{\mu_2} \frac{\cos\left(\sqrt{\frac{\alpha_1}{\nu}}(l_1 - l_0)\right)}{\sin\left(\sqrt{\frac{\alpha_2}{\nu}}(l_2 - l_1)\right)} C_{12}$$

and therefore the eigenvector of $\mathcal{L}$ is given by:

$$V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \text{ with } \left\{ \begin{array}{l} V_1(x) = \sin\left(\sqrt{\frac{\alpha_1}{\nu}}(x - l_0)\right), \ x \in (l_0, l_1) \\ V_2(x) = \frac{\mu_1}{\mu_2} \frac{\cos\left(\sqrt{\frac{\alpha_1}{\nu}}(l_1 - l_0)\right)}{\sin\left(\sqrt{\frac{\alpha_2}{\nu}}(l_2 - l_1)\right)} \cos\left(\sqrt{\frac{\alpha_2}{\nu}}(l_2 - x)\right), \ x \in (l_1, l_2). \end{array} \right.$$

- if $\cos\left(\sqrt{\frac{\alpha_2}{\nu}}(l_2 - l_1)\right) \neq 0$
  According to (3.19) if $\cos\left(\sqrt{\frac{\alpha_2}{\nu}}(l_2 - l_1)\right) \neq 0$ then $\cos\left(\sqrt{\frac{\alpha_2}{\nu}}(l_1 - l_0)\right) \neq 0$ because $C_{21} \neq 0$. Consequently the equality (3.20) is checked for all $C_{21} \neq 0$ and $C_{12} \neq 0$.
  We know that :
  $\left\{ \begin{array}{l} \sin\left(\sqrt{\frac{\alpha_1}{\nu}}(l_1 - l_0)\right) \neq 0 \\ \cos\left(\sqrt{\frac{\alpha_2}{\nu}}(l_1 - l_0)\right) \neq 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} \cos\left(\sqrt{\frac{\alpha_1}{\nu}}(l_1 - l_0)\right) = 0 \\ \sin\left(\sqrt{\frac{\alpha_2}{\nu}}(l_1 - l_0)\right) = 0 \end{array} \right.$
  So we assume that:

$$C_{21} = \frac{\sin\left(\sqrt{\frac{\alpha_1}{\nu}}(l_1 - l_0)\right)}{\cos\left(\sqrt{\frac{\alpha_2}{\nu}}(l_2 - l_1)\right)} C_{12}$$

and therefore the eigenvector of $\mathcal{L}$ is given by:

$$V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \text{ with } \left\{ \begin{array}{l} V_1(x) = \sin\left(\sqrt{\frac{\alpha_1}{\nu}}(x - l_0)\right), \ x \in (l_0, l_1) \\ V_2(x) = \frac{\sin\left(\sqrt{\frac{\alpha_1}{\nu}}(l_1 - l_0)\right)}{\cos\left(\sqrt{\frac{\alpha_2}{\nu}}(l_2 - l_1)\right)} \cos\left(\sqrt{\frac{\alpha_2}{\nu}}(l_2 - x)\right), \ x \in (l_1, l_2). \end{array} \right.$$

□

3.3. **Existence and uniqueness of solutions to the general problem.** First, we will reduce the general non-homogeneous problem (2.1) to a homogeneous problem. Second, we use the theory developed in subsection 3.2. So, we assume that:

$$T_k(x,t) = U_k(x,t) + W_k(x,t), \ x \in (l_{k-1}, l_k), \ k = 1, 2, \ t > 0.$$

Hypothesis
We suppose that:

(i) G_1 and G_2 are differentiable on $(0, +\infty)$.
(ii) G_1' and G_2' are locally integrable on $(0, +\infty)$.

Determine W_k and U_k, $k = 1, 2$ so that T_k, $k = 1, 2$ verify the problem (2.1) and U_k, $k = 1, 2$ verify a problem with homogeneous edges. Indeed by calculating the partial derivatives of $T_k = U_k + W_k$, $k = 1, 2$ and by replacing in (2.1) we obtain:

$$(3.22)\quad \begin{cases} \dfrac{\partial U_1(x,t)}{\partial t} - \dfrac{1}{\alpha_1}\dfrac{\partial^2 U_1(x,t)}{\partial x^2} = -\dfrac{\partial W_1(x,t)}{\partial t} + \dfrac{1}{\alpha_1}\dfrac{\partial^2 W_1(x,t)}{\partial x^2} + F_1(x,t) \\ U_1(l_0,t) + W_1(l_0,t) = G_1(t) \\ \dfrac{\partial U_2(x,t)}{\partial t} - \dfrac{1}{\alpha_2}\dfrac{\partial^2 U_2(x,t)}{\partial x^2} = -\dfrac{\partial W_2(x,t)}{\partial t} + \dfrac{1}{\alpha_2}\dfrac{\partial^2 W_2(x,t)}{\partial x^2} + F_2(x,t) \\ U_2(l_1,t) + W_2(l_1,t) = U_1(l_1,t) + W_1(l_1,t) \\ \mu_2\dfrac{\partial U_2(l_1,t)}{\partial x} + \mu_2\dfrac{\partial W_2(l_1,t)}{\partial x} = \mu_1\dfrac{\partial U_1(l_1,t)}{\partial x} + \mu_1\dfrac{\partial W_1(l_1,t)}{\partial x} \\ -\mu_2\dfrac{\partial U_2(l_2,t)}{\partial x} - \mu_2\dfrac{\partial W_2(l_2,t)}{\partial x} = \sigma_2 G_2(t) \end{cases}$$

hence by posing

$$K_k(x,t) := F_k(x,t) - \frac{\partial W_k(x,t)}{\partial t} + \frac{1}{\alpha_1}\frac{\partial^2 W_k(x,t)}{\partial x^2},\ x \in (l_{k-1}, l_k),\ t > 0$$

and

$$(3.23)\quad W_1(l_0,t) = G_1(t)$$

$$(3.24)\quad W_2(l_1,t) = W_1(l_1,t)$$

$$(3.25)\quad \mu_2\frac{\partial W_2(l_1,t)}{\partial x} = \mu_1\frac{\partial W_1(l_1,t)}{\partial x}$$

$$(3.26)\quad -\mu_2\frac{\partial W_2(l_2,t)}{\partial x} = \sigma_2 G_2(t)$$

we obtain that the functions U_k, $k = 1, 2$ satisfy

$$\begin{cases} \dfrac{\partial U_1(x,t)}{\partial t} - \dfrac{1}{\alpha_1} \dfrac{\partial^2 U_1(x,t)}{\partial x^2} = K_1(x,t) \\ U_1(l_0, t) = 0 \\ \dfrac{\partial U_2(x,t)}{\partial t} - \dfrac{1}{\alpha_2} \dfrac{\partial^2 U_2(x,t)}{\partial x^2} = K_2(x,t) \\ U_2(l_1, t) = U_1(l_1, t) \\ \mu_2 \dfrac{\partial U_2(l_1,t)}{\partial x} = \mu_1 \dfrac{\partial U_1(l_1,t)}{\partial x} \\ -\mu_2 \dfrac{\partial U_2(l_2,t)}{\partial x} = 0 \end{cases}$$

with the initial conditions

$$U_k(x, 0) = T_{k0}(x) - W_k(x, 0), \forall x \in (l_{k-1}, l_k),\ k = 1, 2.$$

We will now determine the W_k, $k = 1.2$. To do this we pose:

$$W_1(x,t) = (C_1(t) - x)C_2(t) \ \text{ and } \ W_2(x,t) = (C_3(t) - x)C_4(t)$$

with $C_1(t),\ C_2(t),\ C_3(t)$ and $C_4(t)$ functions to be determined such as $W_1(x,t)$ and $W_2(x,t)$ verify (3.22). So we have:

$$W_1(l_0, t) = C_1(t)C_2(t) \tag{3.27}$$

$$W_2(l_1, t) = W_1(l_1, t) \tag{3.28}$$

$$\mu_2 \frac{\partial W_2(l_1,t)}{\partial x} = \mu_1 \frac{\partial W_1(l_1,t)}{\partial x} \tag{3.29}$$

$$-\mu_2 \frac{\partial W_2(l_2,t)}{\partial x} = \mu_2 C_4(t). \tag{3.30}$$

Using (3.26) and (3.30) we have:

$$C_4(t) = \frac{\sigma_2}{\mu_2} G_2(t).$$

Then (3.23) and (3.27) give:

$$C_1(t)C_2(t) = G_1(t). \tag{3.31}$$

The combinations (3.24) and (3.28) also give us:

$$(C_1(t) - l_1)C_2(t) = (C_3(t) - l_1)C_4(t),$$

which implies:

$$G_1(t) - l_1C_2(t) = \frac{\sigma_2}{\mu_2}G_2(t)C_3(t) - \frac{\sigma_2 l_1}{\mu_2}G_2(t).$$

Finally by combining (3.25) and (3.29) we get:

$$: \mu_2 l_1 C_4(t) = \mu_1 l_1 C_2(t) \Longrightarrow C_2(t) = \frac{\sigma_2}{\mu_1}G_2(t).$$

$$C_2(t) = \frac{\sigma_2}{\mu_1}G_2(t) \Longrightarrow C_1(t) = \frac{\mu_1}{\sigma_2 G_2(t)}G_1(t).$$

