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AN EMPIRICAL APPROACH TO STUDY THE STABILITY OF GENERALIZED LOGISTIC MAP IN SUPERIOR ORBIT

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ABSTRACT. The standard logistic map and its variants are one of the best and simplest form of a dynamical system which plays an important role in various fields of science like biology, engineering, electronics, cryptography, etc. The generalization of the logistic map is assumed with freedom of an extra degree of parameter β and then the variants of the logistic system are produced. This article is concerned about the stability of generalized logistic map with the help of superior orbit using time series representation. In literature of logistic map, it is observed that the stabilization in standard logistic map exists for the parameter $0 < r \leq 3.2$ in Picard orbit but in superior orbit, we examine that the range of stability in generalized logistic map increases for the larger range of the parameter r depending on the control parameter α and β.

1. INTRODUCTION

Chaos is a word which shows aperiodicity, unstability and sensitivity towards initial conditions and now it becomes a subject of study and is called "chaos theory". It is believed that this concept emerged when Poincare [17] studied the qualitative theory of non-linear dynamical systems on celestial mechanics. But, unfortunately, this subject was not researched much after his demise until Henry Lorenz picked it back up in 1960's. In 1960's, H. Lorenz [12] and R. May [14] took important arithmetical footsteps and after that, almost every scientific field

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has been saturated with arithmetic computation of non-linear differential equations and difference maps. One of the most researched and exploited discrete map is logistic map $rx(1-x)$, which is also known as population growth model, proposed by P. F. Verhulst [10]. The great personality Feigenbaum [9] in 1978 analyzed the analytical as well as experimental analysis of logistic map and also studied its generic dynamical properties. For a brief overview of logistic map, one may also go through [5–8, 10].

The standard logistic map and various modified logistic maps have been introduced in last few decades which has various applications in Physics, Biology, Electronics and many other subjects. In present time, the logistic map and its various modified versions are studied intensively as a prototype of dynamical system. Song and Meng [23] proposed an error value feedback method for synchronization of logistic map. Molina et al. [15] suggested embedding dimensions of time series in logistic map and other chaotic maps. In 2010, Singh and Sinha [22] proposed a communication system and developed chaotic signals via logistic map. Radwan [18] presented a paper in which he studied some generalized discrete maps and analyzed their dynamics. Next year, Prasad and Katiyar [16] presented a paper in which they studied the stability of logistic model. Kumar and Rani [11] proposed a comparative analysis of logistic map in Picard orbit and that of Norlund orbit. In 2005, Rani and Kumar [20] analyzed the logistic map by new technique to study the stability of this map. They proposed the superior orbit as a tool of two- step iterative method and exhibited that the logistic map is convergent for extra-large value of parameter r as compared to Picard orbit. Also, Rani and Goel [21] examined the convergent behavior and stable behavior of the logistic map by I-Superior approach. They showed that the logistic map is more stable in I-superior orbit than that of Picard orbit. In 1953, Mann [13] proposed a fixed-point method to prove superiority in functional analysis and various branches of mathematics. In 2009, Rani and Agarwal [19] used this fixed-point feedback procedure of Mann to study the convergence behavior of this discrete logistic map. Again in 2012, Chugh, Rani and Ashish [4] studied the convergence of logistic map in Noor orbit which is a four-step iterative method. In 2015, Rani and Yadav [24] analyzed and examined the stability of modified and extended logistic map for elaborating the dynamics of multi-scaled population.

Presently, Ashish, et. al. [2] studied the chaotic behavior of logistic map using superior orbit. They investigated chaotic properties using Mann orbit and examined the Lyapunov exponent of standard logistic map. They showed that the new parameter α in Mann orbits is treated as control variable which may enhance the stability performance in traffic control models. Again in 2018, Ashish et al. [1] proposed a new fixed-point feedback method for studying the dynamical behavior of logistic map via superior fixed-point method. This new method gives extra freedom to control parameter that enhances the performance of many applications. In 2019, Ashish et al. [3] again proposed a superior technique for stabilizing the chaos via superior feedback method and also visualized its applications in discrete traffic control model. The article is divided into four sections. In Section 2, we mention the basic entities which have been taken into account during the analysis. In Section 3, the main results of the article are studied followed by conclusion in Section 4.

2. Preliminaries

In this section, we recall some definitions, facts and notions about the study of stability of logistic map.

