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M/M/1 NON-PREEMPTIVE PRIORITY MODEL WITH SYSTEM BREAKDOWN AND REPAIR TIMES

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ABSTRACT. In this paper, the performance analysis of non – preemptive priority M/M/1 queuing model with system breakdown and repair time is examined. We have used the complementary variable method to work out this vector Markov process and investigate the steady state changes that takes place in the equations of the queuing model. The customer length about the two class of customers is also analyzed in this paper based on the system breakdown and repair time. Using little's formula, the waiting time can also be calculated and the graphical representation is also presented.

1. INTRODUCTION

To give quality service to the customers who approach for service, we need to control the queue with some priority discipline. This situation is common in queuing system. In hospitals, telecommunications, railway stations, to give service to different class of customers through IP packages are introduced. There are two class of customers, high priority and low priority customers. In the non-preemptive priority policy, if a low class priority customer is getting serviced, even if the high priority customer comes during the service of the low priority customer, the high priority customer has to wait for service. But the case for the preemptive case , it is the other way round, where the second class customer is

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being services, when the high priority customers comes and squeezes inside to get service is called preemptive priority. Cohen J.W [3] examined a two class M/G/1 queue with non-preemptive priority but the end result was just an approximate one. Miller [6] analyzed a M/M/1 queue with non-preemptive priority queue, Edward P.C. Kao [9] analyzed M/M/N queuing model with preemptive priority queue under two classes of customers. Matrix geometric method is and efficient method to solve these equations but the process of computing these equations becomes too complicated. G.V. Krishna Reddy [4] and Asha Seth Kapadia [5] have done their research papers on non-preemptive priority queues. Bong Dae Choi [8] analyzed M/M/1 queue with deadline until the end of service, formatted a Markov vector process which made the problem simpler. Leonard Cleinrock [1] has his prominent work in Queuing theory which is a branch of Operations research. Shi Ding-hua [2] has his probability analysis on queuing models. B. D. Choi, Y.Chang [7] has his work on single server retrial queues in priority.

(i) Mathematical model

Considering a single server M/M/1 queuing system model which serves two class of customers (i.e) first class of customers and second class of customers. The arrival of these two class of customers are state independent based on the investigation made on the following assumptions:

(a) Arrivals for two class of customers are state independent stationary poisson processes, with arrival rate λ_1, λ_2 respectively. **(b)** Service patterns are distributed exponentially with an average service rate of $1/\mu$. **(c)** Class I customers adopt the non – pre-emptive priority policy over class II customers. **(d)** System supports breakdown with the rate of α during the service and the repair time follows the exponential distribution with the average of $1/\beta$.

(ii) Model description

Considering $N_i\left(t\right),\ (i = 1.2)$ to mean that my system has two types of customers under non preemptive priority policy which has no memoryless property and hence it is not a Markov process. Let me consider ξ =0, 1, 2, 3 respectively which constitute the four states that the server is free , or the server is serving the class I customers, or the server is serving the class II customers, or it is under repair process. This N(t) = $\{N_1\left(t\right),\ N_2\left(t\right),\xi\}$ is a markov process, and the state space structure can be written as follows. (0, 0, 0), (1, 0, 1), (1, 1, 1), (1, 2, 1), (1, 3, 1),. . ., . . . (2, 0, 1), (2, 1, 1), (2, 2, 1),. . .,(0, 1, 2), (0, 2, 2) , (0, 3, 2), . . ., (1, 1, 2), (1, 2, 2), (1, 3, 2), ...

(iii) State Independent equations for this mathematical model When both the class of customers are of the same kind, then it is a regular M/M/1 model. By the by the input process is a characterized Poisson process with parameters $(\lambda_1 + \lambda_2)$, the system busy state is given by $\rho = \lambda_1 + \lambda_2/\mu$, where $\mu < 1$ When $\rho < 1$ let us fix $P_{ijk} = \lim_{t\to\infty} Pr\{N_1(t) = i, N_2(t) = j,\ \xi = k\ \}$, the necessary state equations can be written as follows:

When typing labeled expression, you should do it like this:

$$P_{000}(\lambda_1 + \lambda_2) = P_{101}\mu + P_{012}\mu \tag{1.1}$$

$$P_{012}(\lambda_1 + \lambda_2 + \mu + \alpha) = P_{111}\,\mu + P_{022}\,\mu + P_{000}\lambda_2 + P_{013}\beta \tag{1.2}$$

$$P_{0j2}(\lambda_1 + \lambda_2 + \mu + \alpha) = P_{1j1}\,\mu + P_{0(j+1)2}\,\mu + P_{0(j-1)2}\lambda_2 + P_{0j3}\beta \tag{1.3}$$

$$P_{112}(\lambda_1 + \lambda_2 + \mu + \alpha) = P_{012}\,\lambda_1 + P_{102}\,\lambda_2 \tag{1.4}$$

$$P_{1j2}(\lambda_1 + \lambda_2 + \mu + \alpha) = P_{0j2}\,\lambda_1 + P_{1(j-1)2}\,\lambda_2 \tag{1.5}$$

$$P_{i12}(\lambda_1 + \lambda_2 + \mu + \alpha) = P_{(i-1)12}\,\lambda_1 \tag{1.6}$$

$$P_{ij2}(\lambda_1 + \lambda_2 + \mu + \alpha) = P_{(i-1)j2}\,\lambda_1 + P_{1(j-1)2}\,\lambda_2 \tag{1.7}$$

$$P_{101}(\lambda_1 + \lambda_2 + \mu + \alpha) = P_{201}\mu + P_{112}\,\mu + P_{000}\,\lambda_1 + P_{103}\beta \tag{1.8}$$

$$P_{111}(\lambda_1 + \lambda_2 + \mu + \alpha) = P_{211}\mu + P_{122}\,\mu + P_{101}\,\lambda_2 + P_{113}\beta \tag{1.9}$$

$$P_{1j1}(\lambda_1 + \lambda_2 + \mu + \alpha) = P_{2j1}\mu + P_{1(j+1)2}\,\mu + P_{1(j-1)1}\,\lambda_2 + P_{1j3}\beta \tag{1.10}$$

$$P_{i01}(\lambda_1 + \lambda_2 + \mu + \alpha) = P_{(i+1)01}\mu + P_{i12}\mu + P_{(i-1)01}\lambda_1 + P_{i03}\beta \tag{1.11}$$

$$P_{i11}(\lambda_1 + \lambda_2 + \mu + \alpha) = P_{(i+1)11}\mu + P_{i22}\mu + P_{(i-1)11}\lambda_1 + P_{i01}\lambda_2 + P_{i13}\beta \tag{1.12}$$

$$\begin{aligned} P_{ij1}(\lambda_1 + \lambda_2 + \mu + \alpha) = &P_{(i+1)j1}\mu + P_{(i-1)j1}\,\lambda_1 + P_{i(j-i)1}\lambda_2 \\ &+P_{i(j+1)2}\mu + P_{ij3}\beta \end{aligned} \tag{1.13}$$

$$P_{0j3}(\lambda_1 + \lambda_2 + \beta) = \alpha P_{0j2} \tag{1.14}$$

$$P_{ij3}(\lambda_1 + \lambda_2 + \beta) = \alpha P_{ij1} \tag{1.15}$$

(iv) Solution of stationary equations In order to solve the stationary state equations above, we need some initial substitutions to be made in this to get the solution.

$$F_i^1(z_1) = \sum_{j=0}^{\infty} P_{ij1} z_2^j \; i \geq 1 \tag{1.16}$$

$$F_i^2(z_2) = \sum_{j=1}^{\infty} P_{ij2} z_2^j \; i \geq 0 \tag{1.17}$$

$$F^1(z_1,\; z_2) = \sum_{i=1}^{\infty} F_i^1(z_2)\, z_1^i \; i \geq 1 \tag{1.18}$$

$$F^2(z_1,\; z_2) = \sum_{i=0}^{\infty} F_i^2(z_2)\, z_1^i \; i \geq 0 \tag{1.19}$$

$$F_i^3(z_2) = \sum_{j=1}^{\infty} P_{ij3} z_2^j \; i \geq 0 \tag{1.20}$$

$$F^3(z_1,\; z_2) = \sum_{i=0}^{\infty} F_i^3(z_2)\, z_1^i \; i \geq 0. \tag{1.21}$$

Clearly, the united distribution generating function of these two class of customers length can be given by,

$$(z_1,\; z_2) = F^1(z_1,\; z_2) + F^2(z_1,\; z_2) + F^3(z_1,\; z_2) + P_{000}. \tag{1.22}$$

