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STABILITY OF ADDITIVE AND QUADRATIC FUNCTIONAL EQUATIONS

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ABSTRACT. In this paper, we construct an additive and a quadratic functional equations, which are combinations of some existing functional equations. The Hyers-Ulam stability of these functional equations are obtained using the fixed point method. Moreover, a stability result in the sense of Rassias is obtained for an orthogonally quadratic functional equation using the direct method.

1. INTRODUCTION

The stability problem of functional equations arose from the question of S. M. Ulam in 1940 concerning the stability of group homomorphisms [26].

For a given $\varepsilon > 0$*, a group* X *and a metric group* (Y, d)*, does there exists a* $\delta > 0$ *such that if a mapping* $f : X \longrightarrow Y$ *satisfies*

$$d\Big(f(st), f(s)f(t)\Big) \leq \delta, \qquad \textit{for all} \;\; s, t \in X,$$

then a homomorphism $h : X \longrightarrow Y$ *exists with* $d\Big(f(s), h(s)\Big) \leq \varepsilon$ *for all* $s \in X$*?* If it exists, then the homomorphism $f(st) = f(s)f(t)$ is said to be *stable*.

A partial affirmative answer to this problem, for Banach spaces was given by D. H. Hyers [11] in 1941 in relation to addition. Again in 1978, the result of Hyers was extended by R. M. Rassias [22] with unbounded Cauchy differences

$$\Big\| f(s+t) - \Big(f(s) + f(t)\Big)\Big\| \leq \varepsilon\Big(\|s\|^p + \|t\|^p\Big),$$

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for $\varepsilon > 0$ and $0 \leq p < 1$. In 1994, this result was generalized by Gavruta [8] by replacing the bound $\varepsilon \left(\|s\|^p + \|t\|^p \right)$ with a general control function $\varphi(s,t)$ with $\sum_{k=0}^{\infty} 2^{-k} \varphi \left(2^k s, 2^k t \right) < \infty$.

Since then, different stability problems have been studied by various authors during the last few decades. For instance, one may refer to [6], [7], [15], [16], [17], [18], [19], [20], [24] and the references therein. Skof [23], in 1983, first discussed the stability of the quadratic functional equation (1.2), restricting to the case where f maps a normed space to a Banach space. Cholewa [1] extended the domain to Abelian groups and it was further generalized by Czerwik [2] and Jung [12].

The functional equation

$$f(s+t) = f(s) + f(t) \tag{1.1}$$

is called the *additive Cauchy functional equation*, while the functional equation

$$f(s+t) + f(s-t) = 2f(s) + 2f(t) \tag{1.2}$$

is called a *quadratic functional equation*. Every solution of (1.1) is called an *additive function* [13] and, every solution of (1.2) is called a *quadratic function* [13].

The proof of Hyers [11] involves a direct method. However, other approaches for deriving the stability results are also available in the literature, for instance, the method of invariant means and the fixed point method. Since the famous Banach fixed point theorem of 1922, many generalizations and refinement of the fixed point results are seen in the literatures of metric fixed point theory. For more details on fixed point theory, one may refer, for instance, to [3], [4], [9], [10], [14], [21], and the references therein.

The following definition of a generalized metric is needed for the introduction of a fundamental result in fixed point theory which is used extensively in this paper in obtaining the stability results.

Definition 1.1. *[5] Let X be a set. A function $d : X \times X \longrightarrow [0,\infty]$ is called a generalized metric on X if d satisfy the following conditions:*

(1) $d(s,t) = 0$ *if and only if* $s = t$*;*
(2) $d(s,t) = d(t,s)$ *for all* $s,t \in X$*;*
(3) $d(s,r) \leq d(s,t) + d(t,r)$ *for all* $r,s,t \in X$.

The only difference of the generalized metric with that of the usual metric is that the point at infinity, ∞ is included in the range of the former. The notion of *Lipscitzian mappings* defined on metric spaces naturally extends to generalized metric spaces.

Let (X, d) be a generalized complete metric space. A mapping $T : X \longrightarrow X$ is said to be *Lipschitzian* [14] if there exists a constant $k \geq 0$ such that $d\big(Tu, Tv\big) \leq kd(u, v)$ for all $u, v \in X$. The smallest number k for which the above relation holds true is called the *Lipschitz's constant* of T. A Lipschitzian with the Lipschitz's constant $k < 1$ is called a *contraction mapping*.