According to (3.31) we have:

$$C_3(t) = \frac{\mu_2}{\sigma_2 G_2(t)}(G_1(t) - l_1\frac{\sigma_2}{\mu_1}G_2(t) + \frac{\sigma_2 l_1}{\mu_2}G_2(t))$$

from where

$$W_1(x,t) = G_1(t) - \frac{\sigma_2}{\mu_1}xG_2(t) \text{ et } W_2(x,t) = G_1(t) + (\frac{\sigma_2 l_1}{\mu_2} - \frac{\sigma_2 l_1}{\mu_1} - \frac{\sigma_2}{\mu_2}x)G_2(t).$$

So our problem becomes:

$$\begin{cases} \dfrac{\partial U_1(x,t)}{\partial t} - \dfrac{1}{\alpha_1}\dfrac{\partial^2 U_1(x,t)}{\partial x^2} = F_1(x,t) - \dfrac{\partial G_1(t)}{\partial t} + \dfrac{\sigma_2}{\mu_1}x\dfrac{\partial G_2(t)}{\partial t} \\ U_1(l_0,t) = 0 \\ \dfrac{\partial U_2(x,t)}{\partial t} - \dfrac{1}{\alpha_2}\dfrac{\partial^2 U_2(x,t)}{\partial x^2} = F_2(x,t) - \dfrac{\partial G_1(t)}{\partial t} - (\dfrac{\sigma_2 l_1}{\mu_2} - \dfrac{\sigma_2 l_1}{\mu_1} - \dfrac{\sigma_2}{\mu_2}x)\dfrac{\partial G_2(t)}{\partial t} \\ U_2(l_1,t) = U_1(l_1,t) \\ \mu_2\dfrac{\partial U_2(l_1,t)}{\partial x} = \mu_1\dfrac{\partial U_1(l_1,t)}{\partial x} \\ -\mu_2\dfrac{\partial U_2(l_2,t)}{\partial x} = 0. \end{cases}$$

3.3.1. *Abstract formulation and main result:* Consider the linear operator $\mathcal{A} : D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$ with

$$D(\mathcal{A}) = \{T \in \mathbf{H} : T_1(l_0) = T_2'(l_2) = 0,\ T_1(l_1) = T_2(l_2),\ \mu_1 T_1'(l_1) = \mu_2 T_2'(l_2)\},$$

and

$$\mathcal{A}T = \begin{pmatrix} \dfrac{1}{\alpha_1}T_1'' \\ \dfrac{1}{\alpha_2}T_2'' \end{pmatrix},\ \forall T \in D(\mathcal{A}).$$

By posing

$$\mathbf{U}(t) = \begin{pmatrix} U_1(t,\cdot) \\ U_2(t,\cdot) \end{pmatrix}, \ \mathbf{U}_0 = \begin{pmatrix} U_{10}(\cdot) \\ U_{20}(\cdot) \end{pmatrix} \ \mathbf{K}(t) = \begin{pmatrix} K_1(t,\cdot) \\ K_2(t,\cdot). \end{pmatrix}$$

The problem (2.1) is rewritten in the following abstract form

$$\frac{d\mathbf{U}(t)}{dt} = \mathcal{A}\mathbf{U}(t) + \mathbf{K}(t), \ t > 0, \ \mathbf{U}(0) = \mathbf{U}_0 \in \overline{D(\mathcal{A})}. \tag{3.32}$$

The main result of this part is the following theorem:

Theorem 3.2. *[7] The linear operator $\mathcal{A}$ is a generator of an analytical semi-group $\{T_{\mathcal{A}}(t)\}_{t\geq 0}$. In particular for all $\mathbf{U}_0 \in \mathbf{H}$ there is a single solution of* (3.32) *(or* (2.1)*) given by*

$$\mathbf{U}(t) = T_{\mathcal{A}}(t)\mathbf{U}_0 + \int_0^t T_{\mathcal{A}}(t-s)\mathbf{K}(s)ds, \ t \geq 0.$$

Remark 3.3. *To show the existence and the uniqueness of solution of the system* (2.1) *amounts to showing the existence and the uniqueness of solution of the system* (3.32). *More precisely, we get T as the sum of U and W.*

Remark 3.4. *We can observe that by definition of $\mathcal{L}$ in* (3.18) *and of $\mathcal{A}$ we have the following relation*

$$\mathcal{L}f \in D(\mathcal{A}), \ \forall f \in \mathbf{H} \ \text{ and } \ \mathcal{A}\mathcal{L}f = -f, \ \forall f \in \mathbf{H}.$$

Therefore we have the following proposition:

Proposition 3.3. *Let $\{V_n\}_{n\geq 1}$ be the eigenvectors of $\mathcal{L}$ associated with the eigen-values $(\nu_n)_{n\geq 1}$ obtained in Theorem 3.1. Then $\{V_n\}_{n\geq 1}$ is a family of eigenvectors of $\mathcal{A}$ associated with the eigenvalues $(-\nu_n^{-1})_{n\geq 1}$.*

Proof. Let $V \in \mathbf{H}^2$ be an eigenvector of $\mathcal{L}$ associated with the eigenvalue $\nu > 0$. According to the remark 3.4 we have

$$\mathcal{L}V \in D(\mathcal{A}) \Rightarrow V = \frac{1}{\nu}\mathcal{L}V \in D(\mathcal{A})$$

from where

$$\mathcal{A}V = \frac{1}{\nu}\mathcal{A}\mathcal{L}V = -\frac{1}{\nu}V.$$

Consequently all the eigenvectors of $\mathcal{L}$ are eigenvectors of $\mathcal{A}$ associated with the point eigenvalues $(-\nu_n^{-1})_{n\geq 1}$ of $\mathcal{A}$. The result follows. □

Let's define the projector family $P_n : \mathbf{H} \to \mathbf{H}$

$$P_n f := \frac{\langle\langle f, V_n\rangle\rangle}{\langle\langle V_n, V_n\rangle\rangle} V_n,\ \forall n \geq 1,\ \forall f \in \mathbf{H}$$

in such a way that

$$f = \sum_{n=1}^{\infty} P_n f,\ \forall f \in \mathbf{H}.$$

We have the following property:

Lemma 3.6. *The operator $\mathcal{A} : D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$ satisfied:*

$$\mathcal{A} P_n f = P_n \mathcal{A} f = -\frac{1}{\nu_n} P_n f,\ \forall f \in\ D(\mathcal{A}),\ \forall n \geq 1$$

and

$$\mathcal{A} f = \sum_{n=1}^{\infty} -\frac{1}{\nu_n} P_n f,\ \forall f \in D(\mathcal{A}).$$

Proof. Let $f \in D(\mathcal{A})$ and $n \geq 1$. Then, we have

$$P_n \mathcal{A} f = \frac{\langle\langle \mathcal{A} f, V_n\rangle\rangle}{\langle\langle V_n, V_n\rangle\rangle} V_n.$$

Using the same arguments as in the proof of Lemma 3.2 we have

$$\langle\langle \mathcal{A} f, V_n\rangle\rangle = \langle\langle f, \mathcal{A} V_n\rangle\rangle$$

As a consequence of this previous equality we get

$$P_n \mathcal{A} f = \frac{\langle\langle f, \mathcal{A} V_n\rangle\rangle}{\langle\langle V_n, V_n\rangle\rangle} V_n = -\frac{1}{\nu_n} \frac{\langle\langle f, V_n\rangle\rangle}{\langle\langle V_n, V_n\rangle\rangle} V_n = \frac{\langle\langle f, V_n\rangle\rangle}{\langle\langle V_n, V_n\rangle\rangle} \mathcal{A} V_n = \mathcal{A} P_n f.$$

Finally we note that for all $f \in D(\mathcal{A})$ we have:

$$\mathcal{A} f = \sum_{n=1}^{\infty} P_n \mathcal{A} f \Rightarrow \mathcal{A} f = \sum_{n=1}^{\infty} -\frac{1}{\nu_n} P_n f.$$

□

The Proposition 3.3 and the Lemma 3.6 combined allow us to apply the results of Leiva [6, Lemma 2.1 et Lemma 2.2] to deduce that $\mathcal{A}$ is an analytical semi-group generator $\{T_{\mathcal{A}}(t)\}_{t\geq 0} \subset \mathcal{L}(\mathbf{H})$ to deduce that $\mathcal{A}$ is an analytical semi-group generator given by

$$T_{\mathcal{A}}(t) f = \sum_{n=1}^{\infty} e^{-\frac{1}{\nu_n} t} P_n f,\ \forall f \in \mathbf{H},\ \forall t \geq 0$$

and the spectrum of $\mathcal{A}$ is given by $\{0, \nu_n^{-1} : n \geq 1\}$.

Recalling that $K(t) \in \mathbf{H}$ for all $t \geq 0$ and $t \mapsto K(t)$ is locally integrable on $(0, \infty)$, we can apply the theory of C_0 - semi-group to deduce the formula

$$\mathcal{A}f = \sum_{n=1}^{\infty} -\frac{1}{\nu_n} P_n f, \ \forall f \in D(\mathcal{A}).$$

We refer to Pazy's book [7] for more details on this subject.

4. Numerical method on the heat transfer problem

In this part we use the finite difference method for the search of approximate solutions of the coupled continuous model.

