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CONFORMAL RICCI SOLITON ON ALMOST CO-KÄHLER MANIFOLD

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ABSTRACT. In this paper, we study almost coKähler manifolds admitting the conformal Ricci soliton and determine the value of the soliton constant λ and hence the condition for the soliton to be shrinking, steady or expanding. Then we find the condition on the conformal pressure p under which, a conformal Ricci soliton on a (k,μ)-almost coKähler manifold becomes expanding. Finally we show that a (k,μ)-almost coKähler manifold, with the potential vector field V pointwise collinear with the Reeb vector field ξ, does not admit conformal gradient Ricci soliton.

1. INTRODUCTION

A Riemannian metric g defined on a smooth manifold M^n,of dimension n, is said to be a Ricci soliton if for some constant λ, there exists a smooth vector field X on M satisfying the equation

$$Ric + \frac{1}{2}\mathcal{L}_V g = \lambda g, \tag{1.1}$$

where $\mathcal{L}_V$ denotes the Lie derivative in the direction of V and Ric is the Ricci tensor. The Ricci soliton is called shrinking if $\lambda > 0$, steady if $\lambda = 0$ and expanding if $\lambda < 0$. In 1982, R.S. Hamilton [11] first studied the Ricci soliton as a self similar solution to the Ricci flow equation given by: $\frac{\partial}{\partial t}(g(t)) = -2Ric(g(t))$, where $g(t)$ is a one parameter family of metrics on M^{2n+1}.

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Ricci solitons can also be viewed as natural generalizations of Einstein metrics which moves only by a one-parameter group of diffeomorphisms and scaling [12]. Again a Ricci soliton is called a gradient Ricci soliton [3] if the concerned vector field V in the equation (1.1), is the gradient of some smooth function f, i.e; if $V = Df$, where D is the gradient operator of g. This function f is called the potential function of the Ricci soliton.

A.E. Fisher, in 2005, introduced [9] conformal Ricci flow equation which is a modified version of the Hamilton's Ricci flow equation that modifies the volume constraint of that equation to a scalar curvature constraint. The conformal Ricci flow equations on a smooth closed connected oriented manifold M^n, of dimension n, are given by

$$\frac{\partial g}{\partial t} + 2\left(Ric + \frac{g}{n}\right) = -pg, \tag{1.2}$$

$$r(g) = -1,$$

where p is a non-dynamical(time dependent) scalar field and $r(g)$ is the scalar curvature of the manifold. The term $-pg$ acts as the constraint force to maintain the scalar curvature constraint. Thus these evolution equations are analogous to famous Navier-Stokes equations in fluid mechanics where the constraint is divergence free. That is why sometimes p is also called the conformal pressure.

Recently, in 2015, N. Basu and A. Bhattacharyya [2] introduced the concept of conformal Ricci soliton as a generalization of the classical Ricci soliton.

Definition 1.1. *A Riemannian metric g on a smooth manifold M^n,of dimension n, is called a conformal Ricci soliton if there exists a constant λ and a vector field V such that*

$$\mathcal{L}_V g + 2S = \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g, \tag{1.3}$$

where $S = Ric$ is the Ricci tensor, λ is a constant and p is the conformal pressure.

It can be easily checked that the above soliton equation satisfies the conformal Ricci flow equation (1.2). Later, T. Dutta. et.al. [7] studied this conformal Ricci soliton in the framework of Lorentzian α-Sasakian manifolds. Moreover, if the vector field V is the gradient of some smooth function f on M^n, we call the soliton a conformal gradient Ricci soliton and then the soliton equation (1.2) becomes

$$S + \nabla\nabla f = \left[\lambda - \left(\frac{p}{2} + \frac{1}{n}\right)\right] g, \tag{1.4}$$

where ∇ is the Riemannian connection on the manifold M^n.

Motivated by the above studies, here we study conformal Ricci soliton in the framework of almost coKähler manifold and on its various versions. We find conditions to determine the nature of the soliton for different cases. The paper is organised as follows: in section-2, we discuss some preliminary concepts of almost coKähler manifolds. Then in section-3, we study almost coKähler manifolds admitting the conformal Ricci soliton and we calculate the value of the soliton constant λ and hence we find the condition for the soliton to be shrinking, steady or expanding. After that in section-4, we find the condition on the conformal pressure p under which, a conformal Ricci soliton on a (k, μ)-almost coKähler manifold becomes expanding. Finally in section-5, we show that a (k, μ)-almost coKähler manifold, with the potential vector field V pointwise collinear with the Reeb vector field ξ, does not admit conformal gradient Ricci soliton.

