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MINIMUM DOMINATING MODIFIED SCHULTZ ENERGY

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ABSTRACT. In this article we have defined a new matrix called minimum dominating modified Schlutz matrix and hence minimum dominating modified Schlutz energy. Upper and lower bounds for minimum dominating modified Schlutz energy are presented. At the end of this article minimum dominating modified Schlutz energies for some standard graphs like star graph, complete graph, crown graph, cocktail graph, complete bipartite graph and friendship graphs are computed.

1. INTRODUCTION

Study on energy of graphs goes back to the year 1978, when I. Gutman [3] defined this while working with energies of conjugated hydrocarbon containing carbon atoms. All graphs considered in this paper are assumed to be simple without loops and multiple edges. Let $A = (a_{ij})$ be the adjacency matrix of the graph G with its eigenvalues $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n$. The sum of the absolute eigenvalues values of G is called the energy $\mathcal{E}(G)$ of G, i.e., $\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|$.

1.1. **Modified Schultz index.** In 1997 S. Klavžar and I. Gutman introduced modified Schlutz index [5] which is defined by

$$S^*(G) = \sum_{\{v_i,v_j\}\subseteq V(G)} d(v_i)d(v_j)d(v_i,v_j).$$

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Rajesh Kanna et.al [7] defined minimum dominating energy of a graph. Motivated by this we now define minimum dominating modified Schultz energy of a graph.

1.2. **Minimum dominating modified Schultz energy.** Let G be a simple graph of order n with vertex set $V = \{v_1, v_2, ..., v_n\}$ and edge set E. A subset D of V is called a dominating set of G if every vertex of V-D is adjacent to some vertex in D. Any dominating set with minimum cardinality is called a minimum dominating set. Let D be a minimum dominating set of a graph G. The minimum dominating modified Schlutz matrix of G is the $n \times n$ matrix defined by $A^D_{S^*}(G) := (x_{ij})$,

where $x_{ij} = \begin{cases} d(v_i)d(v_j)d(v_i, v_j) & \text{if } v_iv_j \in E(G) \\ 1 & \text{if } i = j \text{ and } v_i \in D \\ 0 & \text{otherwise.} \end{cases}$

The characteristic polynomial of $A^D_{S^*}(G)$ is denoted by $f_n(G, \lambda) = det(\lambda I - A^D_{S^*}(G))$. The minimum dominating modified Schlutz eigenvalues of the graph G are the eigenvalues of $A^D_{S^*}(G)$. Since $A^D_{S^*}(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$. The minimum dominating modified Schlutz energy of G is defined as $E^D_{S^*}(G) : = \sum_{i=1}^{n} |\lambda_i|$.

Note that the trace of $A^D_{S^*}(G)$ = Domination Number = k.

2. Main results and discussion

2.1. Properties of minimum dominating modified Schultz eigenvalues.

Theorem 2.1. *Let G be a graph with vertex set* $V = \{v_1, v_2, ..., v_n\}$*, edge set* E*. If* $\lambda_1, \lambda_2, ..., \lambda_n$ *are the eigenvalues of minimum dominating modified Schultz matrix* $A^D_{S^*}(G)$ *then*

(i) $\sum_{i=1}^{n} \lambda_i = |D|$

(ii) $\sum_{i=1}^{n} \lambda_i^2 = |D| + 2\sum_{i<j} {d_i}^2 {d_j}^2 d_{ij}^2$.

Proof. (i) We know that the sum of the eigenvalues of $A^D_{S^*}(G)$ is the trace of $A^D_{S^*}(G)$ $\quad \therefore \sum_{i=1}^{n}\lambda_i = \sum_{i=1}^{n} x_{ii} = |D|$.

(ii) Similarly the sum of squares of the eigenvalues of $A^D_{S^*}(G)$ is trace of $[A^D_{S^*}(G)]^2$

$$\begin{aligned}
\therefore \sum_{i=1}^{n}\lambda_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}x_{ji} \\
&= \sum_{i=1}^{n}(x_{ii})^2 + \sum_{i\neq j} x_{ij}x_{ji} = \sum_{i=1}^{n}(x_{ii})^2 + 2\sum_{i<j}(x_{ij})^2 \\
&= \sum_{i=1}^{n}(x_{ii})^2 + 2\sum_{i<j} d(v_i)^2 d(v_j)^2 d(v_i,v_j)^2 \\
&= |D| + 2\sum_{i<j} {d_i}^2 {d_j}^2 d_{ij}^2 = 2M, \text{ where } M = \frac{|D|}{2} + \sum_{i<j} {d_i}^2 {d_j}^2 d_{ij}^2.
\end{aligned}$$

□

2.2. **Bounds for minimum dominating modified Schlutz energy.** MClelland's [8] gave upper and lower bounds for ordinary energy of a graph. Similar bounds for ${\mathcal{E}^{\mathcal{D}}}_{S^*}(G)$ are given in the following theorem.