4.1. **Approximation by the finite difference method.** In numerical analysis, the finite difference method is a common technique for finding approximate solutions of partial differential equations which consists in solving a system of relations (numerical scheme) linking the values of the unknown functions at certain points sufficiently close to each other.

4.2. **Discretization of domains $(l_0, l_1) \times \mathbb{R}_+^*$ and $(l_1, l_2) \times \mathbb{R}_+^*$.** So in this subsection we will divide our work into two stages. The first step will consist in discretizing the domain $(l_0, l_1) \times \mathbb{R}_+^*$ and the second in discretizing the domain $(l_1, l_2) \times \mathbb{R}_+^*$.

To discretize the domain $(l_0, l_1) \times \mathbb{R}_+^*$ we introduce a space step $h_1 = \frac{l_1 - l_0}{N}$ (N is the number of intervals) and a time step Δt all strictly positive.

We define the nodes of a regular mesh:

$$x_j = jh_1 \text{ and } t_n = n\Delta t, \ \forall j \in \{0, 1, 2, ..., N\} \text{ and } \forall n \geq 0.$$

Indeed,

$$x_j = x_{j-1} + h_1 \text{ and } t_n = t_{n-1} + \Delta t \Rightarrow x_j = x_0 + jh_1 \text{ and } t_n = t_0 + n\Delta t$$

and since $x_0 = t_0 = 0$ then:

$$x_j = jh_1 \text{ and } t_n = n\Delta t, \ \forall j \in \{0, 1, 2, ..., N\} \text{ and } \forall n \geq 0.$$

To discretize the domain $(l_1, l_2) \times \mathbb{R}_+^*$ we introduce a space step $h_2 = \frac{l_2 - l_1}{M}$ (M is the number of intervals) and a time step Δt all strictly positive.

We define the nodes of a regular mesh:

$$x_i = ih_2 + l_1 \text{ and } t_n = n\Delta t,\ \forall i \in \{0, 1, 2, ..., M\} \text{ and } \forall n \geq 0.$$

Indeed,

$$x_i = x_{i-1} + h_2 \text{ and } t_n = t_{n-1} + \Delta t \Rightarrow x_i = x_0 + ih_2 \text{ and } t_n = t_0 + n\Delta t$$

and since $x_0 = l_1$ and $t_0 = 0$ then:

$$x_i = l_1 + ih_2 \text{ et } t_n = n\Delta t,\ \forall i \in \{0, 1, 2, ..., M\} \text{ and } \forall n \geq 0.$$

Let (T_1, T_2) be the exact solution of the continuous problem. We note $T^n_{1,,j}$ an approximation of T_1 at the point (x_j) at the instant $t_n)$ and $T^n_{2,i}$ an approximation of T_2 at the point $(x_i$ at the instant $t_n)$.

By replacing in the continuous problem the partial derivatives by their approximation we obtain the following discrete problem:

$$\begin{cases}
T^{n+1}_{1,j} = \frac{\Delta t}{(h_1)^2\alpha_1} T^n_{1,j+1} + (\frac{-2\Delta t}{(h_1)^2\alpha_1} + 1) T^n_{1,j} + \frac{\Delta t}{(h_1)^2\alpha_1} T^n_{1,j-1} + \Delta t f^n_{1,j}, \\
\forall j \in \{1, 2, ..., N-1\},\ \forall n > 0 \\
T^0_{1,j} = T^0_1(x_j),\ \forall j \in \{1, 2, ..., N-1\}, \\
T^{n+1}_{2,i} = \frac{\Delta t}{(h_2\alpha_2)^2} T^n_{2,i+1} + (\frac{-2\Delta t}{(h_2)^2\alpha_2} + 1) T^n_{2,i} + \frac{\Delta t}{(h_2)^2\alpha_2} T^n_{2,i-1} + \Delta t f^n_{2,i}, \\
\forall i \in \{0, 1, 2, ..., M\},\ \forall n > 0 \\
T^0_{2,i} = T^0_2(x_i),\ \forall i \in \{0, 1, 2, ..., M\} \\
T^n_{2,M+1} = T^n_{2,M} - \dfrac{h_2\sigma_2}{\mu_2} G_2(t_n),\ \forall n \geq 0 \\
T^n_{2,-1} = (1 - \frac{h_2\mu_1}{h_1\mu_2}) T^n_{2,0} + \frac{h_2\mu_1}{h_1\mu_2} T^n_{1,N-1},\ \forall n \geq 0.
\end{cases}$$

