

FUZZY MAXIMAL, MINIMAL OPEN AND CLOSED SETS

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ABSTRACT. By means of fuzzy maximal and fuzzy minimal open sets, we obtain few conditions for fuzzy disconnectedness; we obtain some identical results by fuzzy minimal and maximal closed sets in connection with how these related with others. It is shown that if a fuzzy space has a set which is fuzzy minimal and fuzzy maximal then, either this fuzzy set is the only nontrivial fuzzy open set in the fuzzy space or the fuzzy space is fuzzy disconnected.

1. INTRODUCTION

After the introduction of fuzzy sets by the L.A. Zadeh [8], the concept of fuzzy topology introduced by C.L.Chang [2] in 1968. The notions of minimal open [7] and maximal open [6] sets explored by F. Nakaoka and N. Oda. Nakaoka and N.Oda studied analogous of minimal and maximal closed sets in [6]. In order to extend and compare these ideas Ajoy Mukherjee established few results in [1]. The idea of fuzzy minimal and fuzzy maximal open sets studied by B. M. Ittanagi and R. S. Wali in [4].

In this paper, (X, τ) or X stands for fuzzy topological space. The symbols λ , μ , γ , η ,... are used to denote fuzzy sets and all other symbols have their usual meaning unless explicitly stated.

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In this papers we study some properties and interrelations of fuzzy minimal and fuzzy maximal open and closed sets.

2. PRELIMINARIES

Let us recall some known results.

Definition 2.1. A fuzzy topological space X is said to be fuzzy disconnected [3] if it has no proper fuzzy clopen set. A fuzzy set of X which is not fuzzy connected is called as fuzzy disconnected.

Definition 2.2. A proper nonzero fuzzy open set α of X is said to be a fuzzy minimal open [4] set if α and 0_X are only fuzzy open sets contained in α .

Definition 2.3. A proper nonzero fuzzy open set α of X is said to be a fuzzy maximal open [4] set if 1_X and α are only fuzzy open sets containing α .

Theorem 2.1. ([4]) If μ is a fuzzy maximal open set and λ be a fuzzy open subset, then either $\mu \vee \lambda = 1_X$ or $\lambda < \mu$.

Theorem 2.2. ([4]) If μ is a fuzzy minimal open set and λ be a fuzzy open subset, then either $\mu \wedge \lambda = 0_X$ or $\mu < \lambda$.

3. FUZZY MAXIMAL AND MINIMAL OPEN SETS

Theorem 3.1. If μ is a fuzzy maximal open set and λ is a fuzzy minimal open set of a fuzzy topological space X , then either $\lambda < \mu$ or the space is fuzzy disconnected.

Proof. On deploying the maximality of μ by Theorem 2.1, we have either $\mu \vee \lambda = 1_X$ or $\lambda < \mu$. On deploying the minimality of λ by Theorem 2.2, we have either $\mu \wedge \lambda = 0_X$ or $\lambda < \mu$. When $\mu \vee \lambda = 1_X$, $\lambda < \mu$ gives $\mu = 1_X$; when $\mu \wedge \lambda = 0_X$, $\lambda < \mu$ gives $\lambda = 0_X$. Therefore the probable occurrences are $\mu \vee \lambda = 1_X$, $\mu \wedge \lambda = 0_X$ and $\lambda < \mu$. If $\mu \vee \lambda = 1_X$, $\mu \wedge \lambda = 0_X$, then the space is fuzzy disconnected. $\square$

Remark 3.1. $\mu \vee \lambda = 1_X$, $\mu \wedge \lambda = 0_X$, imply $\mu = 1_X - \lambda$. In Theorem 3.1, if $\lambda \not< \mu$, then μ and λ are fuzzy closed. Theorem 3.1 may be stated as follows: if μ is a fuzzy maximal open set and λ is a fuzzy minimal open set of a fuzzy topological space X , then either $\lambda < \mu$ or $\mu = 1_X - \lambda$.

Theorem 3.2. *If a fuzzy topological space X has a fuzzy set which is both fuzzy maximal and fuzzy minimal open, then either this fuzzy set is the only nontrivial fuzzy open set in the space or the fuzzy space is fuzzy disconnected.*

Proof. Let μ be both fuzzy maximal and fuzzy minimal open, and λ be any fuzzy open set. Then we get $\mu < \mu \vee \lambda$. By the fuzzy maximality of μ , we have the following two cases.

