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MODELING OF DEVELOPABLE SURFACES USING HERMITE SPLINE INTERPOLATION CURVES

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ABSTRACT. Developable surfaces design is used mainly in manufactures based on cloth, leather, plywood, and metal. This paper aims to model the developable surfaces by applying the quadratic and cubic Hermite spline interpolation curves. For the purpose, we review the mathematical formulation of quadratic and cubic Hermite curves, quadratic and cubic Hermite spline interpolation curves, and the definition of a developable surface. We then present a method to construct the developable Hermite surfaces with boundary curves defined by the quadratic, cubic Hermite spline interpolation curve and the Hermite curve segments' connection G^1. As a result, the method is straightforward and handy to design these developable surfaces.

1. INTRODUCTION

Consider a parametric quadratic polynomial Hermite curve $\mathbf{R}_2(u) = \mathbf{a}_2.u^2 + \mathbf{b}_2.u + \mathbf{c}_2$ with u in interval $0 \leq u \leq 1$. If we control the curve with the conditions $\mathbf{R}_2(0) = \mathbf{R}_o$, $\mathbf{R}_2(1) = \mathbf{R}_1$ and the tangent vector $\mathbf{R}_2^u(0) = \mathbf{R}_o^u$, then we discover the equations $\mathbf{c}_2 = \mathbf{R}_o$; $\mathbf{a}_2 + \mathbf{b}_2 + \mathbf{c}_2 = \mathbf{R}_1$; and $\mathbf{b}_2 = \mathbf{P}_o^u$. The solution of the unknown coefficients $\mathbf{a}_2$, $\mathbf{b}_2$, and $\mathbf{c}_2$ in the curve equation are $\mathbf{a}_2 = \mathbf{R}_1 - \mathbf{R}_o - \mathbf{R}_o^u$; $\mathbf{b}_2 = \mathbf{R}_o^u$, and $\mathbf{c}_2 = \mathbf{R}_o$. Therefore, it can represent $\mathbf{R}_2(u)$ in the algebraic and geometric formulations, respectively [1]

$$\mathbf{R}_2(u) = (\mathbf{R}_1 - \mathbf{R}_o - \mathbf{R}_o^u).u^2 + \mathbf{R}_o^u.u + \mathbf{R}_o \tag{1.1}$$

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$$\mathbf{R}_2(u) = H_1(u)\mathbf{R}_o + H_2(u)\mathbf{R}_1 + H_3(u)\mathbf{R}_o^u \tag{1.2}$$

with $H_1(u) = (-u^2+1)$; $H_2(u) = u^2$; and $H_3(u) = (-u^2+u)$.

On the other hand, let a cubic polynomial Hermite curve $\mathbf{R}_3(u) = \mathbf{a}_3.u^3 + \mathbf{b}_3.u^2 + \mathbf{c}_3.u + \mathbf{d}_3$ with u in interval $0 \leq u \leq 1$. When we restrict the curve $\mathbf{R}_3(0) = \mathbf{R}_o$, $\mathbf{R}_3(1) = \mathbf{R}_1$, $\mathbf{R}_3^u(0) = \mathbf{R}_o^u$, and $\mathbf{R}_3^u(1) = \mathbf{R}_1^u$, then, we discover four equations $\mathbf{d}_3 = \mathbf{R}_o$; $\mathbf{a}_3 + \mathbf{b}_3 + \mathbf{c}_3 + \mathbf{d}_3 = \mathbf{R}_1$; $\mathbf{c}_3 = \mathbf{R}_o^u$; $3.\mathbf{a} + 2.\mathbf{b} + \mathbf{c} = \mathbf{R}_1^u$. Using the same computation of that quadratic Hermite curve, it can calculate four unknown coefficients $\mathbf{a}_3$, $\mathbf{b}_3$, $\mathbf{c}_3$, and $\mathbf{d}_3$ in the form $\mathbf{a}_3 = 2\mathbf{R}_o - 2\mathbf{R}_1 + \mathbf{R}_o^u + \mathbf{R}_1^u$, $\mathbf{b}_3 = -3\mathbf{R}_o + 3\mathbf{R}_1 - 2\mathbf{R}_o^u - \mathbf{R}_1^u$, $\mathbf{c}_3 = \mathbf{R}_o^u$, and $\mathbf{d}_3 = \mathbf{R}_o$. Thus, the cubic Hermite curve $\mathbf{R}_3(u)$ can be formulated in the algebraic and geometric representations, correspondingly [1,2]

$$\mathbf{R}_3(u) = [2(\mathbf{R}_o - \mathbf{R}_1) + \mathbf{R}_o^u + \mathbf{R}_1^u].u^3 + [3(\mathbf{R}_1 - \mathbf{R}_o) - 2\mathbf{R}_o^u - \mathbf{R}_1^u].u^2 + \mathbf{R}_o^u.u + \mathbf{R}_o \tag{1.3}$$

$$\mathbf{R}_3(u) = H_1(u)\mathbf{R}_o + H_2(u)\mathbf{R}_1 + H_3(u)\mathbf{R}_o^u + H_4(u)\mathbf{R}_1^u \tag{1.4}$$

with $H_1 = 2u^3 - 3u^2 + 1$; $H_2 = -2u^3 + 3u^2$; $H_3 = u^3 - 2u^2 + u$; and $H_4 = u^3 - u^2$.

