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NUMERICAL SOLUTION OF ELEVENTH ORDER BOUNDARY VALUE PROBLEMS BY GALERKIN METHOD WITH SEXTIC B-SPLINES

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ABSTRACT. In this paper, we present a finite element method involving Galerkin method with sextic B-splines as basis functions to solve a general eleventh order boundary value problem. The basis functions are modified into a new set of basis functions which vanish on the boundary where almost all boundary conditions are prescribed. The solution of a non-linear boundary value problem has been obtained by quasilinearization technique.

1. INTRODUCTION

We consider a general eleventh order linear boundary value problem

$$\begin{aligned} a_0(x)y^{(11)}(x) + a_1(x)y^{(10)}(x) + a_2(x)y^{(9)}(x) + a_3(x)y^{(8)}(x) \\ + a_4(x)y^{(7)}(x) + a_5(x)y^{(6)}(x) + a_6(x)y^{(5)}(x) + a_7(x)y^{(4)}(x) + a_8(x)y^{'''}(x) \\ + a_9(x)y^{''}(x) + a_{10}(x)y^{'}(x) + a_{11}(x)y(x) = b(x), \qquad c < x < d \end{aligned} \tag{1.1}$$

subject to boundary conditions

$$\begin{aligned} y(c) = A_0,\ y(d) = C_0,\ y^{'}(c) = A_1,\ y^{'}(d) = C_1,\ y^{''}(c) = A_2,\ y^{''}(d) = C_2, \\ y^{'''}(c) = A_3,\ y^{'''}(d) = C_3,\ y^{(4)}(c) = A_4,\ y^{(4)}(d) = C_4,\ y^{(5)}(c) = A_5. \end{aligned} \tag{1.2}$$

where A_i's, C_i's are finite real constants and $a_i(x)'s$ and $b(x)\epsilon C[c,d]$.

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Generally, this type of eleventh order boundary value problem arises in the study of hydrodynamics and hydromagnetic stability, mathematical modeling of the viscoelastic flows and other areas of applied mathematics, physics, engineering sciences. The boundary value problems of higher order differential equations have been investigated due to their mathematical importance and the potential for applications in diversified applied sciences. In particular, these types of problems occur in study of eigenvalue problems [1]. The literature on the numerical solutions of eleventh order boundary value problems and associated eigenvalue problems is seldom. The existence and uniqueness of solutions of these problems have been discussed by Agarwal [2]. Siddiqi et al [3] solved a special case of eleventh order boundary value problems using the Variational iteration method. Amjad Hussain et al [4] applied Differential Transform method to solve a special case of eleventh order boundary value problems. Md. Bellal Hossain et al [5] developed the Galerkin method with Bernstein and Legendre Polynomials as basis functions to solve a general eleventh order boundary value problem. So far, eleventh order boundary value problems have not been solved by using Galerkin method with sextic B-splines. This motivated us to solve a general eleventh order boundary value problem by Galerkin method with sextic B-splines.

2. Description of the method

Sextic B-splines $B_i(x)'s$ are defined by

$$B_i(x) = \begin{cases} \sum_{r=i-3}^{i+4} \frac{(x_r - x)_+^6}{\pi'(x_r)}, & \text{for } x \in [x_{i-3}, x_{i+4}] \\ 0, & \text{otherwise} \end{cases},$$

where

$$\pi(x) = \prod_{r=i-3}^{i+4} (x - x_r),\ (x_r - x)_+^6 \text{ is nonnegative function}$$

$\{B_{-3}(x), B_{-2}(x), \ldots, B_{n+1}(x), B_{n+2}(x)\}$ forms a basis for the space $S_6(\pi)$ of sextic polynomial splines. Sextic B-splines are defined in [6, 7]. Schoenberg [7] has proved that sextic B-splines are the unique nonzero splines of smallest compact support over $[x_{-6}, x_{n+6}]$. We define the approximation for $y(x)$ as

$$y(x) = \sum_{j=-3}^{n+2} \alpha_j B_j(x), \tag{2.1}$$

where α_j's are the nodal parameter to be determined. Since we are approximating the eleventh order boundary value problem by sextic B-spline polynomial, we redefine the basis functions into a new set of basis functions which vanish on all most of all boundary conditions. The procedure for redefining the basis functions is as follows.