From expressions (1.1), (1.2), (1.3) and by using the generating function method, we end up at this equation,

$$\begin{aligned} &[\lambda_1 + \lambda_2(1 - z_2) + \mu(1 - 1/z_2) + \alpha]\, F_0^2(z_2) \\ =&\mu F_1^1(z_2) + \beta F_0^3(z_2) - P_{000}\lambda_1 + \lambda_2 - \lambda_2 z_2. \end{aligned} \tag{1.23}$$

From expressions (1.4) and (1.5) and by using the generating function method, we get,

$$[\lambda_1 + \lambda_2(1 - z_2) + \mu + \alpha]\, F_1^2(z_2) = \; \lambda_1 F_0^2(z_2). \tag{1.24}$$

From expressions (1.6) & (1.7) and by using the generating function method, we get

$$[\lambda_1 + \lambda_2 (1 - z_2) + \mu + \alpha] F_i^2 (z_2) = \lambda_1 F_{i-1}^2 (z_2). \tag{1.25}$$

From expressions (1.8), (1.9) & (1.10) and by using the generating function method, we get

(1.26)
$$[\lambda_1 + \lambda_2 (1 - z_2) + \mu + \alpha] F_1^{'} (z_2) = \mu F_2^1 (z_2) + \frac{\mu}{z_2} F_1^3 (z_2) + \lambda_1 P_{000} + \beta F_1^3 (z_2).$$

From expressions (1.14) & (1.15) and by using the generating function method, we get

$$[\lambda_1 + \lambda_2 + \beta] F_i^3 (z_2) = \alpha F_i^1 (z_2), \tag{1.27}$$

which gives,

$$F_i^3 (z_2) = \frac{\alpha}{\lambda_1 + \lambda_2 + \beta} F_i^1 (z_2). \tag{1.28}$$

Deducing from expressions (1.23), (1.24) and (1.25), we get

$$\begin{aligned} &[\lambda_1 (1 - z_1) + \lambda_2 (1 - z_2) + \mu + \alpha] F^2 (z_1,\ z_2) \\ =& [\lambda_1 + \lambda_2 (1 - z_2) + \mu + \alpha] F_0^2 (z_2). \end{aligned}$$

This gives,

$$F^2 (z_1,\ z_2) = \frac{[\lambda_1 + \lambda_2 (1 - z_2) + \mu + \alpha] F_0^2 (z_2)}{[\lambda_1 (1 - z_1) + \lambda_2 (1 - z_2) + \mu + \alpha]}. \tag{1.29}$$

Deducing from the expressions of (1.26), (1.27), we get,

$$\begin{aligned} &\left[\lambda_1 (1 - z_1) + \lambda_2 (1 - z_2) + \mu \left(1 - \frac{1}{z_1}\right) + \alpha - \frac{\alpha\beta}{\lambda_1 + \lambda_2 + \beta}\right] F^1 (z_1,\ z_2) \\ =& \frac{\mu}{z_2 F^2 (z_1,\ z_2) - \left[\mu F_1^1 (z_2) + \frac{\mu}{z_2} F_0^2 (z_2)\right]} + \lambda_1 z_1 P_{000}. \end{aligned}$$

Substituting the equations (1.23) & (1.29) in the above results, we get,

$$\begin{aligned} &\left[\lambda_1 (1 - z_1) + \lambda_2 (1 - z_2) + \mu \left(1 - \frac{1}{z_1}\right) + \alpha - \frac{\alpha\beta}{\lambda_1 + \lambda_2 + \beta}\right] F^1 (z_1,\ z_2) \\ =& [\lambda_1 + \lambda_2 (1 - z_2) + \mu + \alpha] \left(\frac{\mu}{z_2} \frac{1}{\lambda_1 (1 - z_1) + \lambda_2 (1 - z_2) + \mu} - 1\right) F_0^2 z_2 \\ &-[\ \lambda_1 (1 - z_1) + \lambda_2 (1 - z_2)] P_{000}. \end{aligned}$$

Thus,

$$F^1(z_1,\ z_2) = \tag{1.30}$$
$$\frac{[\lambda_1+\lambda_2(1-z_2)+\mu+\alpha]\,F_0^2(z_2)-[\ \lambda_1(1-z_1)+\lambda_2(1-z_2)]\ \ P_{000}}{\left[\lambda_1(1-z_1)+\lambda_2(1-z_2)+\mu\left(1-\frac{1}{z_1}\right)+\alpha-\frac{\alpha\beta}{\lambda_1+\lambda_2+\beta}\right]}.$$