2. Preliminaries on almost coKähler manifolds

The geometry of coKähler manifolds as a special case of almost contact manifolds was studied primarily as an odd-dimensional analogy of the Kähler manifolds in complex geometry. So, let us first recall some preliminaries on almost coKähler manifolds. A smooth $(2n+1)$ dimensional manifold M^{2n+1} is said to admit an almost contact structure (ϕ, ξ, η) if there exist a $(1,1)$ tensor field ϕ, a vector field ξ and a global 1-form η on M^{2n+1} such that

$$\phi^2 = -I + \eta \otimes \xi \; and \; \eta(\xi) = 1, \tag{2.1}$$

where I is the identity endomorphism on M. Then the manifold M equipped with this almost contact structure (ϕ, ξ, η) is called an almost contact manifold (see [1]) and is denoted as $(M^{2n+1}\phi, \xi, \eta)$. The vector field ξ is called the characteristic vector field or Reeb vector field.

From (2.1) it can easily be seen that, for an almost contact structure the following relations hold; $\phi(\xi) = 0$ and $\eta \circ \phi = 0$.

Furthermore, on an almost contact manifold $(M^{2n+1}, \phi, \xi, \eta)$ if there exists a Riemannian metric g satisfying;

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for all vector fields X, Y in TM, where TM is tangent bundle of M, then the metric g is called compatible with the almost contact structure. The manifold M^{2n+1} together with the almost contact metric structure (ϕ, ξ, η, g) is called an almost contact metric manifold and we denote it as $(M^{2n+1}, g, \phi, \xi, \eta)$.

We define the fundamental 2-form Φ on an almost contact metric manifold as

$$\Phi(X, Y) = g(X, \phi Y) = d\eta(X, Y), \tag{2.2}$$

for all vector fields X, Y in TM. Now, it is known that on the product manifold $M^{2n+1} \times \mathbb{R}$, if we define a structure J as;

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right),$$

for all X in TM, where t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M^{2n+1} \times \mathbb{R}$: then J becomes an almost complex structure and if this almost complex structure J is integrable we say that the almost contact structure $(M^{2n+1}, \phi, \xi, \eta)$ is normal. Again, D.E. Blair [1] expressed the condition for normality of an almost contact structure as: $[\phi, \phi] = -2d\eta \otimes \xi$; where the Nijenhuis tensor $[\phi, \phi]$ is defined as

$$[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y],$$

for all X, Y in TM and [X,Y] is the Lie bracket operation. Now we are in a position to define the concept of coKähler manifold [see [1], [4]] and almost coKähler manifold.

Definition 2.1. *An almost contact metric manifold is called an almost coKähler manifold if both the* 1*-form* η *and the fundamental* 2*-form* Φ *(as defined by equation (2.2)) are closed.*

In particular, if the associated almost contact structure is normal or equivalently $\nabla\phi = 0$ or $\nabla\Phi = 0$: then the almost coKähler manifold is called a coKähler manifold. Also, it is to be noted that, examples (see [5], [13]) of almost coKähler manifolds exist, which are not globally the product of a almost Kähler manifold and the real line.

Next, we set two symmetric operators h and h' given by, $h = \frac{1}{2}\mathcal{L}_\xi \phi$ and $h' = h \circ \phi$ on the almost coKähler manifold $(M^{2n+1}, g, \phi, \xi, \eta)$. Then the following relations can be obtained (see [13], [6])

$$h\xi = 0,\ h\phi + \phi h = 0,\ tr(h) = tr(h') = 0, \tag{2.3}$$

$$\nabla_\xi \phi = 0,\ \nabla \xi = h',\ div\xi = 0, \tag{2.4}$$

$$S(\xi, \xi) + \|h\|^2 = 0, \tag{2.5}$$

$$\phi l \phi - l = 2h^2, \tag{2.6}$$
$$\nabla_\xi h = -h^2\phi - \phi l,$$

where we set $l := R(., \xi)\xi$ and R is the Riemannian curvature tensor defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{2.7}$$

for all vector fields $X, Y, Z \in TM$.