Using the sextic B-splines and the boundary conditions of (1.2), we get the approximate solution at the boundary points as

$$A_i = y^{(i)}(c) = y^{(i)}(x_0) = \sum_{j=-3}^{n+2} \alpha_j B_j^{(i)}(x_0),\ i = 0, 1, 2, 3, 4, \tag{2.2}$$

$$C_i = y^{(i)}(d) = y^{(i)}(x_n) = \sum_{j=-3}^{n+2} \alpha_j B_j^{(i)}(x_n),\ i = 0, 1, 2, 3, 4. \tag{2.3}$$

Eliminating $\alpha_{-3}, \ldots, \alpha_1$ and $\alpha_{n-2}, \ldots, \alpha_{n+2}$ from the equations (2.1) - (2.3), we get

$$y(x) = w(x) + \sum_{j=2}^{n-3} \alpha_j \tilde{B}_j(x),$$

where

$$w(x) = w_4(x) + \frac{A_4 - w_4^{(4)}(x_0)}{S_1^{(4)}(x_0)} S_1(x) + \frac{C_4 - w_4^{(4)}(x_n)}{S_{n-2}^{(4)}(x_n)} S_{n-2}(x)$$

$$w_4(x) = w_3(x) + \frac{A_3 - w_3^{'''}(x_0)}{R_0^{'''}(x_0)} R_0(x) + \frac{C_3 - w_3^{'''}(x_n)}{R_{n-1}^{'''}(x_n)} R_{n-1}(x)$$

$$w_3(x) = w_2(x) + \frac{A_2 - w_2^{''}(x_0)}{Q_{-1}^{''}(x_0)} Q_{-1}(x) + \frac{C_2 - w_2^{''}(x_n)}{Q_n^{''}(x_n)} Q_n(x)$$

$$w_2(x) = w_1(x) + \frac{A_1 - w_1^{'}(x_0)}{P_{-2}^{'}(x_0)} P_{-2}(x) + \frac{C_1 - w_1^{'}(x_n)}{P_{n+1}^{'}(x_n)} P_{n+1}(x)$$

$$w_1(x) = \frac{A_0}{B_{-3}(x_0)} B_{-3}(x) + \frac{C_0}{B_{n+2}(x_n)} B_{n+2}(x)$$

$$\tilde{B}_j(x) = \begin{cases} S_j(x) - \dfrac{S_j^{(4)}(x_0)}{S_1^{(4)}(x_0)} S_1(x), & j = 2 \\ S_j(x), & j = 3, \dots, n-4 \\ S_j(x) - \dfrac{S_j^{(4)}(x_n)}{S_{n-2}^{(4)}(x_n)} S_{n-2}(x), & j = n-3. \end{cases}$$

$$S_j(x) = \begin{cases} R_j(x) - \dfrac{R_j'''(x_0)}{R_0'''(x_0)} R_0(x), & j = 1, 2 \\ R_j(x), & j = 3, \dots, n-4 \\ R_j(x) - \dfrac{R_j'''(x_n)}{R_{n-1}'''(x_n)} R_{n-1}(x), & j = n-3, n-2 \end{cases}$$

$$R_j(x) = \begin{cases} Q_j(x) - \dfrac{Q_j''(x_0)}{Q_{-1}''(x_0)} Q_{-1}(x), & j = 0, 1, 2 \\ Q_j(x), & j = 3, \dots, n-4 \\ Q_j(x) - \dfrac{Q_j''(x_n)}{Q_n''(x_n)} Q_n(x), & j = n-3, n-2, n-1 \end{cases}$$

