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PARTIAL ORDER IN THE NORMAL CATEGORY ARISING FROM NORMAL BANDS

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ABSTRACT. The notion of normal category was introduced by KSS Nambooripad in connection with the study of the structure of regular semigroups using cross connections [4]. It is an abstraction of the category of principal left ideals of a regular semigroup. A normal band is a semigroup B satisfying $a^2 = a$ and $abca = acba$ for all $a, b, c \in B$. Since the normal bands are regular semigroups, the category $\mathcal{L}(B)$ of principal left ideals of a normal band B is a normal category. In this article we reveal some properties of the partial order on each of the hom sets of $\mathcal{L}(B)$ induced by the elements of B.

1. PRELIMINARIES

In this section we discuss some basic concepts of normal categories, normal bands and the normal category derived from normal bands. We follow [3,4] for other notations and definitions which are not mentioned.

1.1. **Normal Categories.** We begin with the category theory, some basic definitions and notations. In a category C, the class of objects is denoted by vC and C itself is used to denote the morphism class. The collection of all morphisms from a to b in C is denoted by $C(a, b)$ and 1_a denote the identity morphism in $C(a, a)$. A morphism $f \in C$ is said to be a monomorphism(epimorphism) if $fg = fh(gf = hf)$ implies $g = h$ for any two morphisms $g, h \in C$. A morphism

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$f \in C(a, b)$ is said to be right(left) invertible if there exists $g \in C(b, a)$ such that $fg = 1_a(gf = 1_b)$. An isomorphism means a morphism which is both left and right invertible.

Let S be a regular semigroup. A right translation of S is a function ϕ from S to S such that $(xy)\phi = x((y)\phi)$ for all $x, y \in S$. Similarly a function $\psi : S \to S$ is called a left translation if $\psi(xy) = (\psi(x))y$ for all $x, y \in S$. For a fixed $u \in S$, the map $\rho_u : x \mapsto xu$ for $x \in S$ is a right translation. This map can be viewed as a function from S to the principal left ideal Su of S. Since S is regular, we can find an idempotent $f \in S$ such that $Su = Sf$. In particular if we choose $u \in eSf$ the restriction of ρ_u to Se, denoted by $\rho(e, u, f)$, can be viewed as a function from Se to Sf. If $\rho(e, u, f) : Se \to Sf$ and $\rho(g, v, h) : Sg \to Sh$ are such that $Sf = Sg$, the composition of these maps will be $\rho(e, uv, h)$. Now we arrive at a category structure, $\mathcal{L}(S)$, with objects are the principal left ideals of S and morphisms are of the form $\rho(e, u, f) : Se \to Sf$ where $u \in eSf$. The usual set inclusion of the principal left ideals will induce a partial order in the objects class of $\mathcal{L}(S)$. Also $Se \subseteq Sf$ implies $ef = e$ and hence the morphism $\rho(e, e, f)$ is an inclusion map from Se to Sf and is denoted by $j(Se, Sf)$. The category $\mathcal{L}(S)$ become a normal category whose definition is given below.

A preorder is a category P such that each of the hom sets contain at most one morphism. We write $a \leq b$ whenever $P(a, b) \neq \emptyset$.

Definition 1.1. *[8] A category with normal factorization is a small category C with the following properties.*

1. *(1) The vertex set vC of C is a partially ordered set such that whenever $a \leq b$ in vC, there is a monomorphism $j(a, b) : a \to b$ in C. This morphism is called the inclusion from a to b.*
2. *(2) $j : (vC, \leq) \to C$ is a functor from the preorder $(vC, \leq)$ to C which maps $a \leq b$ to $j(a, b)$ for $a, b \in vC$.*
3. *(3) For $a, b \leq c$ in vC, if*
$$j(a, c) = fj(b, c)$$
for some $f : a \to b$ then $a \leq b$ and $f = j(a, b)$.
4. *(4) Every morphism $j(a, b) : a \to b$ has a right inverse $q : b \to a$ such that*
$$j(a, b)q = 1_a$$
Such a morphism q is called a retraction in C.

(5) *Every morphism f in C has a factorization*

$$f = quj$$

where q is a retraction, u is an isomorphism and j is an inclusion. Such a factorization is called normal factorization in C.