3. Conformal Ricci Soliton on Almost CoKähler Manifold

Let us consider $(M^{2n+1}, g, \phi, \xi, \eta)$ be an almost coKähler manifold which satisfies the conformal Ricci soliton equation given in equation (1.3); then for all vector fields X, Y in TM i.e; we have

$$(\mathcal{L}_V g)(X, Y) + 2S(X, Y) = \left[2\lambda - (p + \frac{2}{2n+1})\right] g(X, Y). \tag{3.1}$$

Now, let the vector field V be pointwise collinear with the Reeb vector field ξ, i.e; $V = \beta\xi$, where $\beta$is a non-zero smooth function on the corresponding manifold. Then taking covariant differentiation of both sides of $V = \beta\xi$, along the direction of X we get

$$\nabla_X V = X(\beta)\xi + \beta \nabla_X \xi,$$

and using $\nabla\xi = h'$ from equation (2.4) the above equation eventually becomes

$$\nabla_X V = X(\beta)\xi + \beta h' X. \tag{3.2}$$

On the other hand, from the definiton of Lie derivative it follows from equation (3.1) that

$$g(\nabla_Y \beta\xi, Z) + g(Y, \nabla_Z \beta\xi) + 2S(Y, Z) = \left[2\lambda - \left(p + \frac{2}{2n+1}\right)\right] g(Y, Z), \tag{3.3}$$

for all Y, Z in TM. Then using equation (3.2) in the above equation (3.3) we get

$$g(Y\beta\xi + \beta h'Y, Z) + g(Y, Z\beta\xi + \beta h'Z) + 2S(Y, Z) = [2\lambda - (p + \frac{2}{2n+1})]g(Y, Z).$$

Again using from the fact that h' is symmetric and after simplification the above equation finally becomes

$$Y(\beta)\eta(Z) + Z(\beta)\eta(Y) + 2\beta g(h'Y, Z) + 2S(Y, Z) = [2\lambda - (p + \frac{2}{2n+1})]g(Y, Z). \tag{3.4}$$

Next, we consider a local ϕ-basis $\{e_j : 1 \leq j \leq 2n+1\}$ on the tangent space T_pM for each point $p \in M^{2n+1}$. Then putting $Y = Z = e_j$ in the equation (3.4) and taking summation over $1 \leq j \leq 2n+1$ and also using $tr(h') = 0$ from equation (2.3) we get

$$\xi(\beta) + r = [\lambda - (\frac{p}{2} + \frac{1}{2n+1})](2n+1). \tag{3.5}$$

Again putting $Z = \xi$ in the equation (3.4) and using symmetry of h' we have

$$Y(\beta) + \xi(\beta)\eta(Y) + 2S(Y, \xi) = [2\lambda - (p + \frac{2}{2n+1})]\eta(Y). \tag{3.6}$$

Now, combining equations (3.5) and (3.6) and after some calculations we get

$$Y(\beta) + 2S(Y, \xi) = [[\lambda - (\frac{p}{2} + \frac{1}{2n+1})](1-2n) + r]\eta(Y).$$

Thus, from the above it is easily seen that

$$\xi(\beta) + 2S(\xi, \xi) = [\lambda - (\frac{p}{2} + \frac{1}{2n+1})](1-2n) + r. \tag{3.7}$$

Eliminating $\xi(\beta)$ from equations (3.5) and (3.7) and after simplification we arrive at

$$2n[\lambda - (\frac{p}{2} + \frac{1}{2n+1})] - r + S(\xi, \xi) = 0.$$

Using equation (2.5) in the above equation and using the fact that for conformal Ricci soliton the scalar curvature $r = -1$ (see equation(1.2)) and then simplifying we get the value of the soliton constant as

$$\lambda = \frac{\|h\|^2 - 1}{2n} + (\frac{p}{2} + \frac{1}{2n+1}). \tag{3.8}$$

Therefore in view of the fact that the soliton is shrinking, steady or expanding according as $\lambda > 0$, $\lambda = 0$ or, $\lambda < 0$; from the above equation (3.8) we can state the following theorem

Theorem 3.1. *Let $(M^{2n+1}, g, \phi, \xi, \eta)$ be an almost coKähler manifold such that the metric g is a conformal Ricci soliton. If the potential vector field V be non-zero pointwise collinear with the Reeb vector field ξ, then*

(i) *the soliton is shrinking if, $p > \frac{1-(2n+1)\|h\|^2}{(2n^2+n)}$,*
(ii) *the soliton is steady if, $p = \frac{1-(2n+1)\|h\|^2}{(2n^2+n)}$,*
(iii) *the soliton is expanding if, $p < \frac{1-(2n+1)\|h\|^2}{(2n^2+n)}$.*

Again if we have $S = [\frac{\|h\|^2-1}{2n}]g$, then from equation (3.1) and using value of the soliton constant λ from (3.8) we have $\mathcal{L}_V g = 0$. Therefore we can see that $V = \beta\xi$ is a Killing vector field and hence the soliton becomes trivial. Hence we can state the following corollary.