Theorem 2.2. *Let G be a simple graph with n vertices and m edges and* $P = |detA^D_{S^*}(G)|$ *then*

$$\sqrt{|D| + 2\sum_{i<j} {d_i}^2 {d_j}^2 d_{ij}^2 + n(n-1)P^{\frac{2}{n}}} \leq {\mathcal{E}^{\mathcal{D}}}_{S^*}(G) \leq \sqrt{n\Big(|D| + 2\sum_{i<j} {d_i}^2 {d_j}^2 d_{ij}^2\Big)}.$$

Proof. Cauchy Schwarz inequality is

$$\Big(\sum_{i=1}^{n} a_i b_i\Big)^2 \leq \Big(\sum_{i=1}^{n} a_i^2\Big)\Big(\sum_{i=1}^{n} b_i^2\Big).$$

If $a_i = 1$ and $b_i =| \lambda_i |$ then

$$\Big(\sum_{i=1}^{n} | \lambda_i |\Big)^2 \leq \Big(\sum_{i=1}^{n} 1\Big)\Big(\sum_{i=1}^{n} \lambda_i^2\Big)$$

$$[\mathcal{E}^{\mathcal{D}}{}_{S^*}(G)]^2 \leq n\Big(|D| + 2\sum_{i<j} d_i{}^2 d_j{}^2 d_{ij}^2\Big) \quad \text{[From Theorem 2.1]}$$

$$\Longrightarrow \mathcal{E}^{\mathcal{D}}{}_{S^*}(G) \leq \sqrt{n\Big(|D| + 2\sum_{i<j} d_i{}^2 d_j{}^2 d_{ij}^2\Big)}.$$

Since arithmetic mean is greater than or equal to geometric mean we have

$$\frac{1}{n(n-1)}\sum_{i\neq j} | \lambda_i || \lambda_j | \geq \Big[\prod_{i\neq j} | \lambda_i || \lambda_j |\Big]^{\frac{1}{n(n-1)}}$$

$$= \Big[\prod_{i=1}^{n} | \lambda_i |^{2(n-1)}\Big]^{\frac{1}{n(n-1)}}$$

$$= \Big[\prod_{i=1}^{n} | \lambda_i |\Big]^{\frac{2}{n}} = \Big|\prod_{i=1}^{n} \lambda_i\Big|^{\frac{2}{n}}$$

$$= |detA_{S^*}^{D}(G)|^{\frac{2}{n}} = P^{\frac{2}{n}}$$

$$\therefore \sum_{i\neq j} | \lambda_i || \lambda_j | \geq n(n-1)P^{\frac{2}{n}} \tag{2.1}$$

Now consider,

$$[\mathcal{E}^{\mathcal{D}}{}_{S^*}(G)]^2 = \Big(\sum_{i=1}^{n} | \lambda_i |\Big)^2$$

$$= \sum_{i=1}^{n} | \lambda_i |^2 + \sum_{i\neq j} | \lambda_i || \lambda_j |$$

$$\therefore \quad [\mathcal{E}^{\mathcal{D}}{}_{S^*}(G)]^2 \geq |D| + 2\sum_{i<j} d_i{}^2 d_j{}^2 d_{ij}^2 + n(n-1)P^{\frac{2}{n}} \quad \text{[From Theorem 2.1 and Equation (2.1)]}$$

$$i.e., \ \mathcal{E}^{\mathcal{D}}{}_{S^*}(G) \geq \sqrt{|D| + 2\sum_{i<j} d_i{}^2 d_j{}^2 d_{ij}^2 + n(n-1)P^{\frac{2}{n}}}.$$

□

Theorem 2.3. *If* $\lambda_1(G)$ *is the largest minimum dominating modified Schlutz eigenvalue of* $A^D_{S^*}(G)$*,then* $\lambda_1(G) \geq \dfrac{|D| + 2\sum\limits_{i<j} d_i d_j d_{ij}}{n}$.