Base on Equation (1.1) and (1.3), we review the theory of quadratic and cubic spline interpolation curves in this way [1,3]. Given $(n+1)$ consecutive control points $\mathbf{R}_o, \mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_n$. For $i = 1, 2, 3, .., n$, and the value u in interval $0 \leq u \leq 1$, we will define a quadratic Hermite spline curves $\mathbf{R}_i(u)$ bounded by the endpoints $[\mathbf{R}_{i-1}, \mathbf{R}_i]$ with the criteria as follows. The connection between two curves adjacent at the point $\mathbf{R}_i$ must be smoothly C^1 for $i = 1, 2, 3, .., (n-1)$. This means that the curves have to meet $\mathbf{R}_i(1) = \mathbf{R}_{i+1}(0)$ and $\mathbf{R}_i^u(1) = \mathbf{R}_{i+1}^u(0)$ for $i = 1, 2, 3, .., n-1$. If each pair $[\mathbf{R}_{i-1}, \mathbf{R}_i]$ and $[\mathbf{R}_i, \mathbf{R}_{i+1}]$ respectively define the curve segment

$$\mathbf{R}_i(u) = (\mathbf{R}_i - \mathbf{R}_{i-1} - \mathbf{R}_{i-1}^u).u^2 + \mathbf{R}_{i-1}^u.u + \mathbf{R}_{i-1};$$

$$\mathbf{R}_{i+1}(u) = (\mathbf{R}_{i+1} - \mathbf{R}_i - \mathbf{R}_i^u).u^2 + \mathbf{R}_i^u.u + \mathbf{R}_i$$

then, at the control points $\mathbf{R}_i$, it must be $\mathbf{R}_i^u(1) = \mathbf{R}_{i+1}(0)$, and we obtain the equations

$$2(\mathbf{R}_i - \mathbf{R}_{i-1} - \mathbf{R}_{i-1}^u).1 + \mathbf{R}_{i-1}^u = 2(\mathbf{R}_{i+1} - \mathbf{R}_i - \mathbf{R}_i^u).0 + \mathbf{R}_i^u$$

or $\mathbf{R}_{i-1}^u + \mathbf{R}_i^u = 2(\mathbf{R}_i - \mathbf{R}_{i-1})$. When these equations are applied for $i = 1, 2, 3, .., n-1$, it can be expressed in the matrix form equations

$$\begin{pmatrix} 1 & 1 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & ... & 0 & 0 & 0 & 0 \\ . & . & . & . & ... & . & . & . & . \\ 0 & 0 & 0 & 0 & ... & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{R}_o^u \\ \mathbf{R}_1^u \\ ... \\ ... \\ \mathbf{R}_n^u \end{pmatrix} = \begin{pmatrix} 2(\mathbf{R}_1 - \mathbf{R}_o) \\ 2(\mathbf{R}_2 - \mathbf{R}_1) \\ ... \\ ... \\ 2(\mathbf{R}_n - \mathbf{R}_{n-1}) \end{pmatrix}. \tag{1.5}$$

The equation system (1.5) has $(n-1)$ equations of $(n+1)$ unknown tangent vectors. If we determine one value, for example, $\mathbf{R}_o^u$, and compute the values $\mathbf{R}_i^u$ for $i = 1, 2, 3, .., n$, then, it can found the solution of the tangent vectors $\mathbf{R}_i^u$ for $i = 1, 2, 3, \ldots n$. Thus, the quadratic Hermite spline interpolation curves and their first derivatives are as follows

$$\mathbf{R}_i(u) = (\mathbf{R}_i - \mathbf{R}_{i-1} - \mathbf{R}_{i-1}^u).u^2 + \mathbf{R}_{i-1}^u.u + \mathbf{R}_{i-1} \tag{1.6}$$

$$\mathbf{R}_i^u(u) = 2(\mathbf{R}_i - \mathbf{R}_{i-1} - \mathbf{R}_{i-1}^u).u + \mathbf{R}_{i-1}^u \tag{1.7}$$

for $i = 1, 2, 3, .., n$. Then, we apply this computation method to defining the cubic spline Hermite curve.

Let $n+1$ control points $\mathbf{R}_o, \mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_n$. We will formulate, in interval $0 \le u \le 1$, a cubic spline Hermite curves $\mathbf{R}_i(u)$ for $i = 1, 2, 3, .., n$ in which the curve segments $\mathbf{R}_i(u)$ are respectively delimited by two endpoints $[\mathbf{R}_{i-1}, \mathbf{R}_i]$. The curves must be smoothly connected C^2 at the control points $\mathbf{R}_i$ such that $\mathbf{R}_i(1) = \mathbf{R}_{i+1}(0)$, $\mathbf{R}_i^u(1) = \mathbf{R}_{i+1}^u(0)$, and $\mathbf{R}_i^{uu}(1) = \mathbf{R}_{i+1}^{uu}(0)$ for $i = 1, 2, 3, .., n-1$. Using the same calculation method of the quadratic Hermite spline curve, it can calculate

$$\mathbf{R}_i^{uu}(1) = 6.[2(\mathbf{R}_{i-1} - \mathbf{R}_i) + \mathbf{R}_{i-1}^u + \mathbf{R}_i^u] + 2.[3(\mathbf{R}_i - \mathbf{R}_{i-1}) - 2\mathbf{R}_{i-1}^u - \mathbf{R}_i^u]$$

$$\mathbf{R}_{i+1}^{uu}(0) = 2.[3(\mathbf{R}_{i+1} - \mathbf{R}_i) - 2\mathbf{R}_i^u - \mathbf{R}_{i+1}^u$$

and at the control points $\mathbf{R}_i$, it have to be $\mathbf{R}_i^{uu}(1) = \mathbf{R}_{i+1}^{uu}(0)$ or $\mathbf{R}_{i-1}^u + 4\mathbf{R}_i^u + \mathbf{R}_{i+1}^u = 3(\mathbf{R}_{i+1} - \mathbf{R}_i)$. If these equations are applied for $i = 1, 2, 3, .., n-1$, we will find the equation system in the matrix form

$$\begin{pmatrix} 1 & 4 & 1 & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & ... & 0 & 0 & 0 & 0 \\ . & . & . & . & ... & . & . & . & . \\ 0 & 0 & 0 & 0 & ... & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & 1 & 4 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{R}_o^u \\ \mathbf{R}_1^u \\ ... \\ ... \\ \mathbf{R}_n^u \end{pmatrix} = 3 \begin{pmatrix} \mathbf{R}_2 - \mathbf{R}_o \\ \mathbf{R}_3 - \mathbf{R}_1 \\ ... \\ ... \\ \mathbf{R}_n - \mathbf{R}_{n-2} \end{pmatrix}. \tag{1.8}$$