Corollary 3.1. *Let $(M^{2n+1}, g, \phi, \xi, \eta)$ be an almost coKähler manifold such that the metric g is a conformal Ricci soliton. If the potential vector field V be non-zero pointwise collinear with the Reeb vector field ξ and the Ricci tensor S be a constant multiple of the metric g, with the constant $\frac{\|h\|^2-1}{2n}$, (i.e; if $S = [\frac{\|h\|^2-1}{2n}]g$), then the soliton is trivial.*

4. Conformal Ricci Soliton on (k, μ)-Almost CoKähler Manifold

In recent years, many authors studied (k, μ)-contact metric manifolds as a generalization of Sasakian and K-contact metric manifolds. Also R. Sharma [15], and later A. Ghosh [10] proved some interesting results in the field of Ricci solitons on (k, μ)-contact metric manifolds. Let us now give the definition (k, μ)-almost coKähler manifold.

Definition 4.1. *An almost coKähler manifold is said to be a (k, μ)-almost coKähler manifold if the characteristic vector field ξ belongs to the generalised (k, μ)-nullity distribution i.e; if the Riemannian curvature tensor R satisfies*

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \tag{4.1}$$

for all X, Y in TM and for some smooth functions (k, μ).

Remark 4.1. *Here, in this paper, we call a (k, μ)-almost coKähler manifold with $k < 0$, a proper (k, μ)-almost coKähler manifold . Proper almost coKähler manifolds with k and μ being constants were introduced by H. Endo [8] and later Dacko and Olszak [6] further studied it in generalised cases.*

Now, putting $Y=\xi$ in (4.1) we get

$$R(X,\xi)\xi = k[X-\eta(X)\xi] + \mu[hX-\eta(X)h\xi].$$

Then using the definition of $l := R(.,\xi)\xi$ and from equation (2.3) using the fact that $h\xi = 0$, we can write

$$l = -k\phi^2 + \mu h.$$

Combining the equation (2.6) and the above equation and after brief calculations we get $h^2 = k\phi^2$. Thus, it is clear that the manifold M^{2n+1} is K-almost coKähler if and ony if, $k=0$. According to Dacko and Olszak [6] a (k,μ,ν)-almost coKähler manifold with $k<0$ becomes a $(-1,\frac{\mu}{\sqrt{-k}})$-almost coKähler manifold, under some D-homothetic deformation.

Now, we state a lemma [for proof see Lemma 4.1 of [16]] which will be used in the later theorems.

Lemma 4.1. *Let $(M^{2n+1},g,\phi,\xi,\eta)$ be a (k,μ)-almost coKähler manifold of dimension greater than 3 with $k<0$. Then the Ricci operator is given by*

$$Q = \mu h + 2nk\eta\otimes\xi, \tag{4.2}$$

where k is a non-zero constant and μ is a smooth function satisfying $d\mu\wedge\eta = 0$.

Now let us consider the metric g of the (k,μ)-almost coKähler manifold admits a conformal Ricci soliton. Then from the soliton equation (1.3) and using the definition of the Lie derivative we can write

$$g(\nabla_X V, Y) + g(X, \nabla_Y V) + 2S(X,Y) = [2\lambda - (p+\frac{2}{2n+1})]g(X,Y). \tag{4.3}$$

Then, substituting $V=\xi$ in the above equation (4.3) and using the result $\nabla\xi = h'$ from (2.4) we get

$$g(h'X,Y) + g(X,h'Y) + 2S(X,Y) = [2\lambda - (p+\frac{2}{2n+1})]g(X,Y).$$

Again as h' is symmetric the above equation implies

$$g(h'X,Y) + g(QX,Y) = [\lambda - (\frac{p}{2}+\frac{1}{2n+1})]g(X,Y). \tag{4.4}$$

Now, in view of the Lemma 4.1 putting value of the Ricci operator Q, from equation (4.2), in the above equation (4.4) we get

$$g(h'X,Y) + g(\mu hX,Y) + 2nk\eta(X)\eta(Y) = [\lambda - (\frac{p}{2}+\frac{1}{2n+1})]g(X,Y). \tag{4.5}$$