When we consider the left side of the above equation and equate it to zero, we get, $\lambda_1(1-z_1)+\lambda_2(1-z_2)+\mu\left(1-\frac{1}{z_1}\right)+\alpha-\frac{\alpha\beta}{\lambda_1+\lambda_2+\beta}=0$. On simplifying this equation, we get

$$\lambda_1 z_1^2 - z_1\left[\lambda_1+\lambda_2(1-z_2)+\mu+\alpha-\frac{\alpha\beta}{\lambda_1+\lambda_2+\beta}\right]+\mu=0. \tag{1.31}$$

When $|z_1|=1\&|z_2|$<1 , the inequality that we have is

$$|\lambda_1 z_1^2+\mu| <\lambda_1+\mu<|\lambda_1+\lambda_2(1-z_2)+\mu+\alpha-\frac{\alpha\beta}{\lambda_1+\lambda_2+\beta}|$$
$$=|\left(\lambda_1+\lambda_2(1-z_2)+\mu+\alpha-\frac{\alpha\beta}{\lambda_1+\lambda_2+\beta}\right)z_1|.$$

The above expression (1.31) has a unique root by the method of Rouche's theorem, under the area of , $|z_2|<1$, we may even refer it as $f(z_2)$. When $F^1(z_1,\ z_2)$converges with the condition that when $|z_1|<1\&|z_2|<1$, clearly $f(z_2)$ will become the root of the numerator of (1.31), then the equation that follows is appropriate

$$[\lambda_1+\lambda_2(1-z_2)+\mu+\alpha]\left(\frac{\mu}{z_2}\frac{1}{\lambda_1(1-f(z_2))+\lambda_2(1-z_2)+\mu}-1\right)F_0^2(z_2)$$
$$[\ \lambda_1(1-f(z_2))+\lambda_2(1-z_2)]\,P_{000}=0.$$

Followed by

$$F_0^2(z_2)=\frac{[\ \lambda_1(1-f(z_2))+\lambda_2(1-z_2)]\,P_{000}}{[\lambda_1+\lambda_2(1-z_2)+\mu+\alpha]\left(\frac{\mu}{z_2}\frac{1}{\lambda_1(1-f(z_2))+\lambda_2(1-z_2)+\mu}-1\right)} \tag{1.32}$$

which implies

$$F_0^2(z_2)=\frac{[\lambda_1(1-f(z_2))+\lambda_2(1-z_2)+\mu]\,[\ \lambda_1(1-f(z_2))+\lambda_2(1-z_2)]\,P_{000}}{[\lambda_1+\lambda_2(1-z_2)+\mu+\alpha]\,\frac{\mu}{z_2}-(\lambda_1(1-f(z_2))+\lambda_2(1-z_2)+\mu)}.$$

Then equations (1.29) and (1.30) can be expressed formulas including P_{000}. Substituting $z_1=f(z_2)$into expression (1.31), we have

$$\lambda_1 f(z_2)^2-\left[\lambda_1+\lambda_2(1-z_2)+\mu+\alpha-\frac{\alpha\beta}{\lambda_1+\lambda_2+\beta}\right]f(z_2)+\mu=0. \tag{1.33}$$

When z_2 approaches to 1, the above equation can be written as

$$\lambda_1 f\left(1\right)^2 - \left[\lambda_1 + \mu + \alpha - \frac{\alpha\beta}{\lambda_1 + \lambda_2 + \beta}\right] f\left(1\right) + \mu = 0. \tag{1.34}$$

After solving equation , we easily get f(1) = 1 for $f(1) = \frac{\mu\alpha\beta}{\lambda_1+\lambda_2+\beta}$, the latter value is deleted as $\frac{\mu}{\lambda_1} > 1. As f\left(z_2\right)$ is the root of square equation containing $z_1,\ f\left(z_2\right)$ shall be incident function which is differentiable. When z_2 nearing 1, differentiating both sides of the equation (1.33) , we have $f'\left(1\right) = \frac{\lambda_2}{\mu-\lambda_1+\alpha-\frac{\alpha\beta}{\lambda_1+\lambda_2+\beta}}$.