The discrete problem can be written in the following matrix form:

$$X^{n+1} = A(h_1, h_2)X^n + K^n$$

$A \in \mathcal{M}_{N+M}(\mathbb{R}),\ X^{n+1}, X^n \in \mathcal{M}_{N+M,1}(\mathbb{R})\ et\ K^n \in \mathcal{M}_{N+M,1}(\mathbb{R})$

with, $A(h_1, h_2) = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$, $X^n = \begin{pmatrix} T^n_1 \\ T^n_2 \end{pmatrix}$,

$X^{n+1} = \begin{pmatrix} T^{n+1}_1 \\ T^{n+1}_2 \end{pmatrix}$ and $K^n = \begin{pmatrix} F^n \\ G^n \end{pmatrix}$

where,

$$\begin{cases} B_1 = (b^1_{ij}) \in \mathcal{M}_{N-1}(\mathbb{R}), \\ \\ B_2 = (b^2_{ij}) \in \mathcal{M}_{N-1\times M+1}(\mathbb{R}), \\ \\ B_3 = (b^3_{ij}) \in \mathcal{M}_{M+1\times N-1}(\mathbb{R}), \\ \\ B_4 = (b^4_{ij}) \in \mathcal{M}_{M+1\times M+1}(\mathbb{R}). \end{cases}$$

$$T_1^n \in \mathcal{M}_{N-1,1}(\mathbb{R}),\ T_2^n \in \mathcal{M}_{M+1,1}(\mathbb{R}) \text{ et } F^n \in \mathcal{M}_{N-1,1}(\mathbb{R}),\ G^n \in \mathcal{M}_{M+1,1}(\mathbb{R}).$$

These matrices have the following coefficients:

$$T_1^n = \begin{pmatrix} T^n_{1,1} \\ T^n_{1,2} \\ \vdots \\ T^n_{1,N-1} \end{pmatrix}, T_2^n = \begin{pmatrix} T^n_{2,0} \\ T^n_{2,1} \\ \vdots \\ T^n_{2,M} \end{pmatrix},$$

$$F^n = \begin{pmatrix} \Delta t f_1^n(x_1) + \frac{\Delta t \sigma_1}{(h_1)^2 \alpha_1} G_1(t_n) \\ \Delta t f_1^n(x_2) \\ \vdots \\ \Delta t f_{N-1}^n(x_{N-1}) \end{pmatrix} \text{ with } G^n = \begin{pmatrix} \Delta t f_2^n(x_0) \\ \Delta t f_2^n(x_1) \\ \vdots \\ \Delta t f_M^n(x_M) - \frac{\Delta t \sigma_2}{h_2 \alpha_2 \mu_2} G_2(t_n) \end{pmatrix}$$

To highlight our work we will study a real case of heat transfer.

4.3. **Study of the heat transfer of a thermal insulation plate made of Cement-Typha / concrete wall.** Our objective is to find the temperatures T_1, T_2 respectively of the Cement-Typha and the concrete wall. For this we have the following data [1].

Nomenclature	Cement + Typha (T1)	Concrete
thermal conductivity (μ) in $W.cm^{-1}.K^{-1}$	$\mu_1 = 0.00646$	μ_2=0.01
thermal diffusion (α) in $cm^2.s^{-1}$	$\alpha_1 = 3.74 * 10^{-3}$	$\alpha_2 = 5.24 * 10^{-3}$
trade coefficients (σ) in $W.cm^{-2}.K^{-1}$	$\sigma_1 = 1$	$\sigma_2 = 500 \times 10^{-4}$
Edge temperatures	$G_1 = 25$	$G_2 = (2 - t)$
Length of the two compartments in cm	$l_1 = 15$	$l_2 = 7$
Functions of the second member	$F_1 = 0$	$F_2 = 0$
Temperature at time $t = 0$ for all x	$T_1(x, 0) = 25$	$T_2(x, 0) = 25$

4.3.1. *Results and discussions.* After simulation for $8h$ of time between $6h$ to $14h$ with a time step of 0.01, we observe the results of the evolution of the temperatures of the insulator and the concrete as a function of time:

Legend

- Cement+typha, temperature evolution curve
- Concrete wall, temperature evolution curve

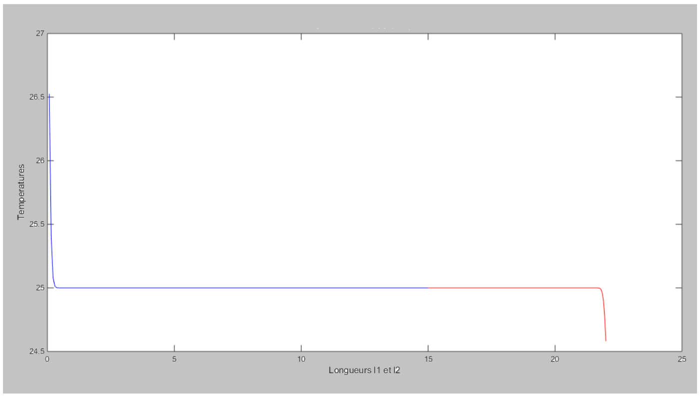

FIGURE 2. Numerical results at $7h$

Interpretation

At $7h$ we observe a slight increase in temperature on the edge of the insulation about 26.5° . This slight difference with the ambient temperature 25° inside the insulation is due to the fact that the material tries to keep itself always at the same initial temperature (25°). We observe a slight increase in the temperature of the edge of the concrete which is exposed to the sun about 24.5°. This temperature remains lower than the temperature inside the materials and the edge of the insulation because the sunrise is not yet very significant.