We get $(n-1)$ equations of $(n+1)$ unknown tangent vectors. If we determine two values of unknown tangent vectors, for example, $\mathbf{R}_o^u$ and $\mathbf{R}_n^u$, and calculate the tangent vector values $\mathbf{R}_i^u$ for $i = 1, 2, 3, .., n-1$ in the equation system (1.8), then it can discover the cubic polynomial Hermite spline curve segments and its first derivative in the forms

$$\begin{aligned} \mathbf{R}_i(u) = & [2(\mathbf{R}_{i-1} - \mathbf{R}_i) + \mathbf{R}_{i-1}^u + \mathbf{R}_i^u]u^3 + [3(\mathbf{R}_i - \mathbf{R}_{i-1}) - 2\mathbf{R}_{i-1}^u - \mathbf{R}_i^u]u^2 \\ & + \mathbf{R}_{i-1}^u u + \mathbf{R}_{i-1} \end{aligned} \tag{1.9}$$

$$\begin{aligned} \mathbf{R}_i^u(u) = & 3[2(\mathbf{R}_{i-1} - \mathbf{R}_i) + \mathbf{R}_{i-1}^u + \mathbf{R}_i^u]u^2 + 2[3(\mathbf{R}_i - \mathbf{R}_{i-1}) \\ & - 2\mathbf{R}_{i-1}^u - \mathbf{R}_i^u]u + \mathbf{R}_{i-1}^u \end{aligned} \tag{1.10}$$

for $i = 1, 2, 3, .., n.$

To discuss the quadratic and cubic developable Hermite surfaces, we initiate with the definition of developable surfaces, and then, we demonstrate the construction of quadratic and cubic developable Hermite patches in this way.

Definition 1.1. *The regular ruled surface $\mathbf{D}(u, v) = \mathbf{R}(u) + v\mathbf{g}(u)$ is developable, if the tangent plane is constant along each generatrix line g(u) or the vectors $[\mathbf{g}'(u), \mathbf{R}'(u), \mathbf{g}(u)]$ must be coplanar [4,5].*

If the generatrix lines $\mathbf{g}(u)$ in the forms $\mathbf{g}(u) = \mathbf{S}(u) - \mathbf{R}(u)$ such that $\mathbf{D}(u, v) = \mathbf{R}(u) + v\mathbf{g}(u) = (1-v)\mathbf{R}(u) + v\mathbf{S}(u)$, then this definition can be stated that $\mathbf{D}(u, v)$ is developable if $\mathbf{g}'(u) = \alpha(u)\mathbf{R}'(u) + \beta(u)\mathbf{g}(u)$ or $\mathbf{S}'(u) = \delta(u)\mathbf{R}'(u) + \beta(u)\mathbf{g}(u)$ with $\delta(u) = [1 + \alpha(u)]$ and $\alpha(u)$, $\beta(u)$, $\delta(u)$ real scalar functions. Because of application reason and the regularity of the surfaces, in this discussion, we limit this developable criteria in the form

$$\mathbf{S}'(u) = \delta\mathbf{R}'(u) \tag{1.11}$$

with δ positive real scalar [5,6]. In this case, if we determine $\delta \neq 1$, then the developable surface type will be a cone, and all generatrixes intersect at a point. In contrast, when the value $\delta = 1$, we find a cylinder, and all generatrixes are parallel. Thus, for any two generatrices selected, it must lay in the same plane (coplanar). Henceforward, we will construct the quadratic and cubic developable Hermite patch shapes by using the quadratic and cubic equations (1.2), (1.4), and the developable criteria of Equation (1.11). The problem is as follows.

Given the quadratic Hermite curves $\mathbf{R}_2(u)$ and cubic Hermite curves $\mathbf{R}_3(u)$ of Equation (1.2) and (1.4), correspondingly. We have to determine a quadratic Hermite curve $\mathbf{S}_2(u)$ and cubic Hermite curve $\mathbf{S}_3(u)$ to construct the quadratic and cubic developable Hermite patches determined by these curves in the formulations

$$\mathbf{D}_2(u,v) = (1-v)\mathbf{R}_2(u) + v\mathbf{S}_2(u) \tag{1.12}$$

$$\mathbf{D}_3(u,v) = (1-v)\mathbf{R}_3(u) + v\mathbf{S}_3(u) \tag{1.13}$$

with the parameters u, and v in interval $0 \leq u, v \leq 1$. From the developable criteria of Equation (1.11), and the formula (1.2) and (1.4) for the quadratic and cubic curves, if $\mathbf{S}_2'(u) = \delta\mathbf{R}_2'(u)$ and $\mathbf{S}_3'(u) = \delta\mathbf{R}_3'(u)$, then

$$H_1^u(u)\mathbf{S}_o + H_2^u(u)\mathbf{S}_1 + H_3^u(u)\mathbf{S}_o^u = \delta[H_1^u(u)\mathbf{R}_o + H_2^u(u)\mathbf{R}_1 + H_3^u(u)\mathbf{R}_o^u]$$

$$H_1^u(u)\mathbf{S}_o + H_2^u(u)\mathbf{S}_1 + H_3^u(u)\mathbf{S}_o^u + H_4^u(u)\mathbf{S}_1^u =$$
$$\delta[H_1^u(u)\mathbf{R}_o + H_2^u(u)\mathbf{R}1 + H_3^u(u)\mathbf{R}_o^u + H_4^u(u)\mathbf{R}_1^u].$$