Case I: $\mu = \mu \vee \lambda$. Then we get $\lambda < \mu$. Since μ is fuzzy minimal, we have $\lambda = 0_X$ or $\lambda = \mu$.

Case II: $\mu \vee \lambda = 1_X$. Considering μ as a fuzzy minimal open set, we get by Theorem 2.2, $\mu \wedge \lambda = 0_X$ or $\mu < \lambda$. Since μ is fuzzy maximal, $\mu < \lambda$ implies $\mu = \lambda$ or $\lambda = 1_X$.

Considering all the cases, we get $\mu = \lambda$ or $\mu \vee \lambda = 1_X$ and $\mu \wedge \lambda = 0_X$. If $\mu \vee \lambda = 1_X$ and $\mu \wedge \lambda = 0_X$, then the space is fuzzy disconnected.

It is trivial that if a fuzzy topological space X has only one proper fuzzy open set, then that fuzzy set is both fuzzy maximal and fuzzy minimal open. If there are only two proper fuzzy open sets in a fuzzy space and the fuzzy open sets are disjoint, then both are fuzzy maximal and fuzzy minimal. If μ and λ are only two proper fuzzy open sets in the fuzzy topological space such that $\mu < \lambda$, then μ is a fuzzy minimal open set and λ is a fuzzy maximal open set in the fuzzy space. However, there may not exist a fuzzy set which is both fuzzy maximal and fuzzy minimal open in a fuzzy disconnected space (see Example 1). $\square$

Corollary 3.1. *If μ is both fuzzy maximal and fuzzy minimal open, and ϑ is a fuzzy closed set in a fuzzy topological space X , then either $\mu = 1_X - \vartheta$ or $\mu = \vartheta$.*

Proof. Given μ is both fuzzy maximal and fuzzy minimal open, and ϑ is a fuzzy closed set. So $1_X - \vartheta$ is a fuzzy open set. Proceeding like the proof of Theorem 3.2, we get $\mu = 1_X - \vartheta$ or $\mu \vee (1_X - \vartheta) = 1_X$ and $\mu \wedge (1_X - \vartheta) = 0_X$. Both $\mu \vee (1_X - \vartheta) = 1_X$ and $\mu \wedge (1_X - \vartheta) = 0_X$ imply $\mu = \vartheta$. $\square$

Corollary 3.2. *If μ is both fuzzy maximal and fuzzy minimal open in a fuzzy topological space X , then either μ is the only proper fuzzy open set in the fuzzy space or proper fuzzy open sets of the fuzzy space are μ and $1_X - \mu$ only.*

Proof. Let λ be any proper fuzzy open set of the fuzzy space. Proceeding like the proof of Theorem 3.2, we get $\mu = \lambda$ or $\mu \vee \lambda = 1_X$ and $\mu \wedge \lambda = 0_X$. Both $\mu \vee \lambda = 1_X$ and $\mu \wedge \lambda = 0_X$ imply $\lambda = 1_X - \mu$. $\square$

Example 1. Let $X = \{a, b, c\}$. Then fuzzy sets $\lambda_1 = \frac{1}{a} + \frac{0}{b} + \frac{0}{c}$; $\lambda_2 = \frac{1}{a} + \frac{1}{b} + \frac{0}{c}$; $\lambda_3 = \frac{0}{a} + \frac{1}{b} + \frac{1}{c}$; $\lambda_4 = \frac{0}{a} + \frac{1}{b} + \frac{0}{c}$; are defined as follows: Consider the fuzzy topology $\tau = \{0_X, \lambda_1, \lambda_2, \lambda_3, \lambda_4, 1_X\}$. The fuzzy topological space (X, τ) is fuzzy disconnected with a separation λ_1 and λ_3 . But the fuzzy space has no fuzzy open set which is both fuzzy maximal and fuzzy minimal open.

Theorem 3.3. If γ_1 and γ_2 are two different fuzzy maximal open sets of a fuzzy topological space X with $\gamma_1 \wedge \gamma_2$ is a fuzzy closed set, then X is fuzzy disconnected.