Proof. For any nonzero vector X, we have by [1], $\lambda_1(A) = \max\limits_{X \neq 0}\left\{\dfrac{X'AX}{X'X}\right\}$

$\therefore \lambda_1(G) \geq \dfrac{J'AJ}{J'J} = \dfrac{|D| + 2\sum\limits_{i<j} d_i d_j d_{ij}}{n}$ where J is a unit column matrix. □

Just like Koolen and Moulton's [4] upper bound for energy of a graph, an upper bound for $\mathcal{E}^{\mathcal{D}}{}_{S^*}(G)$ is given in the following theorem.

Theorem 2.4. *If G is a (n,m) graph with* $|D| + 2\sum\limits_{i<j} {d_i}^2 {d_j}^2 d_{ij}^2 \geq n$ *then*

$$\mathcal{E}^{\mathcal{D}}{}_{S^*}(G) \leq \frac{|D| + 2\sum_{i<j} {d_i}^2 {d_j}^2 d_{ij}^2}{n} + \sqrt{(n-1)\left(|D| + 2\sum_{i<j} {d_i}^2 {d_j}^2 d_{ij}^2 - \left(\frac{|D| + 2\sum_{i<j} {d_i}^2 {d_j}^2 d_{ij}^2}{n}\right)^2\right)}.$$

Proof. Cauchy-Schwartz inequality is: $\left(\sum_{i=2}^n a_i b_i\right)^2 \leq \left(\sum\limits_{i=2}^n a_i^2\right)\left(\sum\limits_{i=2}^n b_i^2\right)$.

Put $a_i = 1$ and $b_i = |\lambda_i|$ then

$$\left(\sum_{i=2}^n |\lambda_i|\right)^2 \leq \sum_{i=2}^n 1 \sum_{i=2}^n |\lambda_i|^2$$

$$\Rightarrow [\mathcal{E}^{\mathcal{D}}{}_{S^*}(G) - \lambda_1]^2 \leq (n-1)\left(|D| + 2\sum_{i<j} {d_i}^2 {d_j}^2 d_{ij}^2 - \lambda_1^2\right)$$

$$\Rightarrow \mathcal{E}^{\mathcal{D}}{}_{S^*}(G) \leq \lambda_1 + \sqrt{(n-1)\left(|D| + 2\sum_{i<j} {d_i}^2 {d_j}^2 d_{ij}^2 - \lambda_1^2\right)}$$

Let $f(x) = x + \sqrt{(n-1)\left(|D| + 2\sum\limits_{i<j} {d_i}^2 {d_j}^2 d_{ij}^2 - x^2\right)}$.

For decreasing function

$$f'(x) \leq 0 \Rightarrow 1 - \frac{x(n-1)}{\sqrt{(n-1)\left(|D|+2\sum\limits_{i<j} {d_i}^2{d_j}^2d_{ij}^2 - x^2\right)}} \leq 0$$

$$\Rightarrow x \geq \sqrt{\frac{|D|+2\sum\limits_{i<j} {d_i}^2{d_j}^2d_{ij}^2}{n}}.$$

Since $|D| + 2\sum\limits_{i<j} {d_i}^2{d_j}^2d_{ij}^2 \geq n,$ we have

$$\sqrt{\frac{|D|+2\sum\limits_{i<j} {d_i}^2{d_j}^2d_{ij}^2}{n}} \leq \frac{|D|+2\sum\limits_{i<j} {d_i}^2{d_j}^2d_{ij}^2}{n} \leq \lambda_1$$

$$\therefore f(\lambda_1) \leq f\left(\frac{|D|+2\sum\limits_{i<j} {d_i}^2{d_j}^2d_{ij}^2}{n}\right)$$

$$i.e., {\mathcal{E}^{\mathcal{D}}}_{S^*}(G) \leq f(\lambda_1) \leq f\left(\frac{|D| + 2\sum\limits_{i<j} {d_i}^2{d_j}^2d_{ij}^2}{n}\right)$$

$$i.e., {\mathcal{E}^{\mathcal{D}}}_{S^*}(G) \leq f\left(\frac{|D| + 2\sum\limits_{i<j} {d_i}^2{d_j}^2d_{ij}^2}{n}\right).$$

The proof of the theorem follows.

□

Milovanović [6] bounds for minimum dominating modified Schlutz energy of a graph are given in the following theorem.

