Advances in Mathematics: Scientific Journal **9** (2020), no.10, 8135–8142
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.10.43

SOLUTION OF BOUNDARY VALUE PROBLEMS USING FIXED POINT ITERATIVE TECHNIQUE FOR A SECOND ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. The foremost objective of the paper is to show the comparison of a new fixed point iteration technique with the Mann iterations. The iteration procedure is effectively discussed as well as applied to solve the Two-Point BVP. Also we support our result by giving some examples and comparing them with those obtained by using exact solution.

1. INTRODUCTION

The Fixed point iteration techniques are intended to be utilized in solving different differential equations originating in physical formulation. There are several methods available which are used in finding the result of various boundary value problems related to approximation of differential equations. This theory finds its origin in the application of approximation of results successively to ascertain existence as well as uniqueness of solution of integral and differential equation. Great mathematicians such as Cauchy, Lipschitz, Liouville, Fredholm, Peano and Picard are associated directly or indirectly with fixed point theory. However, it was pioneering work of Polish mathematician and Stefen Banach (1922) which credited him with introducing underlying thoughts into an abstract frame work for suitable applications. In mathematics, it is of immense

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2010 *Mathematics Subject Classification.* 47H10,34B10, 34B15, 65L10.

Key words and phrases. Mann Iteration, Fixed Point Technique, Boundary Value Problem.

importance to identify which of the specified iteration method converge more rapidly with least error to the desired result. Several researchers are involved in mounting different techniques of fixed point for improved approximation results to solutions of the boundary value problems analytically ([3,10,11]).

Recently Fixed Point Theory for finding the approximated results of second order boundary value questions in differential equations is being developed very fast. Particularly more interest is towards finding the approximate answers of two or three point boundary value problem along with the existence of the same ([1,2,5,6,7]). Khuri and Sayfy [8] has given an algorithm depending upon Green's functions into fixed point iterations similar to Mann's and Picard's iterative techniques to approximately find the results of boundary value problems. Further they generalized the variational iterative schemes in [9] for suitable dealing with the boundary value problems. The main objective of this paper is to develop a fixed point iteration process by which we can approximate the results of two-point boundary value problem similar to Mann iteration scheme. This process transforms a boundary value problem in differential equation into an algorithm which is further used to solve and process the results using numerical computational tool (graph using open access software).

2. METHOD

Consider the given below two-point Boundary Value Problem.

Let

$$y'' = f\left(x, y, y'\right) \ where\ x \in [t_1, t_2] \tag{2.1}$$

$$\begin{aligned} \lambda_0 y\left(t_1\right) + \ \beta_0 y'\left(t_1\right) &= \ \gamma_0 \\ \lambda_1 y\left(t_2\right) + \ \beta_1 y'\left(t_2\right) &= \ \gamma_1 \end{aligned} \tag{2.2}$$

where λ_j and β_j, are real numbers and $\lambda_j^2 + \ \beta_j^2 > 0,\ j = 0, 1.$

This new developed fixed point scheme is useful for constructing an iterative technique that can help us to approximate the results of (2.1), (2.2). Unlike the various residual methods required for estimation whose achievement depends upon the choice of suitable basis function, this projected method under consideration is a self-correcting in itself.

To find the solution of (2.1), (2.2) using the projected fixed point technique, these equations first transformed into

$$y''_{n+1} = (1-\alpha_n)\, y''_n + \alpha_n y''_n = (1-\alpha_n)\, y''_n + \alpha_n f(x, y_n, y'_n)$$

(2.3)
$$\lambda_0 y_{n+1}(t_1) + \beta_0 y'_{n+1}(t_1) = \gamma_0$$

$$\lambda_1 y_{n+1}(t_2) + \beta_1 y'_{n+1}(t_2) = \gamma_1,$$

or

$$y''_{n+1} = \alpha_n y''_n + (1-\alpha_n) y''_n = \alpha_n y''_n + (1-\alpha_n)\, f(x, y_n, y'_n)$$

(2.4)
$$\lambda_0 y_{n+1}(t_1) + \beta_0 y'_{n+1}(t_1) = \gamma_0$$

$$\lambda_1 y_{n+1}(t_2) + \beta_1 y'_{n+1}(t_2) = \gamma_1,$$

assuming $0 \leq \alpha_n \leq 1$ and $\sum_{n=0}^{n=\infty} \propto_n = \infty$ (were n varies from o to ∞).
Then, it can be shown that $y(x)$is the required solution of (2.1),(2.2) if and only if we can prove that $y(x)$gives us solution of the given below equivalent integral equation:

$$y(x) = \int_{t_1}^{t_2} G(x,s)\,(f(s, y(s), y'(s))ds + w(x) \quad on\ [t_1, t_2],$$

where, for the related boundary value problem G(x, s) is the Green's function.