Therefore, the developable condition of patch $\mathbf{D}_2(u,v)$ must meet $\mathbf{S}_o = \delta\mathbf{R}_o$; $\mathbf{S}_1 = \delta\mathbf{R}_1$, $\mathbf{S}_o^u = \delta\mathbf{R}_o^u$ and, the patch $\mathbf{D}_3(u,v)$ have to fulfill $\mathbf{S}_o = \delta\mathbf{R}_o$; $\mathbf{S}_1 = \delta\mathbf{R}_1$; $\mathbf{S}_o^u = \delta\mathbf{R}_o^u$; $\mathbf{S}_1^u = \delta\mathbf{R}_1^u$. Due to the generatrix lines, $\mathbf{S}_o\mathbf{R}_o$ and $\mathbf{S}_1\mathbf{R}_1$ of the cone and cylinder patches have to lay in the same plane, these equations of developable criteria can be simplified, respectively, as follows

$$[\mathbf{S}_1 - \mathbf{S}_o] = \delta[\mathbf{R}_1 - \mathbf{R}_o];\ \mathbf{S}_o^u = \delta\mathbf{R}_o^u \tag{1.14}$$

$$[\mathbf{S}_1 - \mathbf{S}_o] = \delta[\mathbf{R}_1 - \mathbf{R}_o];\ \mathbf{S}_o^u = \delta\mathbf{R}_o^u;\ \mathbf{S}_1^u = \delta\mathbf{R}_1^u. \tag{1.15}$$

Thus, to construct the quadratic dan cubic developable Hermite patches of Equation (1.12) and (1.13) need the steps, first, we determine the control points $\mathbf{S}_o$, $\mathbf{S}_1$, $\mathbf{R}_o$, and $\mathbf{R}_1$ such that $[\mathbf{R}_1 - \mathbf{R}_o]//[\mathbf{S}_1 - \mathbf{S}_o]$, $\delta = |\mathbf{S}_1 - \mathbf{S}_o|/|\mathbf{R}_1 - \mathbf{R}_o|$. Second, we calculate $\mathbf{S}_o^u = \delta\mathbf{R}_o^u$ and $\mathbf{S}_1^u = \delta\mathbf{R}_1^u$ via Equation (1.14) and (1.15).

Figure 1a simulate the quadratic developable Hermite patch by determining the control points $\mathbf{R}_o =< 10, -45, 15 >$, $\mathbf{R}_1 =< 10, 35, 30 >$, $\mathbf{S}_o =< -20, -90,$

$15 >$, $\mathbf{S}_1 =< -20, 110, 52.5 >$, and the tangent vector $\mathbf{R}_o^u =< 0, 20, 80 >$. From the equation (1.14), it can found $\delta = |\mathbf{S}_1 - \mathbf{S}_o|/|\mathbf{R}_1 - \mathbf{R}_o| = 2.5$ and $\mathbf{S}_o^u = \delta\mathbf{R}_o^u =< 0, 50, 200 >$. Therefore, using these results and Equation (1.2), it can design the patch of Equation (1.12). Figure 1b present the cubic developable Hermite patch of the data $\mathbf{R}_o =< 10, -42, 8 >$, $\mathbf{R}_1 =< 10, 28, 23 >$, $\mathbf{S}_o =< -20, -90, 15 >$, $\mathbf{S}_1 =< -20, 110, 52.5 >$ and the tangent vector $\mathbf{R}_o^u = < 0, 20, 80 >$ and $\mathbf{R}_1^u =< 0, 40, 50 >$. Applying Equation (1.15), we can calculate $\delta = 2.5$, $\mathbf{S}_o^u =< 0, 50, 200 >$ and $\mathbf{S}_1^u =< 0, 100, 125 >$. Thus, using these calculation results and Equation (1.4), it can draw the developable Hermite patch of Equation (1.13).

Base on this construction method of the quadratic and cubic developable Hermite patches, in the next section, we will develop a new approach to design the developable surfaces that are evaluated by Equation (1.6), (1.7), (1.9) and (1.10), and the developable surface' criteria equation (1.11) or, explicitly, Equation (1.14) and (1.15). Hereafter, these surface types are called the quadratic and cubic developable Hermite surfaces.

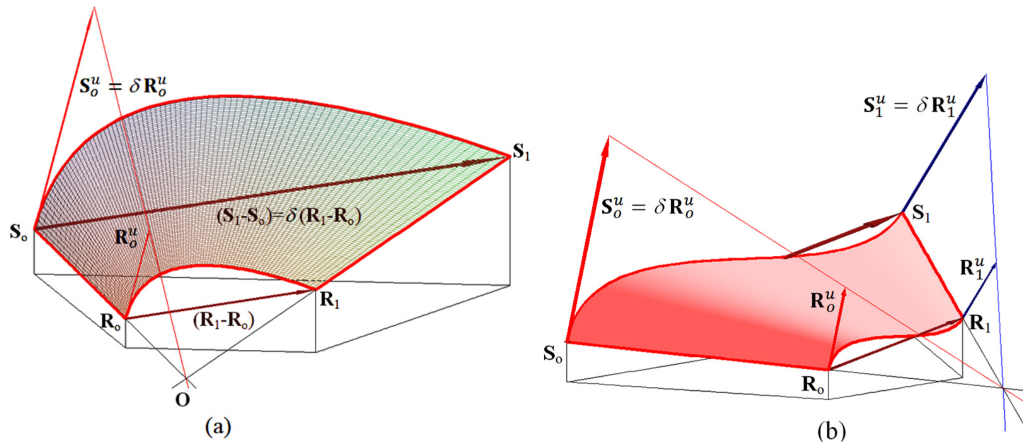

FIGURE 1. Quadratic (a) and cubic (b) developable Hermite patches

2. COMPUTATION OF QUADRATIC AND CUBIC DEVELOPABLE HERMITE SURFACES

There were introduced some methods to construct the developable surfaces by using their boundary curves. Aumann [6], Frey and Bindschadler [7], and