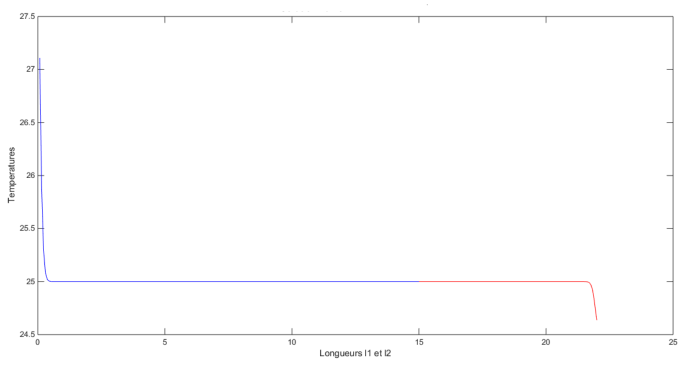

FIGURE 3. Numerical results at $8h$

Interpretation

At $8h$ there is a slight increase in the outside temperature. This brings the temperature on the one hand to the edge of the insulator (Cement + Typha) to 27.2° which always represents a small variation because of the insulator which admits a relatively small exchange coefficient. on the other hand the increase in temperature at the edge of the concrete which goes from 24.6° to 24.7°. This variation is due to the fact that the outside temperature increases with the rising of the sun.

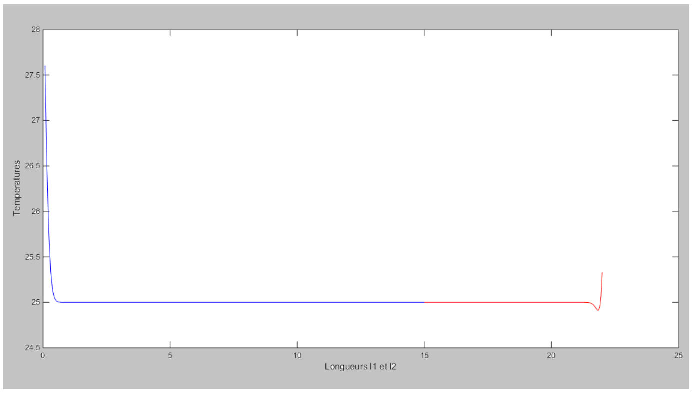

FIGURE 4. Numerical results at $9h$

Interpretation

At $9h$ with the rising of the sun we witness a slightly more consequent increase in the outside temperature which makes the positive flow causing the temperature increase at the edge of the concrete which passes to 25.3°. At the edge of the insulator we are still witnessing the slight variation in temperature which drops to 27.6°.

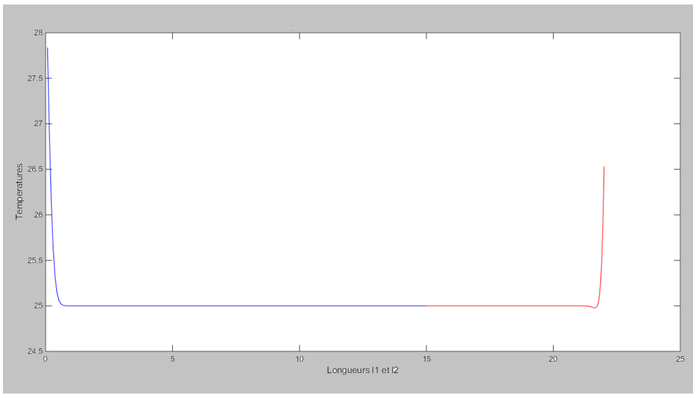

FIGURE 5. Numerical results at $11h$

Interpretation

At $11h$ with a much higher temperature increase than at $9h$, we are witnessing the same phenomenon as before but with higher temperatures: the temperature at the edge of the insulation drops to 27.8° and the one at the edge of the concrete at around 26.6°. a clear difference in growth on both sides due in large part to the nature of the material compartments (insulating for Cement + Typha and non-insulating for concrete).