Theorem 1.1. *[5] Let (X, d) be a complete generalized metric space and $T : X \longrightarrow X$ be a contraction mapping with the Lipschitz constant $\alpha < 1$. Then for each $u \in X$, either $d(T^n u, T^{n+1}u) = \infty$ for all nonnegative integers n, or there exist a positive integer n_0 such that*

(1) $d(T^n u, T^{n+1}u) < \infty$ *for all* $n \geq n_0$*;*
(2) *The sequence* $\{T^n u\}$ *converges to a fixed point* v^* *of* X*;*
(3) v^* *is the unique fixed point of* T *in the set* $Y = \{v \in X | d(T^{n_0}u, v) < \infty\}$*;*
(4) $d(v, v^*) \leq \frac{1}{1-\alpha} d(v, Tv)$ *for all* $v \in Y$.

Motivated by [19] and [22], in this paper, we use the fixed point method, as well as the direct method, to obtain some stability results for some functional equations.

2. Main results

In this section we discuss the stability of three functional equations.

2.1. An additive functional equation.

Combining the Jensen type quadratic functional equation [25]

$$3f\left(\frac{s+t+r}{3}\right) + f(s) + f(t) + f(r) = 2\left[f\left(\frac{s+t}{2}\right) + f\left(\frac{t+r}{2}\right) + f\left(\frac{r+s}{2}\right)\right]$$

and the Jensen's functional equation

$$2\left(\frac{s+t}{2}\right) = f(s) + f(t),$$

first we construct the following additive equation.

$$3f\left(\frac{s+t+r}{3}\right) = 2\Big[f\left(\frac{s+r}{2}\right) + f\left(\frac{t+r}{2}\right)\Big] - f(r). \tag{2.1}$$

Analogous to the proof of Theorem 2.1 in [25], the following existence result of the general solution to the functional equation (2.1) can be obtained.

Theorem 2.1. *Let X and Y be real linear spaces. A function $f : X \longrightarrow Y$ satisfies (2.1) if and only if there exists an element $t_0 \in Y$ and an additive mapping $A : X \longrightarrow Y$ such that $f(s) = A(s) + t_0$, for all s in X.*

Next, we establish a stability result for the functional equation (2.1).

Theorem 2.2. *Let X be a real normed linear space and Y be a real Banach space. Let $\varphi : X^3 \longrightarrow [0, \infty)$ be a function such that there exists $\alpha < 1$ with*

$$\psi(2s) \leq 2\alpha\psi(s) \qquad \textit{where} \;\; \psi(s) = \varphi(-s, 0, s)$$

for all $s \in X$. If $f : X \longrightarrow Y$ satisfies the inequality

$$\left\|3f\left(\frac{s+t+r}{3}\right) + f(r) - 2\Big[f\left(\frac{s+r}{2}\right) + f\left(\frac{t+r}{2}\right)\Big]\right\| \leq \varphi(s,t,r)$$

for all $s, t, r \in X$, then there exists a unique additive mapping $A : X \longrightarrow Y$ such that for all s in X,

$$\big\|f(s) - f(0) - A(s)\big\| \leq \frac{\alpha}{1-\alpha}\psi(s).$$

If in addition, $f(\lambda s)$ is continuous in λ for each $s \in X$, then A is linear.

Proof. Consider the function $g : X \longrightarrow Y$ defined by $g(s) = f(s) - f(0)$. Then $g(0) = 0$ and, if

$$\mathcal{D}f(s,t,r) = 3f\left(\frac{s+t+r}{3}\right) + f(r) - 2\Big[f\left(\frac{s+r}{2}\right) + f\left(\frac{t+r}{2}\right)\Big]$$

then we see that $\mathcal{D}g(s,t,r) = \mathcal{D}f(s,t,r)$ for all $s, t, r \in X$.

Therefore,

$$\left\|3g\left(\frac{s+t+r}{3}\right) + g(r) - 2\Big[g\left(\frac{s+r}{2}\right) + g\left(\frac{t+r}{2}\right)\Big]\right\| \leq \varphi(s,t,r). \tag{2.2}$$

Making the substitutions $r = -s - t$, $s = -s$, $t = -t$, $t = 0$ and $s = 2s$ successively in (2.2) and the resulting inequalities, we get

$$\left\|g(s) - \frac{1}{2}g(2s)\right\| \leq \alpha\psi(s), \qquad \forall\, s \in X.$$