Substituting the above expression $f'\left(1\right) = \frac{\lambda_2}{\mu-\lambda_1+\alpha-\frac{\alpha\beta}{\lambda_1+\lambda_2+\beta}}$ into expression (1.32) when z_2 converges to 1, numerator and denominator of expression (1.32), all converge to zero

$$F_0^2\left(z_2\right)/z_2 = 1 = \left[\frac{\lambda_1 f'\left(1\right) -\ \lambda_2}{\left(\lambda_1 + \mu + \alpha\right)\left(-\mu + \lambda_2 f'\left(1\right) - \lambda_2\right)}\right] P_{000}. \tag{1.35}$$

Hence by applying the value of $f'\left(1\right)$ and finding the successive derivatives using the mathematica software we are able to find the value of $F_0^2\left(1\right)$.

2. Performance measures

2.1. State probability of server. Considering equations (1.18) and (1.19) are the generating functions of the class I and class II customers,

$$F^1\left(z_1,\ z_2\right) = \sum_{i=1}^{\infty} F_i^1\left(z_2\right) z_1^i \ i \geq 1 \qquad \text{from} \qquad (1.18)$$

$$F^2\left(z_1,\ z_2\right) = \sum_{i=0}^{\infty} F_i^2\left(z_2\right) z_1^i \ i \geq 0 \qquad \text{from} \qquad (1.19)$$

$$F^1\left(z_1,\ z_2\right) =$$

$$\left\{\left[\lambda_1 + \lambda_2\left(1 - z_2\right) + \mu + \alpha\right]\left(\frac{\mu}{z_2}\frac{1}{\lambda_1\left(1 - z_1\right) + \lambda_2\left(1 - z_2\right) + \mu} - 1\right)\right.$$

$$\left. F_0^2\left(z_2\right) - \left[\ \lambda_1\left(1 - z_1\right) + \lambda_2\left(1 - z_2\right)\right]\ P_{000}\right\}$$

$$\left[\lambda_1\left(1 - z_1\right) + \lambda_2\left(1 - z_2\right) + \mu\left(1 - \frac{1}{z_1}\right) + \alpha - \frac{\alpha\beta}{\lambda_1 + \lambda_2 + \beta}\right]^{-1}$$

$$F^1\left(z_1, z_2\right) = \frac{\lambda_1 + \mu + \alpha}{\alpha - \frac{\alpha\beta}{\lambda_1+\lambda_2+\beta}}\left[\frac{\lambda_1 f'\left(1\right) - \lambda_2}{\left(\lambda_1 + \mu + \alpha\right)\left(-\mu + \lambda_2 f'\left(1\right) - \lambda_2\right)}\right] P_{000}$$

$$F^2(1,1) = \frac{[\lambda_1 + \lambda_2(1 - z_2) + \mu + \alpha] F_0^2(1)}{[\lambda_1(1 - z_1) + \lambda_2(1 - z_2) + \mu + \alpha]}$$

implying

$$F^2(1,1) = \frac{[\lambda_1 + \mu + \alpha]}{[\mu + \alpha]} \left[\frac{\lambda_1 f'(1) - \lambda_2}{(\lambda_1 + \mu + \alpha)(-\mu + \lambda_2 f'(1) - \lambda_2)} \right] P_{000},$$

i.e.,

$$\left[\frac{\lambda_1 f'(1) - \lambda_2}{[\mu + \alpha](-\mu + \lambda_2 f'(1) - \lambda_2)} \right] P_{000}.$$

Due to : $F^1(z_1, z_2) + F^2(z_1, z_2) + F^3(z_1, z_2) + P_{000} = 1$, From the above theorem, we can bring out the value of P_{000}

2.2. **Theorem on non-preemptive M/M/1 queuing with class I customers over class II customers.** In a non-pre-emptive priority M/M/1 queuing model where the class I customers are served over the class II customers and the server is free with the following expressions given by, $P_{classI=} \left[\frac{\lambda_1 f'(1)}{[\mu+\alpha](-\mu+f'(1))} \right]$, $P_{classII=} \left[\frac{-\lambda_2}{[\mu+\alpha](-\mu+\lambda_2 f'(1)-\lambda_2)} \right]$. And hence average length of the customers at the queue and the system can be calculated by the mathematica software since it is very complicated to deduce it.

3. Conclusion

In the recent queuing theory models with wide range of practical applications in the computer field, we see that to have different type of service facility for different kinds of customers is mandatory. Sometimes priority cases need not be divided only as tow. It can be even more. Hence by using this vector process and using complementary method, we are able to find the stability equation and the number of customers at the system and the queue. These result will help us in optimizing the performance measures and the efficiency level.

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