Theorem 2.5. *Let G be a graph with n vertices and m edges. Let $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$ be a non-increasing order of Schultz eigenvalues of $A^D_{S^*}(G)$ then ${\mathcal{E}^{\mathcal{D}}}_{S^*}(G) \geq \sqrt{n\left(|D| + 2\sum\limits_{i<j} {d_i}^2{d_j}^2d_{ij}^2\right) - \alpha(n)(|\lambda_1| - |\lambda_n|)^2}$ where $\alpha(n) = n[\frac{n}{2}](1 - \frac{1}{n}[\frac{n}{2}])$ and $[x]$ denotes the integral part of a real number.*

Proof. For real numbers $a, a_1, a_2, \ldots, a_n, A$ and $b, b_1, b_2, \ldots, b_n, B$ with $a \leq a_i \leq A$ and $b \leq b_i \leq B \; \forall \; i = 1, 2, \ldots, n$ the following inequality is valid.

$\left|n\sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i\right| \leq \alpha(n)(A-a)(B-b)$ where $\alpha(n) = n[\frac{n}{2}](1-\frac{1}{n}[\frac{n}{2}])$ and equality holds if and only if $a_1 = a_2 = \ldots = a_n$ and $b_1 = b_2 = \ldots = b_n$.
If $a_i = |\lambda_i|$, $b_i = |\lambda_i|$, $a = b = |\lambda_n|$ and $A = B = |\lambda_1|$, then

$$\left|n\sum_{i=1}^{n} |\lambda_i|^2 - \Big(\sum_{i=1}^{n} |\lambda_i|\Big)^2\right| \leq \alpha(n)(|\lambda_1| - |\lambda_n|)^2.$$

But $\sum_{i=1}^{n} |\lambda_i|^2 = |D| + 2\sum_{i<j} d_i^2 d_j^2 d_{ij}^2$ and $\mathcal{E}^{\mathcal{D}}{}_{S^*}(G) \leq \sqrt{n\Big(|D| + 2\sum_{i<j} d_i^2 d_j^2 d_{ij}^2\Big)}$ then the above inequality becomes

$$n\Big(|D| + 2\sum_{i<j} d_i^2 d_j^2 d_{ij}^2\Big) - (\mathcal{E}^{\mathcal{D}}{}_{S^*}(G))^2 \leq \alpha(n)(|\lambda_1| - |\lambda_n|)^2$$

$$\text{i,e., } \mathcal{E}^{\mathcal{D}}{}_{S^*}(G) \geq \sqrt{n\Big(|D| + 2\sum_{i<j} d_i^2 d_j^2 d_{ij}^2\Big) - \alpha(n)(|\lambda_1| - |\lambda_n|)^2}.$$

□

Theorem 2.6. *Let G be a graph with n vertices and m edges. Let $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n| > 0$ be a non-increasing order of eigenvalues of $A^{\mathcal{D}}_{S^*}(G)$ then*

$$\mathcal{E}^{\mathcal{D}}{}_{S^*}(G) \geq \frac{|D| + 2\sum_{i<j} d_i^2 d_j^2 d_{ij}^2 + n|\lambda_1||\lambda_n|}{(|\lambda_1| + |\lambda_n|)}.$$

Proof. Let $a_i \neq 0$, b_i, r and R be real numbers satisfying $ra_i \leq b_i \leq Ra_i$, then the following inequality holds. [Theorem 2, [6]]

$$\sum_{i=1}^{n} b_i^2 + rR\sum_{i=1}^{n} a_i \leq (r+R)\sum_{i=1}^{n} a_i b_i \, .$$

Put $b_i = |\lambda_i|$, $a_i = 1$, $r = |\lambda_n|$ and $R = |\lambda_1|$ then

$$\sum_{i=1}^{n} |\lambda_i|^2 + |\lambda_1||\lambda_n|\sum_{i=1}^{n} 1 \leq (|\lambda_1| + |\lambda_n|)\sum_{i=1}^{n} |\lambda_i|$$

$$i.e., \ |D| + 2\sum_{i<j} d_i^2 d_j^2 d_{ij}^2 + |\lambda_1||\lambda_n| n \leq (|\lambda_1| + |\lambda_n|)\mathcal{E}^{\mathcal{D}}{}_{S^*}(G)$$

$$\therefore \mathcal{E}^{\mathcal{D}}{}_{S^*}(G) \geq \frac{|D| + 2\sum_{i<j} d_i^2 d_j^2 d_{ij}^2 + n|\lambda_1||\lambda_n|}{(|\lambda_1| + |\lambda_n|)}.$$

□

The question of when does the graph energy becomes a rational number was answered by Bapat and S.Pati in their paper [2]. Similar result for minimum dominating modified Schlutz energy is obtained in the following theorem.