$$Q_j(x) = \begin{cases} P_j(x) - \dfrac{P_j'(x_0)}{P_{-2}'(x_0)} P_{-2}(x), & j = -1, 0, 1, 2 \\ P_j(x), & j = 3, \dots, n-4 \\ P_j(x) - \dfrac{P_j'(x_n)}{P_{n+1}'(x_n)} P_{n+1}(x), & j = n-3, n-2, n-1, n \end{cases}$$

$$P_j(x) = \begin{cases} B_j(x) - \dfrac{B_j(x_0)}{B_{-3}(x_0)} B_{-3}(x), & j = -2, -1, 0, 1, 2 \\ B_j(x), & j = 3, \dots, n-4 \\ B_j(x) - \dfrac{B_j(x_n)}{B_{n+2}(x_n)} B_{n+2}(x), & j = n-3, n-2, n-1, n, n+1. \end{cases}$$

Now the new set of basis functions for the approximation $y(x)$ is $\{\tilde{B}_j(x), j = 2, 3, \dots, n-3\}$. Applying the Galerkin method with the new set of basis functions

$$\text{(2.4)} \quad \int_{x_0}^{x_n} \Big[a_0(x)y^{(11)}(x) + a_1(x)y^{(10)}(x) + a_2(x)y^{(9)}(x) + a_3(x)y^{(8)}(x) + a_4(x)y^{(7)}(x) + a_5(x)y^{(6)}(x) + a_6(x)y^{(5)}(x) + a_7(x)y^{(4)}(x) + a_8(x)y'''(x) + a_9(x)y''(x) + a_{10}(x)y'(x) + a_{11}(x)y(x)\Big]\tilde{B}_i(x)\,dx = \int_{x_0}^{x_n} b(x)\tilde{B}_i(x)\,dx \quad \text{for} \quad i = 2, 3, 4, \dots, n-3.$$

Integtating by parts terms the first six terms on the left hand side of (2.4), we get term after applying the boundary conditions prescribed in (1.2), After rearrange the system of equations in the matrix form as

$$\mathbf{A}\alpha = \mathbf{B}$$

where $\mathbf{A} = [a_{ij}]$, $\mathbf{B} = [b_i]$,

$$a_{ij} = \int_{x_0}^{x_n} \Bigg\{ \bigg[\frac{d^4}{dx^4}\Big[a_1(x)\tilde{B}_i(x)\Big] - \frac{d^3}{dx^3}\Big[a_2(x)\tilde{B}_i(x)\Big] + \frac{d^2}{dx^2}\Big[a_3(x)\tilde{B}_i(x)\Big]\bigg]$$
$$-\frac{d}{dx}\Big[a_3(x)\tilde{B}_i(x)\Big]\bigg]\ \tilde{B}_j^{(6)}(x) + \bigg[a_6(x)\tilde{B}_i(x) - \frac{d}{dx}\Big[a_5(x)\tilde{B}_i(x)\Big] - \frac{d^6}{dx^6}\Big[a_0(x)\tilde{B}_i(x)\Big]\bigg]$$
$$\tilde{B}_j^{(5)}(x) + a_7(x)\tilde{B}_i(x)\tilde{B}_j^{(4)}(x) + a_8(x)\tilde{B}_i(x)\tilde{B}_j^{'''}(x) + a_9(x)\tilde{B}_i(x)\tilde{B}_j^{''}(x) + a_{10}(x)\tilde{B}_i(x)$$
$$\tilde{B}_j^{'}(x) + a_{11}(x)\tilde{B}_i(x)\tilde{B}_j(x)\Bigg\}\ dx - \frac{d^5}{dx^5}\Big[a_0(x)\tilde{B}_i(x)\Big]\tilde{B}_j^{(5)}(x)\Big|_{x_n} \quad i,j = 2,3,\ldots,n-3.$$