Kusno [5] defined developable surfaces in the form of Bezier patches. Chalfant [8] proposed a method to construct a B-spline developable surface. Then, Fernándes and Pérez [9] designed the developable surfaces bounded by two rational or NURBS curves. This latter method requires that the endpoints of patch' boundary curves have to be coplanar, and meet the developability criteria. A different approach from these methods will be discussed to define the developable surfaces. The steps are as follows. We will formulate the first boundary curve using the quadratic and cubic Hermite spline interpolation curve. Then, via the developability condition of Hermite patches in Equation (1.14) and (1.15), it can compute the second boundary curves of the developable surfaces in the form of Hermite curve segments.

Given $(n+1)$ consecutive control points $\mathbf{R}_o, \mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_n$, and one tangent vector $\mathbf{R}_o^u$. Using the matrix equation (1.5) for calculating the tangent vectors $\mathbf{R}_i^u$ for $i = 1, 2, \ldots, n$, it can determine the first boundary curve in the form of the quadratic Hermite spline interpolation (Equation (1.6)), that is $\mathbf{R}_i(u) = (\mathbf{R}_i - \mathbf{R}_{i-1} - \mathbf{R}_{i-1}^u).u^2 + \mathbf{R}_{i-1}^u.u + \mathbf{R}_{i-1}$ for $i = 1, 2, 3, .., n$. From this spline curves result, we have to calculate the second boundary curve segments that are defined by the quadratic Hermite curves connection G^1 in the formulations

$$\mathbf{S}_i(u) = (\mathbf{S}_i - \mathbf{S}_{i-1} - \mathbf{S}_{i-1}^u).u^2 + \mathbf{S}_{i-1}^u.u + \mathbf{S}_{i-1} \tag{2.1}$$

for i = 1, 2, 3, .., n that can construct the quadratic developable Hermite surface

$$\mathbf{D}_{2i}(u, v) = (1 - v)\mathbf{R}_i(u) + v\mathbf{S}_i(u) \tag{2.2}$$

with u, and v in interval $0 \leq u, v \leq 1$. Using Equation (1.14), the numerical solution is as follows:

(i) Arrange the consecutive elected control points $\mathbf{S}_o, \mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_n$ such that the control points' pairs $[\mathbf{R}_{i-1}, \mathbf{R}_i]$ and $[\mathbf{S}_{i-1}, \mathbf{S}_i]$ can respectively define the vectors $[\mathbf{R}_i - \mathbf{R}_{i-1}]//[\mathbf{S}_i - \mathbf{S}_{i-1}]$ and the values $\delta_i = |\mathbf{S}_i - \mathbf{S}_{i-1}|/|\mathbf{R}_i - \mathbf{R}_{i-1}|$ for $i = 1, 2, \ldots, n$.

(ii) To meet smoothly G^1 the second boundary curve of the surface at the point $\mathbf{S}_i$ for $i = 0, 1, 2, 3, .., (n-1)$, it must calculate the tangent vectors $\mathbf{S}_i^u(0) = \delta_{i+1}\mathbf{R}_i^u$.

Therefore, if the quadratic Hermite spline interpolation data $\mathbf{R}_i(u)$ is given, then the quadratic Hermite curve segments $\mathbf{S}_i(u)$ of Equation (2.1) can be determined smoothly G^1. Consequently, using these quadratic Hermite spline interpolation curves $\mathbf{R}_i(u)$ and the quadratic Hermite curve segments $\mathbf{S}_i(u)$, it can construct the quadratic developable Hermite surface of Equation (2.2).

On the other hand, let $(n+1)$ consecutive control points $\mathbf{R}_o, \mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_n$ and two tangent vectors $\mathbf{R}_o^u$ and $\mathbf{R}_n^u$. Via Equation (1.8), it can find the tangent vectors $\mathbf{R}_i^u$ for $i = 1, 2, \ldots, n$, and then, it can define the cubic Hermite spline interpolation $\mathbf{R}_i(u) = [2(\mathbf{R}_{i-1} - \mathbf{R}_i) + \mathbf{R}_{i-1}^u + \mathbf{R}_i^u].u^3 + [3(\mathbf{R}_i - \mathbf{R}_{i-1}) - 2\mathbf{R}_{i-1}^u - \mathbf{R}_i^u].u^2 + \mathbf{R}_{i-1}^u.u + \mathbf{R}_{i-1}$ for $i = 1, 2, 3, .., n$. After that, we must compute the second boundary curve of the surface

$$\begin{aligned}\mathbf{S}_i(u) = {} & [2(\mathbf{S}_{i-1} - \mathbf{S}_i) + \mathbf{S}_{i-1}^u + \mathbf{S}_i^u]u^3 + [3(\mathbf{S}_i - \mathbf{S}_{i-1}) - 2\mathbf{S}_{i-1}^u - \mathbf{S}_i^u]u^2 \\ & + \mathbf{S}_{i-1}^u u + \mathbf{S}_{i-1}\end{aligned} \tag{2.3}$$

for i = 1, 2, 3, .., n that can be used to define the cubic developable Hermite surface in the form

$$\mathbf{D}_{3i}(u, v) = (1 - v)\mathbf{R}_i(u) + v\mathbf{S}_i(u) \tag{2.4}$$

with u, and v in interval $0 \leq u, v \leq 1$. Applying Equation (1.15), the numerical procedure is in this way:

(i) Arrange the determined control points $\mathbf{S}_o, \mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_n$ such that, for $i = 1, 2, \ldots, n$, the pairs $[\mathbf{R}_{i-1}, \mathbf{R}_i]$ and $[\mathbf{S}_{i-1}, \mathbf{S}_i]$ define the vectors $[\mathbf{R}_i - \mathbf{R}_{i-1}]//[\mathbf{S}_i - \mathbf{S}_{i-1}]$ and compute the values $\delta_i = |\mathbf{S}_i - \mathbf{S}_{i-1}|/|\mathbf{R}_i - \mathbf{R}_{i-1}|$.

(ii) To fulfill smoothly G^1 the second boundary curve, we determine two tangent vector $\mathbf{S}_o^u = \delta_1 \mathbf{R}_o^u$ and $\mathbf{S}_n^u = \delta_n \mathbf{R}_n^u$, and then, at the point $\mathbf{S}_i$ for $i = 1, 2, 3, .., (n-1)$, it can calculate the tangent vectors $\mathbf{S}_{i1}^u = \mathbf{S}_i^u(1) = \delta_i \mathbf{R}_i^u$ and $\mathbf{S}_{i2}^u = \mathbf{S}_{i+1}^u(0) = \delta_{i+1}\mathbf{R}_i^u$.

Thus, the cubic Hermite splines interpolation curves $\mathbf{R}_i(u)$ and the cubic Hermite curve segments $\mathbf{S}_i(u)$ of Equation (2.3) can construct the cubic developable Hermite surface of Equation (2.4).

3. Simulation Of The Developable Hermite Surfaces Computation

Developable surfaces are broadly applied in many manufactures, such as the shape design of shoes and clothing [10]. It is also used in plat-metal-based industries for modeling ship hulls, aircrafts trains, and automobile parts [7,8,11, 12,13,14]. For this application purposes, we will discuss to design the developable surfaces of Equation (2.2) and (2.4) by following criteria from Frey and

Bindschadler [7] that both boundary Hermite spline curves $\mathbf{R}_i(u)$ and the Hermite curve segments $\mathbf{S}_i(u)$ must respectively be laid in the plane $\Psi_1//\Psi_2$. In this case, to make more straightforward calculations, we restrict $\Psi_1//\Psi_2//YOZ$.

Consider the data of the control points $\mathbf{R}_o =< 20, -175, 15 >$, $\mathbf{R}_1 =< 20, -135, 20 >$, $\mathbf{R}_2 =< 20, -50, 50 >$, $\mathbf{R}_3 =< 20, 30, 60 >$, $\mathbf{R}_4 =< 20, 80, 80 >$, $\mathbf{R}_5 =< 20, 130, 55 >$, and one tangent vector $\mathbf{R}_o^u =< 0, 5, 40 >$. Using Equation (1.5), it will find the tangent vectors $\mathbf{R}_1^u =< 0, 40, -50 >$, $\mathbf{R}_2^u =< 0, 130, 110 >$, $\mathbf{R}_3^u =< 0, 30, -90 >$, $\mathbf{R}_4^u =< 0, 35, 110 >$, $\mathbf{R}_5^u =< 0, 65, -160 >$, and we obtain the quadratic Hermite spline curve with the curve segments in the form

$\mathbf{R}_1(u) =< 20, 35u^2 + 5u - 175, -35u^2 + 40u + 15 >$;
$\mathbf{R}_2(u) =< 20, 10u^2 + 75u - 135, 60u^2 - 30u + 20 >$;
$\mathbf{R}_3(u) =< 20, -15u^2 + 95u - 50, -80u^2 + 90u + 50 >$;
$\mathbf{R}_4(u) =< 20, -15u^2 + 65u + 30, 90u^2 - 70u + 60 >$;
$\mathbf{R}_5(u) =< 20, 15u^2 + 35u + 80, -135u^2 + 110u + 80 >$.

Then, we fixe the control points $\mathbf{S}_o =< -20, -200, 15 >$, $\mathbf{S}_1 =< -20, -112, 26 >$, $\mathbf{S}_2 =< -20, -86.5, 35 >$, $\mathbf{S}_3 =< -20, -6.5, 45 >$, $\mathbf{S}_4 =< -20, 53.5, 69 >$, $\mathbf{S}_5 =< -20, 73.5, 59 >$, such that $\mathbf{R}_o\mathbf{R}_1//\mathbf{S}_o\mathbf{S}_1$, $\mathbf{R}_1\mathbf{R}_2//\mathbf{S}_1\mathbf{S}_2$, $\mathbf{R}_2\mathbf{R}_3//\mathbf{S}_2\mathbf{S}_3$, $\mathbf{R}_3\mathbf{R}_4//\mathbf{S}_3\mathbf{S}_4$, $\mathbf{R}_4\mathbf{R}_5//\mathbf{S}_4\mathbf{S}_5$, and we compute $\delta_1 = |\mathbf{S}_1 - \mathbf{S}_o|/|\mathbf{R}_1 - \mathbf{R}_o| = 2.2$; $\delta_2 = |\mathbf{S}_2 - \mathbf{S}_1|/|\mathbf{R}_2 - \mathbf{R}_1| = 0.3$; $\delta_3 = |\mathbf{S}_3 - \mathbf{S}_2|/|\mathbf{R}_3 - \mathbf{R}_2| = 1$; $\delta_4 = |\mathbf{S}_4 - \mathbf{S}_3|/|\mathbf{R}_4 - \mathbf{R}_3| = 1.2$; $\delta_5 = |\mathbf{S}_5 - \mathbf{S}_4|/|\mathbf{R}_5 - \mathbf{R}_4| = 0.4$. Thus, it can determine