Thus putting $Y = \xi$ in the above (4.5) and using $h\phi + \phi h = 0$ from equation (2.3) we finally get

$$2nk = [\lambda - (\frac{p}{2} + \frac{1}{2n+1})]. \tag{4.6}$$

Now, as it is mentioned in the Lemma 4.1 that $k < 0$, so from the above relation (4.6) we can conclude that $[\lambda - (\frac{p}{2} + \frac{1}{2n+1})] < 0$ that is; $\lambda < (\frac{p}{2} + \frac{1}{2n+1})$. Thus if $(\frac{p}{2} + \frac{1}{2n+1}) \leq 0$, i.e; if, $p \leq \frac{-2}{2n+1}$ then $\lambda < 0$ and therefore the soliton is expanding. So, in view of the above we have the following theorem.

Theorem 4.1. *Let $(M^{2n+1}, g, \phi, \xi, \eta)$ be a (k, μ)-almost coKähler manifold of dimension greater than 3 with $k < 0$ and the metric g admits a conformal Ricci soliton. Then the soliton is expanding if the conformal pressure p satisfy the inequality $p \leq \frac{-2}{2n+1}$.*

5. Conformal gradient Ricci soliton on (k, μ)-Almost CoKähler manifold

This section is devoted to the study of conformal gradient Ricci soliton on (k, μ)-almost coKähler manifold. So, let us first give the statement of our main theorem of this section.

Theorem 5.1. *Let $(M^{2n+1}, g, \phi, \xi, \eta)$ be a (k, μ)-almost coKähler manifold of dimension greater than 3 with $k < 0$. Then there exist no conformal gradient Ricci soliton on the manifold, with the potential vector field V pointwise collinear with the Reeb vector field ξ.*

Proof. We prove this theorem by the method of contradiction. So, let us assume that the manifold admits a conformal gradient Ricci soliton. Then from equation (1.4) we have

$$S + \nabla\nabla f = [\lambda - (\frac{p}{2} + \frac{1}{2n+1})]g.$$

Now as the soliton is gradient, i.e; $V = Df$ for some smooth function f and here D is the gradient operator. Thus for any vector field $X \in TM$, the above equation is equivalent to

$$\nabla_X Df + QX = [\lambda - (\frac{p}{2} + \frac{1}{2n+1})]X. \tag{5.1}$$

Replacing X by Y in the above (5.1) we get

$$\nabla_Y Df + QY = [\lambda - (\frac{p}{2} + \frac{1}{2n+1})]Y. \tag{5.2}$$

Similarly replacing X by $[X,Y]$ in the equation (5.1) we get

$$\nabla_{[X,Y]} Df + Q[X,Y] = [\lambda - (\frac{p}{2} + \frac{1}{2n+1})][X,Y]. \tag{5.3}$$

Now from the well-known formula for Riemannian curvature, using (2.7) we can write

$$R(X,Y)Df = \nabla_X \nabla_Y Df - \nabla_Y \nabla_X Df - \nabla_{[X,Y]} Df. \tag{5.4}$$

Using equations (5.1), (5.2) and (5.3) in the equation (5.4) and after some simple calculations we get

$$R(X,Y)Df = (\nabla_Y Q)X - (\nabla_X Q)Y. \tag{5.5}$$

Again for any vector fields X, Y in TM, using equation (4.2) of Lemma 4.1 we obtain

$$\begin{aligned}(\nabla_Y Q)X - (\nabla_X Q)Y = \mu((\nabla_Y h)X - (\nabla_X h)Y) \\ + 2nk(\eta(X)h'Y - \eta(Y)h'X) + Y(\mu)hX - X(\mu)hY.\end{aligned} \tag{5.6}$$

Now we shall use an equation from Proposition-9 of the paper [14]. The result is, for any vector fields X, Y in TM,

$$\begin{aligned}(\nabla_X h)Y - (\nabla_Y h)X = k(\eta(Y)\phi X - \eta(X)\phi Y \\ + 2g(\phi X, Y)\xi) + \mu(\eta(X)h'Y - \eta(Y)h'X).\end{aligned} \tag{5.7}$$