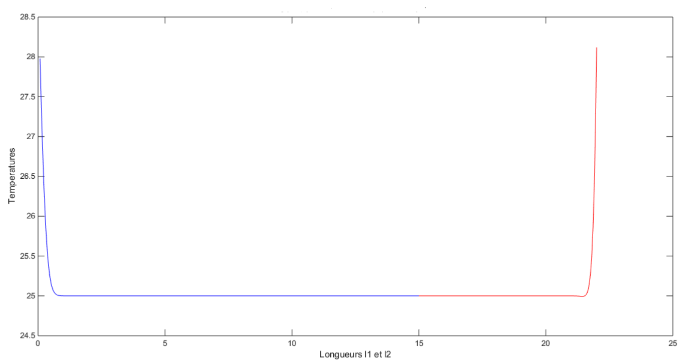

FIGURE 6. Numerical results at $13h$

Interpretation

At $13h$ with a sun at its zenith we note a significant rise in temperature at the edge of the concrete which drops to 28.2° . This significant increase is also due to the fact that concrete is not an insulator therefore admits a relatively large exchange coefficient. And on the other side of the insulation we are still witnessing the slight variation in temperature at the edge of the insulation which drops to 27.9° . We also find that the temperature at the edge of the concrete that is exposed to the sun is higher than the temperature of the rest of the material.

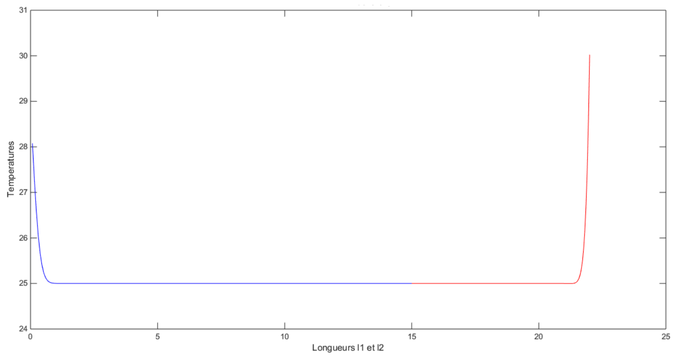

FIGURE 7. Numerical results at $14h$

Interpretation

At $14h$, the hour when it is still very hot, we see much higher temperatures. At the edge of the concrete the temperature goes up to around 30° justifying once again its relatively large exchange coefficient. And at the edge of the insulation we notice a slight increase in fact a temperature which goes from 27.9° to 28.1°. This small variation also justifies its relatively low exchange coefficient. We also find that the temperature at the edge of the concrete exposed to the sun is higher than the temperature of the rest of the material.

5. CONCLUSION

In this work we study a phenomenon of heat transfer between two compartments, one made of insulation and the other of concrete. This transfer phenomenon is modeled by the coupling of equations reflecting the propagation of heat along the materials.

First, we showed the existence and the uniqueness of the solution of the coupled problem through mathematical tools. Then, a numerical diagram based on the finite difference method is established as well as a calculation code under Matlab. Finally, we proceeded to the numerical simulation under Matlab and the interpretation of the numerical results with a specialist in the field of the origin of our coupled problem. To highlight our work, we simulated a real case of heat transfer from a thermal insulation plate made of Cement-Typha / concrete wall.

REFERENCES

[1] A. S. DIAW, H. M. BAL, M. WADE, S. GAYE: *Use of Typha Australis in the Habitat for the Improvement of Energy Efficiency of Buildings*, Journal of Scientific and Engineering Research, **5**(1) (2018), 164–171.

[2] A. S. DIAW, D. SOW, M. B. NDIAYE, A. O. ABDELAKH, M. WADE, S. GAYE: *Valorization of Typha Australis by its Integration in Building Construction Materials*, International Journal of Emerging Technology and Advanced Engineering, **6**(1) (2016), 34–37.

[3] S. BOULKROUNE, M. CHAOUR, O. KHOLAI: *Simulation du transfert de chaleur dans un mur multicouche application à un four cubilot*, La 2ième Conférence Internationale sur la Maintenance et la Sécurité Industrielle CIMSI 2013 le 28 et 29/10/2013 Skikda.

[4] H. BREZIS: *Functional analysis, Sobolev spaces and partial differential equations*, Springer New York Dordrecht Heidelberg London, 2010.

[5] G. ALLAIRE: *Analyse numérique et optimisation*, Éditions de l'École Polytechnique, 2012.

[6] H. LEIVA: *A Lemma On C_0-semigroups And Applications,* Quaestiones Mathematicae, **26**(3) (2003), 247–265.

[7] A. PAZY: *Semigroups of Linear Operators and Applications to Partial Differential Equations*, https://doi.org/10.1007/978-1-4612-5561-1, Springer-Verlag New York, 1983.

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