$\mathbf{S}_o^u = \delta_1\mathbf{R}_o^u =< 0, 11, 88 >$, $\mathbf{S}_1^u = \delta_2\mathbf{R}_1^u =< 0, 12, -15 >$,
$\mathbf{S}_2^u = \delta_3\mathbf{R}_2^u =< 0, 130, 110 >$, $\mathbf{S}_3^u = \delta_4\mathbf{R}_3^u =< 0, 36, -108 >$,
$\mathbf{S}_4^u = \delta_5\mathbf{R}_4^u =< 0, 14, 44 >$.

Applying Equation (2.1), it can obtain the quadratic Hermit curve segments

$\mathbf{S}_1(u) =< -20, 77u^2 + 11u - 200, -77u^2 + 88u + 15 >$;
$\mathbf{S}_2(u) =< -20, 3u2 + 22.5u - 112, 18u2 - 9u + 26 >$;
$\mathbf{S}_3(u) =< -20, -15u^2 + 95u - 86.5, -80u^2 + 90u + 35 >$;
$\mathbf{S}_4(u) =< -20, -18u^2 + 78u - 6.5, 108u^2 - 84u + 45 >$;
$\mathbf{S}_5(u) =< -20, 6u^2 + 14u + 53.5, -54u^2 + 44u + 69 >$.

Therefore, the developable Hermite patches of Equation (2.2) can be constructed, and its graph is shown in Figure 2(a). If the tangent vector $\mathbf{R}_o^u =< 0, 5, 40 >$ is replaced by $\mathbf{R}_o^u =< 0, 40, 60 >$, the boundary curve shape R(u) will change as presented in Figure (2b) and the resulted patches are exposed in Figure (2c).

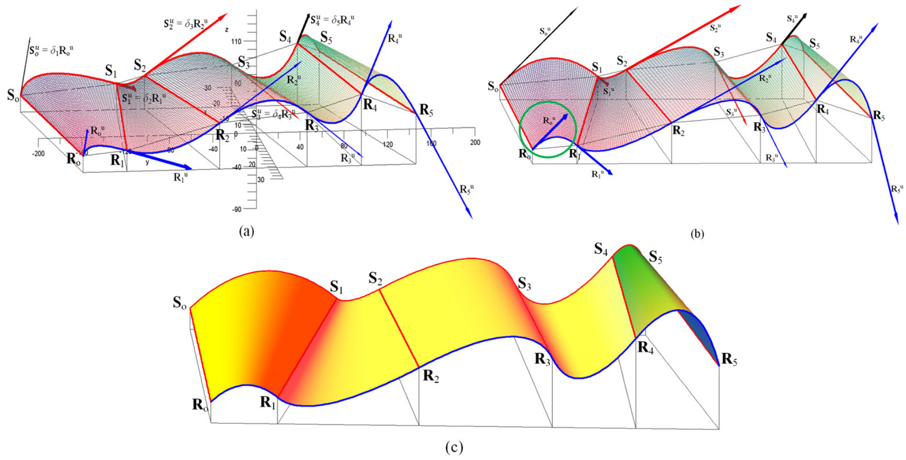

FIGURE 2. Construction of quadratic developable Hermite surface

Let the control points of the quadratic Hermit spline curve $\mathbf{R}_o, \mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3, \mathbf{R}_4$, $\mathbf{R}_5$ and two tangent vector $\mathbf{R}_o^u =< 0, 40, 60 >$, $\mathbf{R}_5^u =< 0, 65, -160 >$. From Equation (1.8), it will obtain the tangent vectors $\mathbf{R}_1^u =< 0, 61, 3 >$, $\mathbf{R}_2^u =< 0, 90, 33 >$, $\mathbf{R}_3^u =< 0, 73, -16 >$, $\mathbf{R}_4^u =< 0, 8, 120 >$, and it can formulate the cubic Hermite spline curve with the curve segments in the form

$\mathbf{R}_1(u) =< 20, 21u^3 - 21u^2 + 40u - 175, 53u^3 - 108u^2 + 60u + 15 >$;

$\mathbf{R}_2(u) =< 20, 19u^3 + 42u^2 + 61u - 135, -24u^3 + 51u^2 + 3u + 20 >$;

$\mathbf{R}_3(u) =< 20, 3u^3 - 13u^2 + 90u - 50, -2.6u^3 - 21u^2 + 33u + 50 >$;

$\mathbf{R}_4(u) =< 20, 19u^3 - 4u^2 + 73u + 30, 17u^3 - 20u^2 - 16u + 60 >$;

$\mathbf{R}_5(u) =< 20, -3u^3 + 69u^2 + 8u + 80, 10u^3 - 112u^2 + 120u + 80 >$.