Then using (5.6) in (5.5) and then using (5.7), a simple computation gives that

$$\begin{aligned}R(X,Y)Df = k\mu(\eta(X)\phi Y - \eta(Y)\phi X + 2g(X,\phi Y)\xi) + Y(\mu)hX \\ - X(\mu)hY - \mu^2(\eta(X)h'Y - \eta(Y)h'X) + 2nk(\eta(X)h'Y - \eta(Y)h'X),\end{aligned} \tag{5.8}$$

for any vector fields X, Y in TM. Putting $X = \xi$ in the above equation (5.8) we get

$$R(\xi, Y)Df = k\mu(\phi Y) - \xi(\mu)hY - \mu^2(h'Y) + 2nk(h'Y).$$

Replacing Y by X in the above equation and then taking inner product with respect to arbitrary vector Y gives us

$$\begin{aligned}g(R(\xi, X)Df, Y) =& k\mu g(\phi X, Y) - \xi(\mu)g(hX, Y) - \mu^2 g(h'X, Y) \\ &+ 2nkg(h'X, Y).\end{aligned} \tag{5.9}$$

Again for a (k, μ)-almost coKähler manifold, using equation (4.1) we can write

$$R(\xi, X)Y = k[g(X, Y)\xi - \eta(Y)X] + \mu[g(hX, Y)\xi - \eta(Y)hX].$$

Taking inner-product of the equation with respect to the vector field Df and using the fact that $g(X, Df) = (Xf)$ we get

$$\begin{aligned} g(R(\xi, X)Y, Df) = k[g(X, Y)(\xi f) - \eta(Y)(Xf)] + \mu[g(hX, Y)(\xi f) \\ - \eta(Y)((hX)f)]. \end{aligned} \tag{5.10}$$

Now combining (5.9) and (5.10) and applying the property $g(R(X, Y)Z, U) = -g(R(X, Y)U, Z)$, for any vector fields X, Y, Z, U in TM, yields

$$\begin{aligned} &k\mu g(\phi X, Y) - \xi(\mu)g(hX, Y) - \mu^2 g(h'X, Y) + 2nkg(h'X, Y) = \\ &- kg(X, Y)(\xi f) + k\eta(Y)(Xf) - \mu g(hX, Y)(\xi f) + \mu\eta(Y)((hX)f). \end{aligned} \tag{5.11}$$

Antisymmetrizing the above equation we get

$$\begin{aligned} k\mu[g(\phi X, Y) - g(X, \phi Y)] = k[\eta(Y)(Xf) - \eta(X)(Yf)] \\ + \mu[\eta(Y)((hX)f)] - \eta(X)((hY)f). \end{aligned} \tag{5.12}$$

Now as per our assumption $V = b\xi$, it is easy to see that $h'(Df) = 0$. This again implies, $(h'X)f = g(h'X, Df) = g(X, h'(Df)) = 0$. Similarly $(h'Y)f = 0$. Thus

$$(h(\phi X))f = 0, \ (h(\phi Y))f = 0. \tag{5.13}$$

Using antisymmetry of ϕ and then putting $X = \phi X$ in equation (5.11) and using (5.12) we get

$$-2\mu g(X, Y) + \mu\eta(X)\eta(Y) = \eta(Y)((\phi X)f). \tag{5.14}$$

Putting $Y = \xi$ in the above (5.13) yields

$$-\mu g(X, \xi) = g(\phi X, Df). \tag{5.15}$$

Then again using $X = \phi X$ in the above equation (5.14) we get $g(X, Df) = g(X, \xi(\xi f))$. This gives us

$$Df = (\xi f)\xi. \tag{5.16}$$

Covariant differentiation of the equation (5.15) along the direction of X we get

$$\nabla_X Df = (X(\xi f))\xi + (\xi f)(h'X). \tag{5.17}$$

Again from the equation (5.1) we have

$$\nabla_X Df = [\lambda - (\frac{p}{2} + \frac{1}{2n+1})]X - QX. \tag{5.18}$$

Thus combining equations (5.16) and (5.17) we get

$$QX = [\lambda - (\frac{p}{2} + \frac{1}{2n+1})]X - (X(\xi f))\xi - (\xi f)(h'X). \tag{5.19}$$

Again, the value of Q from Lemma 4.1 gives us

$$QX = \mu hX + 2nk\eta(X)\xi.$$

Now, compairing right hand sides of (5.18) and (5.19) we get $(X(\xi f)) = -2nk\eta(X)$ i.e; $D(\xi f) = -2nk\xi$ or equivalently, $d^2 f = -2nk$, where d is the exterior derivative of f. Again from the well-known Poincare lemma of exterior differentiation we know that, $d^2 = 0$ and this implies, $-2nk = 0$, which is a contradiction to our assumtion that $k < 0$. This completes the proof. □

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