(2.5)
$$y'' = 0, \quad \lambda_0 y(t_1) + \beta_0 y'(t_1) = \gamma_0, \lambda_1 y'(t_2) + \beta_1 y'(t_2) = \gamma_1$$

and the solution of (2.5) is specified as *w(x)*.

Now consider $T : C^{(1)}[t_1, t_2] \to C^{(1)}[t_1, t_2]$ is such that

(2.6)
$$T[y(x)] = \int_{t_1}^{t_2} G(x,s)\,(f(s, \mathrm{y}(s), y'(s))ds + w(x) \quad on\ [t_1, t_2],$$

where T is an operator on $y(x)$such that solution of (2.1),(2.2) is a fixed point [7].

3. CONVERGENCE

Derivation of the convergence of the fixed point technique:

Consider E as a convex subset which is non empty of a real Banach space S as well as $T : E \to E$ is a mapping. Thus the sequence $\{y_n\}\, E$ defined as

$$y_o \in E,\; y_{n+1} = (1-\alpha_n)\, y_n + \alpha_n T y_n,\; n \geq 0. \tag{3.1}$$

where $\{\alpha_n\}$ is a real sequence in the interval [0, 1] fulfilling $0 \leq \alpha_n \leq 1$ and $\sum_{n=0}^{n=\infty} \propto_n = \infty$ is called Mann Iterative technique [4].

Next, we compare the present method (2.3) with Mann method (3.1) to show their equivalence. For this first differentiate (2.6) we get

$$(Ty_n)' = \int_{t_1}^{t_2} \frac{\partial}{\partial x} G\,(x,s)\,(f(s, y_n\,(s)\,, y'{}_n\,(s))ds + w'(x). \tag{3.2}$$

Also (3.1) is differentiated to get

$$y_{n+1}^{"} = (1-\alpha_n)\, y_n^{"} + \;\alpha_n (Ty_n)" \tag{3.3}$$

and next we differentiate (3.2), to get

$$(Ty_n)" = \int_{t_1}^{t_2} \frac{\partial^2}{\partial x^2} G\,(x,s)\,(f(s, y_n\,(s)\,, y_n^{'}\,(s))ds + w"(x) \;on\; [t_1, t_2]. \tag{3.4}$$

Now on putting value from (3.4) into (3.3), we get

$$y_{n+1}^{"} = (1-\alpha_n)\, y_n^{"} + \alpha_n \{ \int_{t_1}^{t_2} \frac{\partial^2}{\partial x^2} G\,(x,s)\,(f(s, y_n\,(s)\,, y_n^{'}\,(s))ds + w"(x). \tag{3.5}$$

From the given above equivalent integral equation value obtained by (3.5) is equal to $y_n^{"}$, so it can be written as,

$$y_{n+1}^{"} = (1-\alpha_n)\, y_n^{"} + \;\alpha_n y_n^{"} \;=\; (1-\alpha_n)\, y_n^{"} + \;\alpha_n f(x, y_n, y_n^{'}),$$

which is the required fixed point iteration scheme for the result of (2.1),(2.2) .

4. NUMERICAL EXAMPLE

Now we will show the results of one boundary value problem applying the obtained present method, we have shown that the results obtained are comparable with those obtained by exact solution as well as obtained by other existing methods. All the numerical computations are done by using equation (2.4) by means of successive integration.

TABLE 1. Comparison of the Exact Solution for Example with the Present New Method

X	**Present Results**	**Exact Results**
0.00	1.00000	1.00000
0.10	0.79936	0.79891
0.20	0.62673	0.62647
0.30	0.47970	0.47996
0.40	0.35606	0.35690
0.50	0.25374	0.25508
0.60	0.17083	0.17246
0.70	0.10557	0.10721
0.80	0.05633	0.05570
0.90	0.02160	0.02241
1.00	0.00000	0.00000

Example 1. *Let us define the boundary value problem*

$$y'' + y' = 1, where\ y = 1\ when\ x = 0\ as\ well\ y = 0\ when\ x = 1,$$

are the given boundary conditions. On applying the obtained method (2.4)*, the given problem is transformed into*
$y''_{n+1} = (1 - \lambda_n)(1 - y'_n) + \lambda_n y''_n$*, where* $\lambda_n = \frac{n}{n+1}$.