Proposition 1.1. *[4] Let C is a category with normal factorization. If $f = quj = q'u'j'$ are two normal factorizations of $f \in C$ then $j = j'$ and $qu = q'u'$.*
In this case $f^0 = qu$ is called the epimorphic component of f. The codomain of f° is called the image of f and is denoted by im f

Corresponding to a right translation ρ_u from S to Su, we get morphisms from each of the principal left ideals of S to Su by restricting the ρ_u. This phenomena is similar to the concept of cones in the category theory [3]. Now we define normal cones and normal categories.

Definition 1.2. *[4] Let C be a category with normal factorization. A cone γ with vertex $a \in vC$ is a function from vC to C satisfying the following*

(1) *$\gamma(c) \in C(c, a)$ for all $c \in vC$*
(2) *If $c_1 \subseteq c_2$ then $\gamma(c_1) = j(c_1, c_2)\gamma(c_2)$*
(3) *There is an object $b \in vC$ such that $\gamma(b)$ is an isomorphism.*

Definition 1.3. *[6] A normal category is a category with normal factorization such that for each $a \in vC$ there is a normal cone γ with $\gamma(a) = 1_a$.*

The cones induced by a right translation ρ_u is called a principal cone and is denoted by ρ^u. The category $\mathcal{L}(S)$ may contain normal cones other than principal cones.

1.2. **Normal Bands.** Several studies have been carried out for recent years about the normal categories arising from the semigroups satisfying some special conditions [1, 5, 8]. We focus on the normal category of principal left ideals of a normal band. A semigroup B is said to be a normal band if $a^2 = a$ and $abca = acba$ for all $a, b, c \in B$. The following theorem, by Yamada and Kimura [2], states that a semigroup is a normal band if and only if it is a strong semilattice of rectangular bands.

Theorem 1.1. *[2, 9] The following conditions are equivalent for a semigroup B*

(1) *B is a normal band*

(2) *There exists a semilattice Γ, a disjoint family of rectangular bands $\{E_\alpha : \alpha \in \Gamma\}$ such that $B = \bigcup\{E_\alpha : \alpha \in \Gamma\}$ and a family of homomorphisms $\{\phi_{\alpha,\beta} : E_\alpha \to E_\beta,$ where $\alpha, \beta \in \Gamma$ and $\beta \leq \alpha\}$ satisfying the following*
   (a) *$\phi_{\alpha,\alpha}$ is the identity function,*
   (b) *$\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$ for $\gamma \leq \beta \leq \alpha$ and*
   (c) *For $a, b \in B$ with $a \in E_\alpha$ and $b \in E_\beta$*

$$ab = (a\phi_{\alpha,\alpha\beta})(b\phi_{\beta,\alpha\beta})$$

(3) *B is a band and*

$$abcd = acbd \text{ for all } a, b, c, d \in B.$$

1.3. **The category $\mathcal{L}(B)$.** Let B is a normal band. In addition to that the category $\mathcal{L}(B)$ is normal, it possess some special qualities which are not common in general. Several properties of the category $\mathcal{L}(B)$ are described in [5]. Here the normal factorization is unique and there is only one isomorphism between any two isomorphic objects. Also the principal cones respect the isomorphic objects in the sense that

$$\gamma(Ba) = \rho(a, ab, b)\gamma(Bb)$$

for any principal cone γ and isomorphic objects Ba and Bb. This leads to the definition of strong cones by AR Rajan [7].

Definition 1.4. *[7] A normal cone γ in a normal category C is said to be a strong cone if the following holds*

If c and d are two isomorphic objects, then there is a unique isomorphism α from c to d such that $\gamma(c) = \alpha\gamma(d)$

Thus every principal cone is a strong cone. That is for every objects Ba of $\mathcal{L}(B)$ there is a strong cone with vertex Ba. The category of principal left ideals of any normal band is characterized as follows.

Theorem 1.2. *[5] A normal category C satisfies following conditions if and only if it is isomorphic to $\mathcal{L}(B)$ for some normal band B*

SC1 *The isomorphism between any two objects is unique.*
SC2 *The right inverse of an inclusion is unique.*
SC3 *For every object c in C, there is a strong cone with vertex c.*

2. Partial order in $\mathcal{L}(B)$

Let B is a normal band. Let us denote by $[Ba, Bb]$, the collection of all morphisms from Ba to Bb in $\mathcal{L}(B)$. Now we show that there is a partial order on the morphism sets $[Ba, Bb]$ and that $[Ba, Bb]$ is an order ideal with respect to this partial order. In particular there is a maximum element in $[Ba, Bb]$. We note that any normal band B has a natural partial order given by $a \leq b$ if and only if $ab = ba = a$.