Proof. Since γ_1 and γ_2 are fuzzy maximal, we have $\gamma_1 \vee \gamma_2 = 1_X$. We put $\mu = \gamma_1 - \gamma_1 \wedge \gamma_2$, $\lambda = \gamma_2$ or $\mu = \gamma_1$, $\lambda = \gamma_2 - \gamma_1 \wedge \gamma_2$. We note that μ, λ are disjoint fuzzy open sets with $\mu \vee \lambda = 1_X$. Therefore X is fuzzy disconnected. $\square$

Theorem 3.4. If β is a fuzzy maximal open set, then either $Cl(\beta) = 1_X$ or $Cl(\beta) = \beta$.

Theorem 3.5. If there exists a fuzzy maximal open set which is not fuzzy dense in a fuzzy topological space, then the space is fuzzy disconnected.

Proof. Let β be a fuzzy maximal set which is not fuzzy dense in X . By Theorem 3.4, $\beta = Cl(\beta)$. We write $\mu = \beta$ and $\lambda = 1_X - Cl(\beta)$. Therefore (μ, λ) is a fuzzy separation for X . $\square$

4. FUZZY MAXIMAL AND FUZZY MINIMAL CLOSED SETS

Definition 4.1. A proper nonzero fuzzy closed set ϑ of X is said to be a fuzzy maximal closed [4] set if any fuzzy closed set which contains ϑ is 1_X or ϑ .

Theorem 4.1. ([4]) If ϑ is a fuzzy maximal closed set and γ is a fuzzy closed set, then either $\vartheta \vee \gamma = 1_X$ or $\gamma < \vartheta$.

Definition 4.2. A proper nonzero fuzzy closed set ϑ of X is said to be a fuzzy minimal closed [4] set if any fuzzy closed set which is contained in ϑ is 0_X or ϑ .

Theorem 4.2. ([4]) If ϑ is a fuzzy minimal closed set and γ is any fuzzy closed set, then either $\vartheta \wedge \gamma = 0_X$ or $\vartheta < \gamma$.

Theorem 4.3. A fuzzy set ϑ in a fuzzy topological space is both fuzzy minimal and fuzzy maximal closed set, then either of the following is true:

- (i) ϑ is the only proper fuzzy closed set in the fuzzy space.
- (ii) If there exists another proper fuzzy closed set γ , then $\vartheta \vee \gamma = 1_X$ and $\vartheta \wedge \gamma = 0_X$.

Corollary 4.1. *If ϑ is both fuzzy maximal and fuzzy minimal closed set and λ is a fuzzy open set in a fuzzy topological space X , then either $\vartheta = 1_X - \lambda$ or $\vartheta = \lambda$.*

Corollary 4.2. *If ϑ is both fuzzy maximal and fuzzy minimal closed in a fuzzy topological space X , then either ϑ is the only proper fuzzy closed set in the fuzzy space or proper fuzzy closed sets of the fuzzy space are ϑ and $1_X - \vartheta$ only.*

It is trivial that if a fuzzy topological space X has only one proper fuzzy closed set, then that set is both fuzzy maximal and fuzzy minimal closed. If there are only two proper fuzzy closedsets in a fuzzy space and the fuzzy closed sets are disjoint, then both are fuzzy maximal and fuzzyminimal. If γ and δ are only two proper fuzzy closed sets in a fuzzy topological space such that $\gamma < \delta$, then γ is a fuzzy minimal closed set and δ is a fuzzy maximal closed set in the fuzzy space. However, there may not exist a fuzzy set which is both fuzzy maximal and fuzzy minimal closed in a fuzzy disconnected space . It is observed from Example 1 that there exist no disjoint fuzzy closed set in the space which is both fuzzy maximal and fuzzy maximal closed. From this consideration we easily conclude that there may exist fuzzy closed sets λ_1 and λ_2 in X such that $\lambda_1 \vee \lambda_2 = 1_X$ and $\lambda_1 \wedge \lambda_2 = 0_X$ but there may not exist a set ehich is both fuzzy maximal and fuzzy minimal closed .

Theorem 4.4. *If μ is both fuzzy maximal open and fuzzy minimal closed, λ is a fuzzy closed, then either of the following is true:*

- (i) $\lambda < \mu < \vartheta$.
- (ii) $\lambda < \mu$ and $\mu \wedge \vartheta = 0_X$.
- (iii) $\mu \vee \vartheta = 1_X$ and $\mu < \vartheta$.
- (iv) $\mu \vee \vartheta = 1_X, \mu \wedge \vartheta = 0_X$.