Theorem 2.7. *If the minimum dominating modified Schlutz energy* $\mathcal{E}^{\mathcal{D}}{}_{S^*}(G)$ *is a rational number,then* $\mathcal{E}^{\mathcal{D}}{}_{S^*}(G) \equiv 0$*(mod 2).*

Proof. Proof is similar to theorem 5.4 of [7]

□

2.3. **Minimum dominating modified Schultz energy of some standard graphs.**

Theorem 2.8. *For* $n \geq 2$*, the Minimum dominating modified Schultz energy of complete graph* K_n *is* $(n^3 - n^2) + \sqrt{n^6 + 2n^5 + n^4 - 2n^3 + 2n^2 + 1}$.

Proof. Let K_n be a complete graph with vertex set $V = \{v_1, v_2, v_3, ..., v_n\}$. Then the Minimum dominating modified Schultz matrix of complete graph is,

$$A^D_{S^*}(K_n) = \begin{pmatrix} 1 & (n-1)^2 & (n-1)^2 & \dots & (n-1)^2 & (n-1)^2 & (n-1)^2 \\ (n-1)^2 & 0 & (n-1)^2 & \dots & (n-1)^2 & (n-1)^2 & (n-1)^2 \\ (n-1)^2 & (n-1)^2 & 0 & \dots & (n-1)^2 & (n-1)^2 & (n-1)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ (n-1)^2 & (n-1)^2 & (n-1)^2 & \dots & 0 & (n-1)^2 & (n-1)^2 \\ (n-1)^2 & (n-1)^2 & (n-1)^2 & \dots & (n-1)^2 & 0 & (n-1)^2 \\ (n-1)^2 & (n-1)^2 & (n-1)^2 & \dots & (n-1)^2 & (n-1)^2 & 0 \end{pmatrix}_{n\times n}$$

$$\text{Spec}(A^D_{S^*}(K_n)) = \begin{pmatrix} -n^2 & \frac{(n^3-n^2+1)+\sqrt{n^6+2n^5+n^4-2n^3+2n^2+1}}{2} & \frac{(n^3-n^2+1)-\sqrt{n^6+2n^5+n^4-2n^3+2n^2+1}}{2} \\ n-1 & 1 & 1 \end{pmatrix}$$

The Minimum Dominating Modified Schultz energy is,

$$E^D_{S^*}(K_n) = n^3 - n^2 + \sqrt{n^6 + 2n^5 + n^4 - 2n^3 + 2n^2 + 1}.$$

□

Theorem 2.9. *The Minimum dominating modified Schultz energy of star graph* $K_{1,n-1}$ *is* $2n - 4 + \sqrt{4n^3 - 8n^2 - 8n + 21}$.

Proof. Let $K_{1,n-1}$ be a star graph with vertex set $V = \{v_0, v_1, v_2, ..., v_{n-1}\}$. The Minimum dominating modified Schultz matrix of star graph is,

$$A^D_{S^*}(K_{1,n-1}) = \begin{pmatrix} 1 & n & n & n & \ldots & n \\ n & 0 & 2 & 2 & \ldots & 2 \\ n & 2 & 0 & 2 & \ldots & 2 \\ n & 2 & 2 & 0 & \ldots & 2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ n & 2 & 2 & 2 & \ldots & 0 \end{pmatrix}_{n\times n}$$

$$\text{Spec}(A^D_{S^*}(K_{1,n-1})) = \begin{pmatrix} -2 & \frac{2n-3+\sqrt{4n^3-8n^2-8n+21}}{2} & \frac{2n-3-\sqrt{4n^3-8n^2-8n+21}}{2} \\ & & \\ n-2 & 1 & 1 \end{pmatrix}$$

The Minimum dominating modified Schultz energy is

$$E^D_{S^*}(K_{1,n-1}) = 2n - 4 + \sqrt{4n^3 - 8n^2 - 8n + 21}.$$

□

Theorem 2.10. *The Minimum dominating modified Schultz energy of cocktail party graph is* $16n(n-1)^2$.

Proof. For $n > 2$, consider cocktail party graph $K_{n\times 2}$ with vertex set $V = \bigcup_{i=1}^{n}\{u_i, v_i\}$. Then the Minimum dominating modified Schultz matrix of cocktail party is $A^D_{S^*}(K_{n\times 2}) =$