Definition 2.1. *Let f be a one-dimensional map defined on non-empty sets X. Then the Picard orbit which is also known as orbit of function is the set of all iterates of an initial point x_0 and defined as $x_{n+1} = f(x_n)$.*

Definition 2.2. *Let f be a one-dimensional map defined on a non-empty set X. Then the sequence $\{x_n\}$ of all iterates defined by $x_{n+1} = x_n - \alpha(x_n - f(x_n))$, is said to be superior fixed-point feedback system, where $\alpha \in [0, 1]$ and $n \in N$. This sequence of iterates $\{x_n\}$ is also called Superior orbit [13].*

Definition 2.3. *Let f be a one-dimensional map defined on a set X, where X is a non-empty set. A point $x \in X$ is said to be fixed if it satisfies the condition $f(x) = x$ [5].*

Definition 2.4. *Let f be a one-dimensional map defined on a set X, where X is a non-empty set. A point $x \in X$ is said to be periodic fixed point of period-p or cycle-p if it satisfies $f^p(x) = x$, where p is a positive integer [5].*

3. Stability in Generalized Logistic Map using Superior Orbit

In this section, we deal with the stability of the generalized logistic map via Superior Orbit. The generalized logistic map is defined as

$$f_{r,\beta}(x) = rx(1-x)^{\beta}, \tag{3.1}$$

where $x \in [0,1]$, $r > 0$ and $\beta > 1$. For an initial point $x_0 \in [0,1]$, let us consider x_1 as new outcome for the generalized map $f_{r,\beta}(x)$, in superior orbit (3.2) such that

$$x_1 = x_0 - \alpha(x_0 - f_{r,\beta}(x_0)) = S(x_0, \alpha, r, \beta), \tag{3.2}$$

where $\alpha \in [0,1]$. So, the stable behavior of the superior system $S(x_0, \alpha, r, \beta)$ completely relies on the values of α, β and r. Therefore, to understand the stable behavior of the generalized map, we use the numerical simulation with the help of time series. The following cases of simulation results with different value of control parameter α and β are stated below in tables and figures.

In superior orbit, the standard logistic map exhibits stable behavior for larger values of parameter r for all $x \in [0,1]$, where the maximum value of parameter r depends on one control parameter β. But, in superior orbit the generalized logistic map exhibits stable behavior for large values of r for all $x \in [0,1]$, where the maximum value of r depends on the value of two control parameters α and β. In this article, we take both parameters α and β in their prescribed range where $\alpha \in (0,1)$, and $\beta > 1$ always. We choose $\beta > 1$ in this article because the generalized map converts into the standard logistic map for $\beta = 1$. For analyzing the stability in the generalized logistic map, we consider some initial point x_0 in $[0,1]$ and try to get the maximum range of r for which the generalized logistic map exhibits stable behavior. When we study the stability behavior of this generalized logistic map for different value of α and β via superior orbit, we observe that there is drastic increment in the value of parameter r. The results are studied using some tables and time-series representations which show the maximum value of r for specific choices of α and β against some initial values. We also represent some time-series plots which show the cyclic behavior of the function corresponding to various value of α, β and r. The control parameter value considered in this simulation are $\alpha = 0.95, 0.84, 0.8, 0.22$.

3.1 Case-I for $\beta = 2$ and $\alpha = 0.95, 0.84, 0.8, 0.22$

In Table 1, using Equation (3.1) and (3.2) we evaluate the maximum value of the parameter r for $\beta = 2$, $\alpha = 0.95, 0.84, 0.8, 0.22$, and initial values $x_0 = 0.2, 0.4, 0.6, 0.9$. It is observed that for $\alpha = 0.95, 0.84, 0.8, 0.22$ maximum value of r reaches to $5.65, 7.63, 9.31, 29$ respectively. Further, the following results are analyzed from Table 1 and Figures 1 and 2:

(1). For $\alpha = 0.95$, and $0 < r \leq 5.65$ we notice that for $0 < r \leq 4.1$, the generalized logistic map is convergent to a fixed point and for $4.1 < r \leq 5.65$, it shows periodicity or cyclic behavior. The cyclic behavior of the generalized map at $\beta = 2$ and $\alpha = 0.95$ has been shown through blue line in Figure 1, for $r = 5.65$. We also examined that the generalized logistic map loses its stable behavior in periodicity for $r > 5.66$.

(2). For $\alpha = 0.84$, and $0 < r \leq 7.63$, we find that the generalized map is convergent to a fixed point for $0 < r \leq 4.77$ and for $4.77 < r \leq 7.63$, it shows cyclic behavior. Moreover, the generalized map does not create any irregular behavior, that is, chaos at $\beta = 2$ and $\alpha = 0.84$. The cyclic behavior of the generalized map at $\beta = 2$ and $\alpha = 0.84$ has been shown by red line in Figure 1, for $r = 7.63$.

(3). For $\alpha = 0.8$, and $0 < r \leq 9.32$, Table 1 shows the maximum range of r in which the generalized map exhibits stable behavior. Here, we find that the generalized map shows only stable behavior. It does not exhibit chaotic behavior. The map convergent to a fixed point for $0 < r \leq 5.05$ and for $5.05 < r \leq 9.31$, it shows cyclic behavior only. For graphical presentation of the cyclic nature see Figure 2 shown by blue line.