Proof. By Theorem 2.1, if we take μ as a fuzzy maximal open set, we get $\lambda < \mu$ or $\mu \vee \lambda = 1_X$. If we take, by Theorem 4.2, μ as a fuzzy minimal closed set, we get $\mu < \vartheta$ or $\mu \wedge \vartheta = 0_X$. $\lambda < \mu$ and $\mu < \vartheta$ imply $\lambda < \mu < \vartheta$. The remaining probable combinations are $\lambda < \mu, \mu \wedge \lambda = 0_X; \mu \vee \lambda = 1_X, \mu < \vartheta$ and $\mu \vee \lambda = 1_X, \mu \wedge \lambda = 0_X$. □

Corollary 4.3. *If μ is both fuzzy maximal open and fuzzy minimal closed, then μ and $1_X - \mu$ are only proper fuzzy clopen sets in the fuzzy space.*

Proof. Let ϑ be fuzzy clopen in 1_X . Putting $\lambda = \vartheta$ in Theorem 4.4, we get $\mu = \vartheta$ or $\mu = 1_X - \vartheta$. $\square$

Theorem 4.5. *If μ is both fuzzy maximal open and fuzzy maximal closed, ϑ is fuzzy clopen, then either $\vartheta < \mu$ or $\mu \vee \vartheta = 1_X$.*

Proof. Similar to that of proof of Theorem 4.4. $\square$

Theorem 4.6. *If μ is both fuzzy minimal open and fuzzy maximal closed, λ is a fuzzy open and ϑ is fuzzy closed, then either of the following is true:*

- (i) $\vartheta < \mu < \lambda$.
- (ii) $\mu < \lambda$ and $\mu \vee \vartheta = 1_X$.
- (iii) $\mu \wedge \lambda = 0_X$ and $\vartheta < \mu$.
- (iv) $\mu \vee \vartheta = 1_X, \mu \wedge \vartheta = 0_X$.

Corollary 4.4. *If μ is both fuzzy minimal open and fuzzy maximal closed, then μ and $1_X - \mu$ are only proper fuzzy clopen sets in the space.*

Theorem 4.7. *Let α, μ be fuzzy open sets in X such that $\alpha \wedge \mu \neq 0_X, \alpha$. Then $\alpha \wedge \mu$ is a fuzzy minimal open set in (α, τ_α) if μ is a fuzzy minimal open set in (X, τ) .*

Proof. If $\alpha \wedge \mu$ is not a fuzzy minimal open set in (α, τ_α) , there exists a fuzzy open set $\beta \neq 0_X$ in (α, τ_α) such that $\beta \not\leq \alpha \wedge \mu$. Since μ is a fuzzy minimal open set in X and $\alpha \wedge \mu \neq 0_X$ we have by Theorem 2.2, $\mu < \alpha$ which implies $\alpha \wedge \mu = \mu$. α being fuzzy open in X , β is also fuzzy open in X . Hence we get a fuzzy set β fuzzy open in X such that $0_X \neq \beta \not\leq \mu$ which is a contradiction to our assumption that μ is a fuzzy minimal open set in X . $\square$

REFERENCES

- [1] A. MUKHERJEE, K. BHANDHU BAGCHI: *On maximal, minimal open and closed sets*, Commun. Korean Math. Soc., **30**(3) (2015), 277-282.
- [2] C.L. CHANG: *Fuzzy topological spaces*: J. Math. Anal. Appl., **24** (1968), 182-190.
- [3] U.V. FATTEH, D.S. BASSAN: *Fuzzy connectedness and its stronger forms*, J. Math. Anal. Appl., **111**(2) (1985), 440-464.

- [4] B.M. ITTANAGI, R.S. WALI: *On fuzzy minimal open and fuzzy maximal open sets in fuzzy topological spaces*, International J. of Mathematical Sciences and Applications, **1**(2) (2011), 2023- 1037.
- [5] B.M. ITTANAGI, S.S. BENCHALLI: *On paraopen sets and maps in topological spaces*, Kyungpook Math. J., **56**(1) (2016), 301-310.
- [6] F. NAKAOKA, N. ODA: *Some Properties of Maximal Open Sets*, International Journal of Mathematics and Mathematical Sciences, **21** (2003), 1331-1340.
- [7] F. NAKAOKA, N. ODA: *Minimal closed sets and maximal closed sets*, International Journal of Mathematics and Mathematical Sciences, art.id. 018647, (2006), 1-8.
- [8] L.A. ZADEH: *Fuzzy sets*, Information and control, **8** (1965), 338-353.

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