	u_1	u_2	$\ldots$	u_{n-1}	u_n	v_1	v_2	$\ldots$	v_{n-1}	v_n
u_1	1	$4(n-1)^2$	$\ldots$	$4(n-1)^2$	$4(n-1)^2$	$8(n-1)^2$	$4(n-1)^2$	$\ldots$	$4(n-1)^2$	$4(n-1)^2$
u_2	$4(n-1)^2$	0	$\ldots$	$4(n-1)^2$	$4(n-1)^2$	$4(n-1)^2$	$8(n-1)^2$	$\ldots$	$4(n-1)^2$	$4(n-1)^2$
u_3	$4(n-1)^2$	$4(n-1)^2$	$\ldots$	$4(n-1)^2$	$4(n-1)^2$	$4(n-1)^2$	$4(n-1)^2$	$\ldots$	$4(n-1)^2$	$4(n-1)^2$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
u_{n-1}	$4(n-1)^2$	$4(n-1)^2$	$\ldots$	0	$4(n-1)^2$	$4(n-1)^2$	$4(n-1)^2$	$\ldots$	$8(n-1)^2$	$4(n-1)^2$
u_n	$4(n-1)^2$	$4(n-1)^2$	$\ldots$	$4(n-1)^2$	0	$4(n-1)^2$	$4(n-1)^2$	$\ldots$	$4(n-1)^2$	$8(n-1)^2$
v_1	$8(n-1)^2$	$4(n-1)^2$	$\ldots$	$4(n-1)^2$	$4(n-1)^2$	1	$4(n-1)^2$	$\ldots$	$4(n-1)^2$	$4(n-1)^2$
v_2	$4(n-1)^2$	$8(n-1)^2$	$\ldots$	$4(n-1)^2$	$4(n-1)^2$	$4(n-1)^2$	0	$\ldots$	$4(n-1)^2$	$4(n-1)^2$
v_3	$4(n-1)^2$	$4(n-1)^2$	$\ldots$	$4(n-1)^2$	$4(n-1)^2$	$4(n-1)^2$	$4(n-1)^2$	$\ldots$	$4(n-1)^2$	$4(n-1)^2$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
v_{n-1}	$4(n-1)^2$	$4(n-1)^2$	$\ldots$	$8(n-1)^2$	$4(n-1)^2$	$4(n-1)^2$	$4(n-1)^2$	$\ldots$	0	$4(n-1)^2$
v_n	$4(n-1)^2$	$4(n-1)^2$	$\ldots$	$4(n-1)^2$	$8(n-1)^2$	$4(n-1)^2$	$4(n-1)^2$	$\ldots$	$4(n-1)^2$	0

$$\operatorname{Spec}(A^D_{S^*}(K_{n\times 2})) = \begin{pmatrix} 0 & -(8n^2-16n+7) & -(8n^2-16n+8) & \frac{X+\sqrt{Y}}{2} & \frac{X-\sqrt{Y}}{2} \\ \\ n-2 & 1 & n-1 & 1 & 1 \end{pmatrix}$$

where, $X = 8n^3 - 16n^2 + 8n + 1$ and
$Y = 64n^6 - 256n^5 + 384n^4 - 272n^3 + 128n^2 - 80n + 33$. The Minimum dominating modified Schultz energy is,

$$E^D_{S^*}(K_{n\times 2}) = 16n(n-1)^2.$$

□

Theorem 2.11. *For $n \geq 2$, the Minimum dominating modified Schultz energy of the crown graph is*

$$\begin{cases} 2\sqrt{5} & \text{if } n = 2 \\ 7n^3 - 22n^2 + 23n - 7 + \sqrt{n^6 - 4n^5 + 6n^4 - 6n^3 + 9n^2 - 10n + 5} & \text{if } n \geq 3. \end{cases}.$$

Proof. **Case: 1** If $n \geq 3$,
The Minimum dominating modified Schultz matrix of crown graph is $A^D_{S^*}(S^0_n) =$

$$\left(\begin{array}{c|ccccc|ccccc}
 & v_1 & v_2 & v_3 & \dots & v_n & u_1 & u_2 & u_3 & \dots & u_n \\ \hline
v_1 & 1 & 2(n-1)^2 & 2(n-1)^2 & \dots & 2(n-1)^2 & 3(n-1)^2 & (n-1)^2 & (n-1)^2 & \dots & (n-1)^2 \\
v_2 & 2(n-1)^2 & 0 & 2(n-1)^2 & \dots & 2(n-1)^2 & (n-1)^2 & 3(n-1)^2 & (n-1)^2 & \dots & (n-1)^2 \\
v_3 & 2(n-1)^2 & 2(n-1)^2 & 0 & \dots & 2(n-1)^2 & (n-1)^2 & (n-1)^2 & 3(n-1)^2 & \dots & (n-1)^2 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
v_n & 2(n-1)^2 & 2(n-1)^2 & 2(n-1)^2 & \dots & 0 & (n-1)^2 & (n-1)^2 & (n-1)^2 & \dots & 3(n-1)^2 \\ \hline
u_1 & 3(n-1)^2 & (n-1)^2 & (n-1)^2 & \dots & (n-1)^2 & 1 & 2(n-1)^2 & 2(n-1)^2 & \dots & 2(n-1)^2 \\
u_2 & (n-1)^2 & 3(n-1)^2 & (n-1)^2 & \dots & (n-1)^2 & 2(n-1)^2 & 0 & 2(n-1)^2 & \dots & 2(n-1)^2 \\
u_3 & (n-1)^2 & (n-1)^2 & 3(n-1)^2 & \dots & (n-1)^2 & 2(n-1)^2 & 2(n-1)^2 & 0 & \dots & 2(n-1)^2 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
u_n & (n-1)^2 & (n-1)^2 & (n-1)^2 & \dots & 3(n-1)^2 & 2(n-1)^2 & 2(n-1)^2 & 2(n-1)^2 & \dots & 0
\end{array}\right)$$