and

$$b_i = \int_{x_0}^{x_n} \Bigg\{ b(x)\tilde{B}_i(x) + \bigg[-\frac{d^4}{dx^4}\Big[a_1(x)\tilde{B}_i(x)\Big] + \frac{d^3}{dx^3}\Big[a_2(x)\tilde{B}_i(x)\Big]$$
$$-\frac{d^2}{dx^2}\Big[a_3(x)\tilde{B}_i(x)\Big] + \frac{d}{dx}\Big[a_3(x)\tilde{B}_i(x)\Big]\bigg]\ w^{(6)}(x) + \bigg[-a_6(x)\tilde{B}_i(x) + \frac{d}{dx}\Big[a_5(x)\tilde{B}_i(x)\Big]$$
$$+\frac{d^6}{dx^6}\Big[a_0(x)\tilde{B}_i(x)\Big]\bigg]\ w^{(5)}(x) - a_7(x)\tilde{B}_i(x)w^{(4)}(x) - a_8(x)\tilde{B}_i(x)w^{'''}(x) - a_9(x)\tilde{B}_i(x)$$
$$w^{''}(x) - a_{10}(x)\tilde{B}_i(x)w^{'}(x) - a_{11}(x)\tilde{B}_i(x)w(x)\Bigg\}\ dx + \frac{d^5}{dx^5}\Big[a_0(x)\tilde{B}_i(x)\Big]w^{(5)}(x)\Big|_{x_n}$$
$$- A_5\frac{d^5}{dx^5}\Big[a_0(x)\tilde{B}_i(x)\Big]\Big|_{x_0} \quad \text{for} \quad i = 2,3,\ldots,n-3$$

with $\alpha = [\alpha_2\ \alpha_3\ \ldots\alpha_{n-3}]^T$. The stiff matrix $\mathbf{A}$ is a thirteen diagonal band matrix. The nodal parameter vector α has been obtained from the system $\mathbf{A}\alpha = \mathbf{B}$ using a band matrix solution package.

3. Numerical Results

To demonstrate the applicability of the proposed method for solving the eleventh order boundary value problems of the types (1.1) and (1.2), we considered one linear and one nonlinear boundary value problems. The obtained numerical results for each problem by taking moderate step size h=0.1, are presented in tabular forms and compared with the exact solutions available in the literature.

Example 1. *Consider the linear boundary value problem*

$$y^{(11)} + xy = -(10 + x^2)e^x, \quad 0 < x < 1$$

subject to $y(0) = 1$, $y(1) = 0$, $y^{'}(0) = 0$, $y^{'}(1) = -e$, $y^{''}(0) = -1$, $y^{''}(1) = -2e$, $y^{'''}(0) = -2$, $y^{'''}(1) = -3e$, $y^{(4)}(0) = -3$, $y^{(4)}(1) = -4e$, $y^{(5)}(0) = -4e$. *The exact solution for the above problem is* $y(x) = (1 - x)e^x$. *The obtained numerical results for this problem are given in Table 1. The maximum absolute error obtained by the proposed method is* 8.672476×10^{-05}.

Example 2. *Consider the nonlinear boundary value problem*

$$y^{(11)} - e^{-x}y^2 = -e^{-x} - e^{-3x}, \qquad 0 < x < 1 \tag{3.1}$$

subject to $y(0) = 1, y(1) = \frac{1}{e}$, $y^{'}(0) = -1$, $y^{'}(1) = \frac{-1}{e}$, $y^{''}(0) = 1, y^{''}(1) = \frac{1}{e}$, $y^{'''}(0) = -1$, $y^{'''}(1) = \frac{-1}{e}$, $y^{(4)}(0) = 1$, $y^{(4)}(1) = \frac{1}{e}$, $y^{(5)}(0) = -1$. *The exact solution for the above problem is* $y(x) = e^{-x}$. *The nonlinear boundary value problem* (3.1) *is converted into a sequence of linear boundary value problems generated by quasilinearization technique [8] as*

$$y^{(11)}_{(n+1)} - [2y_n e^{-x}]y_{(n+1)} = -y_n^2 e^{-x} - e^{-x} - e^{-3x}, \qquad n = 0, 1, 2, 3, \ldots \tag{3.2}$$