(4). Again, Table 1, shows that for $\beta = 2$ and $\alpha = 0.22$, the generalized logistic map shows only convergent behavior which is shown by red line in Figure 2 for the larger range of parameter r approaches to 29.5. But, for $r > 29.5$, the generalized map cannot be described in superior orbit as x_n is greater than 1.

Remark 3.1. *From the above analysis it is examined that the generalized map for $\beta = 2$ shows that the stable behavior exists only for $\alpha \leq 0.84$, that is no irregular or chaos occurs for $\alpha \leq 0.84$. It only shows the convergent behavior to fixed point and periodic points.*

TABLE 1. Stable cyclic nature of generalized logistic map for maximum value of r for $\alpha \in [0,1]$ and $\beta = 2$

Initiator	$\alpha = 0.95$, $\beta = 2$	$\alpha = 0.84$, $\beta = 2$	$\alpha = 0.8$, $\beta = 2$	$\alpha = 0.22$, $\beta = 2$
x_0	r	r	r	r
0.20	5.65	7.63	9.13	29.5
0.40	5.65	7.63	9.13	29.5
0.60	5.65	7.63	9.13	29.5
0.90	5.65	7.63	9.13	29.5

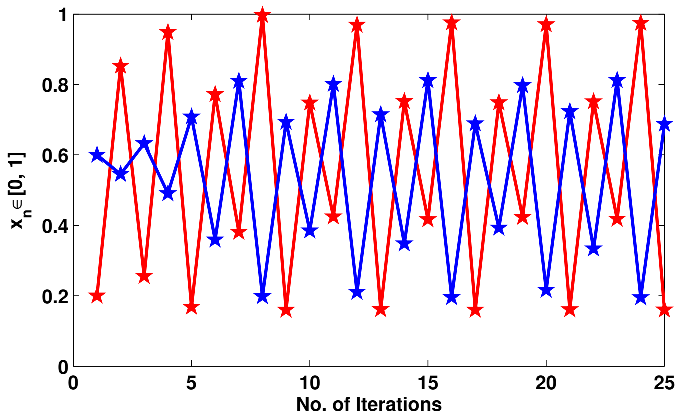

FIGURE 1. Cyclic behavior for $\alpha = 0.95, 0.84, r = 5.65, 7.63$ and $\beta = 2$

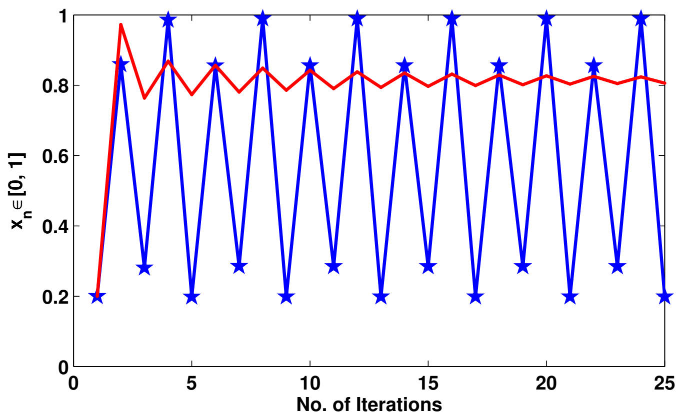

FIGURE 2. Stable behavior for $\alpha = 0.8, 0.22, r = 9.13, 29.5$ and $\beta = 2$

3.2 Case-II for $\beta = 3$ and $\alpha = 0.95, 0.89, 0.8, 0.35$

In Table 2, using Equation (3.1) and (3.2) we evaluate the maximum value of the parameter r for $\beta = 3$, $\alpha = 0.95, 0.89, 0.8, 0.35$, and initial values $x_0 = 0.2, 0.4, 0.6, 0.9$. It is observed that for $\alpha = 0.95, 0.89, 0.8, 0.35$ maximum value of r reaches to $7.2, 10.5, 11, 24.2$ respectively. Further, the following results are analyzed from Table 2 and Figures 3 and 4:

(1). For $\alpha = 0.95$, and $0 < r \leq 7.2$, we notice that for $0 < r \leq 4.8$, the generalized logistic map is convergent to a fixed point and for $4.8 < r \leq 7.2$, it shows periodicity or cycle behavior. The blue line in Figure 3 shows the graphical representation. Further, it is observed that the generalized logistic map loses its stable behavior for $r > 7.2$ and exhibits chaotic nature for $7.2 < r \leq 9.85$.

(2). For $\alpha = 0.89$, and $0 < r \leq 10.5$, we find that the generalized map is convergent to a point for $0 < r \leq 5.15$ and for $5.15 < r \leq 10.5$; it shows only cyclic behavior and does not create chaos. Here this generalized map loses its chaotic nature. The cyclic behavior of the generalized map at $\beta = 3$ and $\alpha = 0.89$ has been shown by red line in time series Figure 3, for $r = 10.5$.