Then, we use the control points data of the quadratic Hermite curves $\mathbf{S}_o =< -20, -200, 15 >$, $\mathbf{S}_1 =< -20, -112, 26 >$, $\mathbf{S}_2 =< -20, -86.5, 35 >$, $\mathbf{S}_3 = < -20, -6.5, 45 >$, $\mathbf{S}_4 =< -20, 53.5, 69 >$, $\mathbf{S}_5 =< -20, 73.5, 59 >$, and it meet $\mathbf{R}_{i-1}\mathbf{R}_i // \mathbf{S}_{i-1}\mathbf{S}_i$, for $i = 1, 2, ..., 5$. We find $\delta_1 = 2.2$; $\delta_2 = 0.3$; $\delta_3 = 1$; $\delta_4 = 1.2$; $\delta_5 = 0.4$. As a result, it can determine $\mathbf{S}_o^u = \delta_1 \mathbf{R}_o^u =< 0, 88, 132 >$; $\mathbf{S}_{11}^u = \delta_1 \mathbf{R}_1^u =< 0, 134.2, 6.6 >$; $\mathbf{S}_{12}^u = \delta_2 \mathbf{R}_1^u =< 0, 18.3, 0.9 >$; $\mathbf{S}_{21}^u = \delta_2 \mathbf{R}_2^u = < 0, 27, 9.9 >$; $\mathbf{S}_{22}^u = \delta_3 \mathbf{R}_2^u =< 0, 90, 33 >$; $\mathbf{S}_{31}^u = \delta_3 \mathbf{R}_3^u =< 0, 73, -16 >$; $\mathbf{S}_{32}^u = \delta_4 \mathbf{R}_3^u =< 0, 87.6, -19.2 >$; $\mathbf{S}_{41}^u = \delta_4 \mathbf{R}_4^u =< 0, 9.6, 144 >$; $\mathbf{S}_{42}^u = \delta_5 \mathbf{R}_4^u = < 0, 3.2, 48 >$; $\mathbf{S}_5^u = \delta 5 R5u =< 0, 26, -64 >$. From the control points data, the calculated tangent vectors, and applying Equation (2.3), it can, respectively, define the cubic Hermite curve segments

$\mathbf{S}_1(u) =< -20, 47u^3 - 47u^2 + 88u - 200, 116u^3 - 237u^2 + 132u + 15 >$;

$\mathbf{S}_2(u) =< -20, 6u^3 + 13u^2 + 18u - 112, -7u^3 + 15u^2 + u + 26 >;$
$\mathbf{S}_3(u) =< -20, 3u^3 - 13u^2 + 90u - 86.5, -2.6u^3 - 21u^2 + 33u + 35 >;$
$\mathbf{S}_4(u) =< -20, -23u^3 - 5u^2 + 88u - 6.5, 77u^3 - 34u^2 - 19u + 45 >;$
$\mathbf{S}_5(u) =< -20, -11u^3 + 28u^2 + 3u + 54, 4u^3 - 62u^2 + 48u + 69 >.$

As a result, the Hermite curve segments are presented in Figure (3a) and the developable Hermite patches of Equation (2.2) can be constructed and shown in Figure (3b). If the tangent vector $\mathbf{R}_o^u =< 0, 40, 60 >$ and $\mathbf{R}_5^u =< 0, 65, -160 >$ is replaced by $\mathbf{R}_o^u =< 0, 20, 80 >$ and $\mathbf{R}_5^u =< 0, 10, -90 >$, the boundary curve $\mathbf{R}(u)$ is modified and the constructed patches are presented in Figure (3c).

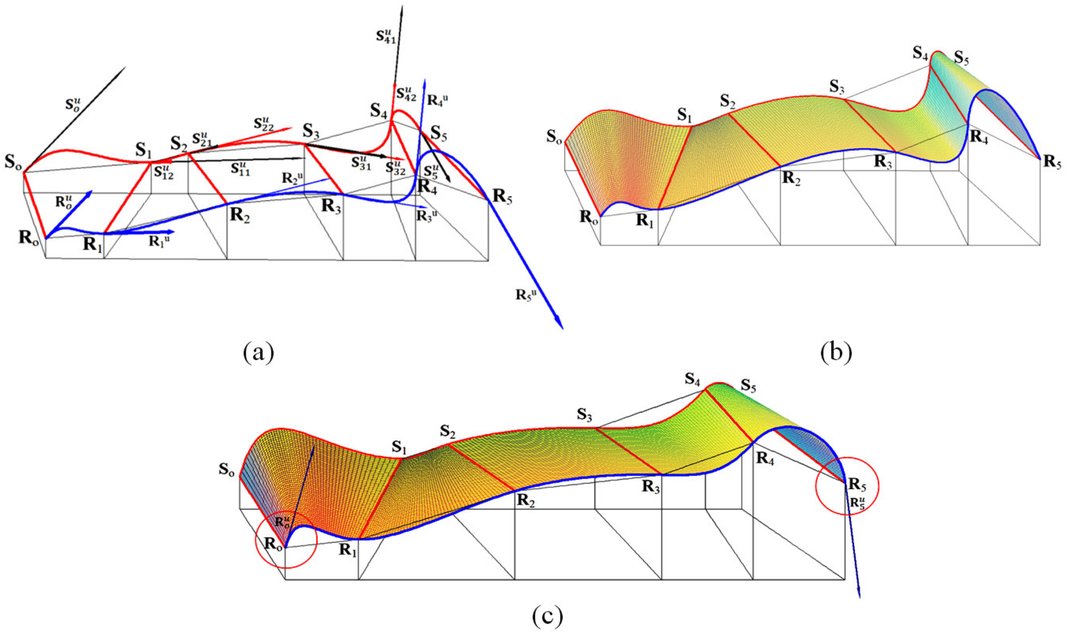

FIGURE 3. Construction of cubic developable Hermite surface

4. CONCLUSION

We presented the method to design the quadratic and cubic developable Hermite surfaces in which their boundary curves are respectively be laid in the parallel plane. Using the control points data and determining the tangent vectors at starting control points and or endpoints of both boundary curves can construct developable surfaces' various shapes. Future exciting discussions should expand to formulate the developable surfaces with their boundaries curves in space position.

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