$$\operatorname{Spec}(A^D_{S^*}(S^0_n)) = \begin{pmatrix} 0 & -(4n^2-8n+4) & \frac{X+\sqrt{Y}}{2} & \frac{X-\sqrt{Y}}{2} & \frac{R+\sqrt{S}}{2} & \frac{R-\sqrt{S}}{2} \\ \\ n-2 & n-2 & 1 & 1 & 1 & 1 \end{pmatrix}$$

where

$$X = \frac{3n^3 - 6n^2 + 3n + 1 + \sqrt{9n^6 - 36n^5 + 54n^4 - 42n^2 + 33n^2 - 30n + 13}}{2}$$

$$Y = \frac{3n^3 - 6n^2 + 3n + 1 - \sqrt{9n^6 - 36n^5 + 54n^4 - 42n^2 + 33n^2 - 30n + 13}}{2}$$

$$R = \frac{n^3 - 10n^2 + 17n - 7 + \sqrt{n^6 - 4n^5 + 6n^4 - 6n^3 + 9n^2 - 10n + 5}}{2}$$

$$S = \frac{n^3 - 10n^2 + 17n - 7 - \sqrt{n^6 - 4n^5 + 6n^4 - 6n^3 + 9n^2 - 10n + 5}}{2}.$$

Then Minimum dominating modified Schultz energy,
$E^D_{S^*}(S^0_n) = 7n^3 - 22n^2 + 23n - 7 + \sqrt{n^6 - 4n^5 + 6n^4 - 6n^3 + 9n^2 - 10n + 5}$.

Case: 2 if $n = 2$
Then the Minimum dominating modified Schultz matrix is Crown graph is,

$$A^D_{S^*}(S^0_n) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}_{4\times 4} \quad ,\text{Spec}(A^D_{S^*}(S^0_n)) = \begin{pmatrix} \frac{\sqrt{5}+1}{2} & \frac{-\sqrt{5}+1}{2} \\ \\ 2 & 2 \end{pmatrix}$$

The Minimum dominating modified Schultz energy, $E^D_{S^*}(S^0_n) = 2\sqrt{5}$. □

Theorem 2.12. *The Minimum dominating modified Schultz energy of friendship graph $E^D_{S^*}(F^n_3)$ is equal to $(16n - 12) + \sqrt{128n^3 + 256n^2 - 416n + 169}]$.*

Proof. For a friendship graph F^n_3 with vertex set $V = \{v_o, v_1, v_2, ..., v_n, v_{n+1}, v_{n+2}, ..., v_{2n}\}$
Then the Schultz matrix of friendship graph is $A^D_{S^*}(F^n_3) =$