$y_{(n+1)}(0) = 1$, $y_{(n+1)}(1) = \frac{1}{e}$, $y^{'}_{(n+1)}(0) = -1$, $y^{'}_{(n+1)}(1) = \frac{-1}{e}$, $y^{''}_{(n+1)}(0) = 1$, $y^{''}_{(n+1)}(1) = \frac{1}{e}$, $y^{'''}_{(n+1)}(0) = -1$, $y^{'''}_{(n+1)}(1) = \frac{-1}{e}$, $y^{(4)}_{(n+1)}(0) = 1$, $y^{(4)}_{(n+1)}(1) = \frac{1}{e}$, $y^{(5)}_{(n+1)}(0) = -1$. *The proposed method is applied to the sequence of a linear problems* (3.2) *and these numerical results are presented in Table 2. The maximum absolute error is* 7.337332×10^{-05}.

4. CONCLUSIONS

In this paper, we have deployed a Galerkin method with sextic B-splines as basis functions to solve a general eleventh order boundary value problem. The proposed method has been tested on one linear and one nonlinear eleventh order boundary value problems. We found that numerical results are close to the exact solutions. The objective of this paper is to present a simple, efficient method to solve a general eleventh order boundary value problem.

TABLE 1. Numerical results for Example 1

0.1	9.946538E-01	2.264977E-06
0.2	9.771222E-01	1.156330E-05
0.3	9.449012E-01	4.512072E-05
0.4	8.950948E-01	7.575750E-05
0.5	8.243606E-01	8.672476E-05
0.6	7.288475E-01	7.891655E-05
0.7	6.041259E-01	4.553795E-05
0.8	4.451082E-01	1.704693E-05
0.9	2.459602E-01	8.359551E-06

TABLE 2. Numerical results for Example 2

0.1	9.048374E-01	4.768372E-07
0.2	8.187308E-01	1.138449E-05
0.3	7.408182E-01	3.165007E-05
0.4	6.703200E-01	6.061792E-05
0.5	6.065307E-01	7.337332E-05
0.6	5.488116E-01	5.125999E-05
0.7	4.965853E-01	2.574921E-05
0.8	4.493290E-01	5.513430E-06
0.9	4.065697E-01	2.592802E-06

REFERENCES

[1] S. CHANDRA SEKHAR: *Hydrodynamics and Hydromagnetic Stability*, New York: Dover, 1981.

[2] R.P. AGARWAL: *Boundary value problems for Higher Order Differential Equations*, World Scientific, Singapore, 1986.

[3] S.S. SIDDIQI, G. AKRAM, I. ZULFIQAR: *Solutions of eleventh order boundary value problems using the Variational iteration technique*, Euro. J. Scient. Res., **30** (2009), 505–525.

[4] A. HUSSAIN, S. TAUSEEF MOHYUD-DIN, A. YILDIRIM: *Comparison of Numerical Solutions of Eleventh Order Two point Boundary Value Problems*, J. Inf. Comp. Sci., **7** (2012), 181–189.

[5] MD. BELLAL HOSSAIN, MD. SHAFIQUL ISLAM, MD. AZIZUR RAHMAN: *Numerical Solutions of Eleventh Order Boundary Value Problems Using Piecewise Polynomials*, IOSR J. Math., **10** (2014), 58–68.

[6] P. M. PRENTER: *Splines and Variational Methods*, John-Wiley and Sons, New York, 1989.

[7] I.J. SCHOENBERG: *On Spline Functions*, MRC Report 625, University of Wisconsin, 1966.

[8] R.E. BELLMAN, R.E. KALABAA: *Quasilinearzation and Nonlinear Boundary Value Problems*, American Elsevier, New York, 1965.

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