Theorem 2.1. *[5] Let B be a normal band and $\rho(a, u, b)$, $\rho(a, v, b) \in [Ba, Bb]$ be morphisms in $\mathcal{L}(B)$. Define*

$$\rho(a, u, b) \leq \rho(a, v, b) \text{ if } u \leq v$$

Then $\leq$ is a partial order on $[Ba, Bb]$.

Proposition 2.1. *Every morphism set $[Ba, Bb]$ contains a maximum with respect to the above partial order.*

Proof. Taking $\delta = \alpha\beta$ where $a \in E_\alpha$ and $b \in E_\beta$ we see that there is a unique $u \in E_{\alpha\beta}$ such that $\rho(a, u, b) \in [Ba, Bb]$.

Since $\omega^l(b) \cap \omega^r(a) = \omega(h)$ for some $h \in E_{\alpha\beta}$ and $\rho(a, h, b) \in [Ba, Bb]$ we conclude that $u = h$. It follows that $\rho(a, u, b)$ is the maximum in $[Ba, Bb]$ □

The following compatibility properties hold for this partial order.

Proposition 2.2. *Let $\rho(a, u, b)$ and $\rho(a, v, b)$ be elements in the morphism set $[Ba, Bb]$ in $\mathcal{L}(B)$ such that $\rho(a, u, b) \leq \rho(a, v, b)$. Then the following compatibility relations hold.*

1. *If $\rho(b, w, c) \in [Bb, Bc]$ then $\rho(a, u, b)\rho(b, w, c) \leq \rho(a, v, b)\rho(b, w, c)$.*
2. *If $\rho(d, x, a) \in [Bd, Ba]$ then $\rho(d, x, a)\rho(a, u, b) \leq \rho(d, x, a)\rho(a, v, b)$.*

Proof. Follows from the compatibility of natural partial order on normal bands. □

Now we describe some properties of the maximum elements in the morphism sets.

Proposition 2.3. *Let $m = m(a, b)$ be the maximum element in the morphism set $[Ba, Bb]$ in $\mathcal{L}(B)$ and $\alpha = \alpha(b, c)$ be an isomorphism from Bb to Bc in $\mathcal{L}(B)$. Then $m\alpha$ is the maximum element in the morphism set $[Ba, Bc]$.*

Proof. Let $f \in [Ba, Bc]$. Then $f\alpha^{-1} \in [Ba, Bb]$. Since m is maximum element in $[Ba, Bb]$ we have $f\alpha^{-1} \leq m$. Now by compatibility given by Proposition 2.2

$$f = f\alpha^{-1}\alpha \leq m\alpha$$

. So $m\alpha$ is the maximum in $[Ba, Bc]$. □

Proposition 2.4. *Let $m = m(a, b)$ be the maximum element in the morphism set $[Ba, Bb]$ in $\mathcal{L}(B)$ and $j = j(a_0, a)$ be an inclusion in $\mathcal{L}(B)$. Then jm is the maximum element in the morphism set $[Ba_0, Bb]$.*

Proof. Let $f \in [Ba_0, Bb]$. Then $f = \rho(a_0, u, b)$ for some $u \leq k$ such that $\omega(k) = a_0Bb$. Now

$$jm = \rho(a_0, a_0, a)\rho(a, h, b) = \rho(a_0, a_0h, b) = \rho(a_0, k, b)$$

where $\omega(h) = aBb$. Therefore $f \leq jm$ so that jm is maximum element in $[Ba_0, Bb]$. □

Proposition 2.5. *Let $m = m(a, b)$ be the maximum element in the morphism set $[Ba, Bb]$ in $\mathcal{L}(B)$ and $q : b \to b_0$ be a retraction in $\mathcal{L}(B)$. Then mq is the maximum element in the morphism set $[Ba, Bb_0]$.*

Proof. Let $f \in [Ba, Bb_0]$. Then $fj(b_0, b)$ is in $[Ba, Bb]$. Since m is the maximum element in $[Ba, Bb]$ we have $fj(b_0, b) \leq m$. Now by compatibility given by Proposition 2.2

$$f = fj(b_0, b)q \leq mq.$$

So mq is the maximum in $[Ba, Bb_0]$. □

As a consequence of this propositions we can see that mf° is always a maximum if m is a maximum.