(3). For $\alpha = 0.8$, and $0 < r \leq 11$, we find that the generalized map again shows only stable behavior with period-2. It does not exhibit chaotic behavior. The map convergent to a fixed point for $0 < r \leq 6$ and for $6 < r \leq 11$, it shows cyclic behavior of period-2 only. The cyclic behavior of the generalized map at $\beta = 3$ and $\alpha = 0.8$ has been shown through blue line in the time series Figure 4, for $r = 11$.

(4). Table 2, shows that for $\beta = 3$ and $\alpha = 0.35$, the generalized logistic map exhibits only stable behavior for all values of r and the maximum value of r is 24.2. For $0 < r \leq 24.2$, this map convergent to a fixed point only and does not show cyclicity. For graphical presentation the red line shows the convergent behavior in Figure 4.

Remark 3.2. *From the above analysis it is studied that the generalized map for $\beta = 3$ shows only stable behavior for $\alpha \leq 0.90$ and it is also observed that for $\alpha \leq 0.90$, it shows only convergent behavior to fixed point and periodic point.*

TABLE 2. Stable regime nature of generalized logistic map for maximum value of r for $\alpha \in [0, 1]$ and $\beta = 3$

Initiator	$\alpha = 0.95$, $\beta = 3$	$\alpha = 0.89$, $\beta = 3$	$\alpha = 0.8$, $\beta = 3$	$\alpha = 0.35$, $\beta = 3$
x_0	r	r	r	r
0.20	7.2	10.5	11	24.2
0.40	7.2	10.5	11	24.2
0.60	7.2	10.5	11	24.2
0.90	7.2	10.5	11	24.2

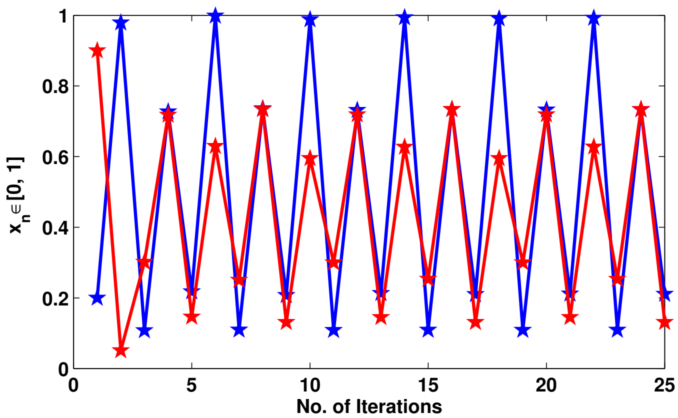

FIGURE 3. Cyclic behavior for $\alpha = 0.95, 0.89, r = 7.2, 10.5$ and $\beta = 3$

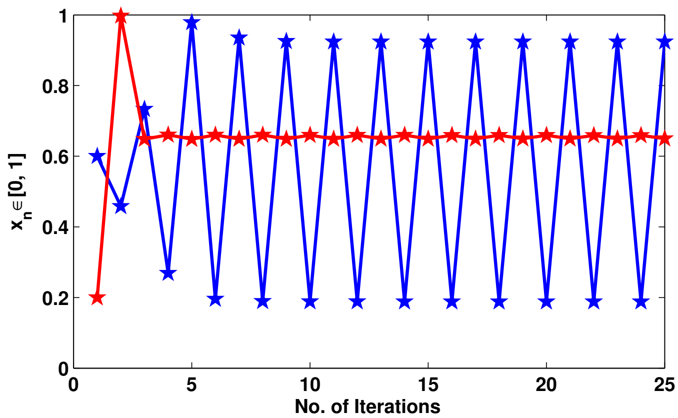

FIGURE 4. Stable behavior for $\alpha = 0.8, 0.35, r = 11, 24.2$ and $\beta = 3$

4. Conclusion

Throughout this article using some computational work on generalized logistic map using Mann fixed point iterative procedure, we achieve some more exciting and outstanding results on stabilization of logistic map as compared to Picard system. From the above analysis it is examined that the generalized logistic map for $\beta = 2$ shows that the stable behavior exists only for $\alpha \leq 0.84$, that is, no irregular or chaos occurs for this range and it shows the convergent behavior to fixed point and periodic points. Further, for $\beta = 3$ the map shows only stable behavior for $\alpha \leq 0.90$ and it is also observed that for $\alpha \leq 0.90$, it shows only convergent behavior to fixed point and periodic point and no irregular behavior occurs in the range. Furthermore, it is concluded that as the value of parameter β increases through 1 the range of the growth rate parameter r also increases.

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