$$y_0(x) = 1 - x,\ y_{n+1}(0) = 1\ and\ y_{n+1}(1) = 0,$$

where y_0 *is the result of* $y'' = 0$ *for the given boundary conditions.*
Exact result of above boundary value problem is obtained as

$$y_E\ (x) = -\frac{2e^{-x}}{e^{-1}\ - 1} + \frac{1 +\ e^{-1}}{e^{-1} - 1},$$

The evaluation of the obtained technique with the exact results considering used for computation numerically is shown below in Table-1.

It is clear from the Graph-1 and Graph-2 below that the present method converges to the exact solutions. As well as in Graph-1 both lines overlap each other which clearly show that the results are approximately same.

FIGURE 1. Graph-1

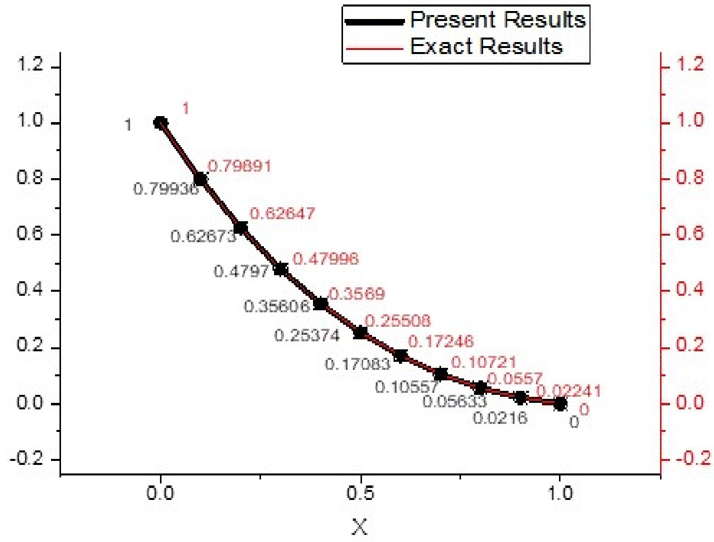
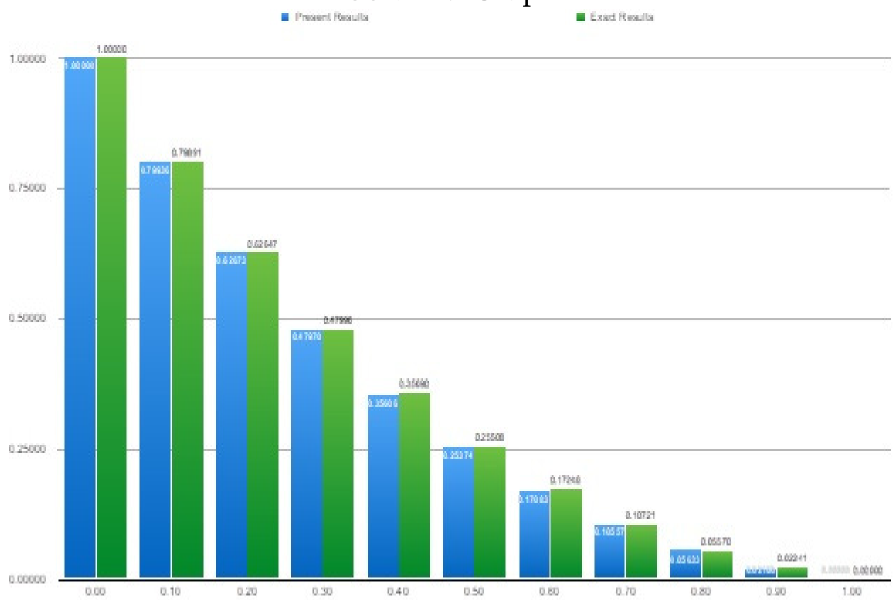


FIGURE 2. Graph-2



5. CONCLUSION

In this paper we present an iterative scheme using fixed point which is similar to Mann iteration scheme and is effectively applied to two-point boundary value problem. Example is given to show that outcomes of the problems compete positively when compared with the exact solution. Moreover as we try to increase n, the accuracy of the results also increases which is the self correcting quality of this proposed scheme. The work can be extended to CUIA iteration and other new iterative schemes [12,13] .

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