Proposition 2.6. *Let $m = m(a, b)$ be the maximum element in the morphism set $[Ba, Bb]$ in $\mathcal{L}(B)$ and $f : b \to c$ be any morphism in $\mathcal{L}(B)$. Then mf° is the maximum element in the morphism set $[Ba, Bc_0]$ where $c_0 \leq c$ is the codomain of f°.*

Proof. Suppose $f = \varrho uj$, where ϱ is a retraction, u is an isomorphism and j is an inclusion. Then by the above propositions 2.3 and 2.5 we can say that $mf^\circ = m\varrho u$ is the maximum in $[Ba, Bc_0]$. □

The following proposition exactly locates the maximum element in each hom set in $\mathcal{L}(B)$.

Proposition 2.7. *The maximum element in* $[Ba, Bb]$ *is* $\rho(a, ab, b)$.

Proof. Suppose $\rho(a, u, b) \in [Ba, Bb]$. Then $u \in aBb$ and hence $u = aub$. Now

$$(ab)(u) = (ab)(aub) = aabub = aubb = aub = u.$$

Similarly we get

$$(u)(ab) = aubab = auab = aub = u.$$

That is $u \leq ab$ for all $u \in aBb$ and this proves that

$$\rho(a, u, b) \leq \rho(a, ab, b).$$

for all $\rho(a, u, b) \in [Ba, Bb]$. □

Note that $\rho(a, ab, b)$ is the unique isomorphism in $[Ba, Bb]$ whenever Ba and Bb are isomorphic. Also the following theorem shows that these morphisms are the building blocks of principal cones in $\mathcal{L}(B)$.

Theorem 2.2. *The component of the principal cone* ρ^a *at* $Bb \in \mathcal{L}(B)$ *is the maximum element in* $[Bb, Ba]$.

Proof. We have the principal cone ρ^a with vertex Ba is defined as

$$\rho^a(Bb) = \rho(b, ba, a) \text{ for all } Bb \in v\mathcal{L}(B).$$

But $\rho(b, ba, a)$ is the maximum in $[Bb, Ba]$ by proposition 2.7. Hence the theorem. □

Now we give another characterization of the partial order in the hom sets of $\mathcal{L}(B)$ using the image of the morphisms.

Theorem 2.3. *For any two morphisms* $\rho(a, u, b), \rho(a, v, h) \in [Ba, Bb]$*,*

$$\rho(a, u, b) \leq \rho(a, v, b) \textit{ if and only if im } \rho(a, u, b) \subseteq \textit{im}, \rho(a, v, b)$$

Proof. To prove the theorem it is enough to show the following

$$\text{im}\, \rho(a, u, b) \subseteq \text{im}, \rho(a, v, b) \text{ if and only if } u \leq v.$$

The unique normal factorization [5] of $\rho(a, u, b)$ is given by

$$\rho(a, u, b) = \rho(a, g, g)\rho(g, u, h)\rho(h, h, b),$$

where $u\mathscr{R}g \leq a$ and $u\mathscr{L}h \leq b$. Therefore $\operatorname{im}\rho(a,u,b) = Bh = Bu$. Similarly we have $\operatorname{im}\rho(a,v,b) = Bv$.

Assume that $u \leq v$. Then $uv = vu = u$. This implies $Bu = Buv \subseteq Bv$. Thus $\operatorname{im}\rho(a,u,b) \subseteq \operatorname{im}\rho(a,v,b)$.

Conversely suppose that $\operatorname{im}\rho(a,u,b) \subseteq \operatorname{im}, \rho(a,v,b)$. That is $Bu \subseteq Bv$. Then $uv = v$.

Note that $u, v \in aBb = aB \cap Bb$. Therefore there exists $p, q, r, s \in B$ such that $u = ar = pb$ and $v = as = qb$. Also $u = u^2 = arpb$ and $v = v^2 = asqb$. Then $uv = u$ implies $arqb = ar$ and $pbqb = pb$. Now we get

$$\begin{aligned} vu &= (asqb)(arpb) \\ &= (aras)(pbqb) && \text{By repeted application of normality} \\ &= (ar)(pb) \\ &= uu \\ &= u \end{aligned}$$

Therefore $\operatorname{im}\rho(a,u,b) \subseteq \operatorname{im}, \rho(a,v,b)$ implies $uv = vu = u$ and hence $u \leq v$. Hence the theorem. □

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