$$\left(\begin{array}{c|cccccccccc}
 & v_0 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & \dots & v_{2n} \\ \hline
v_0 & 1 & 2^2n & 2^2n & 2^2n & 2^2n & 2^2n & 2^2n & 2^2n & \dots & 2^2n \\
v_1 & 2^2n & 0 & 2^2 & (3^2-1) & (3^2-1) & (3^2-1) & (3^2-1) & (3^2-1) & \dots & (3^2-1) \\
v_2 & 2^2n & 2^2 & 0 & (3^2-1) & (3^2-1) & (3^2-1) & (3^2-1) & (3^2-1) & \dots & (3^2-1) \\
v_3 & 2^2n & (3^2-1) & (3^2-1) & 0 & 2^2 & (3^2-1) & (3^2-1) & (3^2-1) & \dots & (3^2-1) \\
v_4 & 2^2n & (3^2-1) & (3^2-1) & 2^2 & 0 & (3^2-1) & (3^2-1) & (3^2-1) & \dots & (3^2-1) \\
v_5 & 2^2n & (3^2-1) & (3^2-1) & (3^2-1) & (3^2-1) & 0 & 2^2 & (3^2-1) & \dots & (3^2-1) \\
v_6 & 2^2n & (3^2-1) & (3^2-1) & (3^2-1) & (3^2-1) & 2^2 & 0 & (3^2-1) & \dots & (3^2-1) \\
v_7 & 2^2n & (3^2-1) & (3^2-1) & (3^2-1) & (3^2-1) & (3^2-1) & (3^2-1) & 0 & \dots & (3^2-1) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
v_{2n} & 2^2n & (3^2-1) & (3^2-1) & (3^2-1) & (3^2-1) & (3^2-1) & (3^2-1) & (3^2-1) & \dots & 0
\end{array}\right)_{(2n+1)\times(2n+1)}$$

$$\text{Spec}(A^D_{S^*}(F^n_3)) = \begin{pmatrix} -4 & -12 & \frac{(16n-11)+\sqrt{128n^3+256n^2-416n+169}}{2} & \frac{(16n-11)-\sqrt{128n^3+256n^2-416n+169}}{2} \\ \\ n & n-1 & 1 & 1 \end{pmatrix}$$

Minimum dominating modified Schultz energy is,

$$E^D_{S^*}(F_3^n) = (16n-12) + \sqrt{128n^3 + 256n^2 - 416n + 169}.$$

□

Theorem 2.13. *The Minimum dominating modified Schultz energy of the complete bipartite graph is* $4mn^2 - 4n^2 + 4m^2n - 4m^2 - m + 2$.

Proof. For the complete bipartite graph $K_{m,n}$ $(m \leq n)$ with vertex set $V = \{u_1, u_2, ..., u_m, v_1, v_2, ..., v_n\}$ The a minimum dominating modified Schlutz matrix of complete bipartite graph is $A^D_{S^*}(K_{m,n}) =$

$$\left(\begin{array}{c|ccccc|ccccc}
 & v_1 & v_2 & v_3 & \dots & v_m & u_1 & u_2 & u_3 & \dots & u_n \\ \hline
v_1 & 1 & 2n^2 & 2n^2 & \dots & 2n^2 & mn & mn & mn & \dots & mn \\
v_2 & 2n^2 & 1 & 2n^2 & \dots & 2n^2 & mn & mn & mn & \dots & mn \\
v_3 & 2n^2 & 2n^2 & 1 & \dots & 2n^2 & mn & mn & mn & \dots & mn \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
v_m & 2n^2 & 2n^2 & 2n^2 & \dots & 1 & mn & mn & mn & \dots & mn \\ \hline
u_1 & mn & mn & mn & \dots & mn & 0 & 2m^2 & 2m^2 & \dots & 2m^2 \\
u_2 & mn & mn & mn & \dots & mn & 2m^2 & 0 & 2m^2 & \dots & 2m^2 \\
u_3 & mn & mn & mn & \dots & mn & 2m^2 & 2m^2 & 0 & \dots & 2m^2 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
u_n & mn & mn & mn & \dots & mn & 2m^2 & 2m^2 & 2m^2 & \dots & 0 \\ \hline
\end{array}\right)$$

$$\text{Spec}(A^D_{S^*}(K_{m,n})) = \begin{pmatrix} -2n^2+1 & -2m^2 & \frac{X+\sqrt{Y}}{2} & \frac{X-\sqrt{Y}}{2} \\ m-1 & n-1 & 1 & 1 \end{pmatrix}$$

where $X = (2m-2)n^2 + 2m^2n - 2m^2 + 1$ and

$$Y = (4m^2 - 8m + 4)n^4 + (8m^2 - 4m^3)n^3 + (4m^4 + 8m^3 - 8m^2 + 4m - 4)n^2 \\ -(8m^4 + 4m^2)n + 4m^4 + 4m^2 + 1.$$

Minimum dominating modified Schultz energy is,

$E^D_{S^*}(K_{m,n})$ $= 4mn^2 - 4n^2 + 4m^2n - 4m^2 - m + 2$. □

3. CONCLUSIONS

In this article we defined minimum dominating modified Schlutz energy of a graph. Upper and lower bounds for minimum dominating modified Schlutz energy are established. A generalized expression for minimum dominating modified Schultz energies for star graph, complete graph, crown graph, complete bipartite graph, cocktail party graph and friendship graphs are also computed.

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