Considering the set $S = \Big\{h : X \longrightarrow Y,\ h(0) = 0\Big\}$ and the generalized metric d on S as

$$d(f,h) = d_\psi(f,h) = \inf S_\psi(f,h), \qquad \forall\ f,h \in S$$

where $$S_\psi(f,h) = \Big\{\mu \in \mathbb{R}_+ : \|f(s) - h(s)\| \le \mu\psi(s),\ \forall\ s \in X\Big\},$$

and $\inf S_\psi(f,h) = \infty$, if $S_\psi(f,h) = \emptyset$, we can show (S,d) is complete, and the rest of the proof follows in an analogous manner, as in the proof of Theorem 2.2 in [19]. □

2.2. **A quadratic functional equation.** The functional equation

$$2f\left(\frac{s}{2} + t\right) + 2f\left(\frac{s}{2} - t\right) = f(s) + 4f(t). \tag{2.3}$$

is a quadratic functional equation. The fact that equation (2.3) is quadratic can be seen easily as follows. Putting $t = 0$ in (2.3) and replacing s by $2s$ in the resulting equation, we get

$$f\,(2s) = 2^2 f(s).$$

Now, replacing s with $2s$ in (2.3), we get (1.2) by virtue of the above relation.

Combining the Jensen's quadratic functional equation and (2.3), we construct the following quadratic equation

$$2f\left(\frac{s+t}{2}\right) + 2f\left(\frac{s-t}{2}\right) + 2f\left(\frac{s}{2} + t\right) + 2f\left(\frac{s}{2} - t\right) = 2f(s) + 5f(t). \tag{2.4}$$

Proposition 2.1. *The functional equation (2.4) is a quadratic equation.*

Proof. It is easy to see that $f(s) = cs^2$ is a solution for (2.4) where c is a real number. Nevertheless, we shall show that (2.4) is equivalent to (1.2) so that every solution of (2.4) is a quadratic function.

Let f be a solution of (1.2). Then putting $s = t = 0$ and $s = t = \frac{s}{2}$ in (1.2), we respectively get

$$f(0) = 0 \qquad \text{and} \qquad f\left(\frac{s}{2}\right) = \frac{1}{2^2} f(s). \tag{2.5}$$

Substituting $s = \frac{s+t}{2}$ and $t = \frac{s-t}{2}$ in (1.2), we get

$$2f\left(\frac{s+t}{2}\right) + 2f\left(\frac{s-t}{2}\right) = f(s) + f(t).$$

Replacing s by $\frac{s}{2}$ in (1.2) and using (2.5), we get

$$2f\left(\frac{s}{2}+t\right)+2f\left(\frac{s}{2}-t\right)=f(s)+4f(t).$$

Adding the last two equations, we get (2.4).

On the other hand, if f is a solution of (2.4), then putting $s=0$, $t=0$ and $t=0$ ($s\neq 0$) in (2.4), we respectively get

$$f(0)=0,\quad f(-t)=f(t)\quad \text{and}\quad f\left(\frac{s}{2}\right)=\frac{1}{2^2}f(s).$$

Therefore, (2.4) can be rewritten as

$$f(s+t)+f(s-t)+f(s+2t)+f(s-2t)=4f(s)+10f(t). \tag{2.6}$$

Putting $s=2s$ in (2.4) and using the result $f\left(\frac{s}{2}\right)=\frac{1}{2^2}f(s)$, we get

$$4f(s+t)+4f(s-t)+f(2s+t)+f(2s-t)=16f(s)+10f(t).$$

Interchanging s and t in the above equation and using $f(-t)=f(t)$ for all $t\in X$, we get

$$4f(s+t)+4f(s-t)+f(s+2t)+f(s-2t)=10f(s)+16f(t). \tag{2.7}$$

Subtracting (2.6) from (2.7), we get (1.2). □

Let X be a normed space over the field F and Y a Banach space over F. The following notion of F-quadratic mapping will be used in the next theorem.

Definition 2.1. *[19] A quadratic mapping $f:X\longrightarrow Y$ is called an F-quadratic mapping if $f(as)=a^2f(s)$ for all $a\in F$ and $s\in X$.*

Now, we derive the stability result for the quadratic functional equation (2.4) using fixed point method.

Theorem 2.3. *Let $\varphi: X^2\longrightarrow[0,\infty)$ be a function such that there exists an $\alpha<1$ with*

$$\varphi(s,t)\leq 8\alpha\varphi\left(\frac{s}{2},\frac{t}{2}\right),\qquad s,t\in X.$$

Let $f:X\longrightarrow Y$ be a mapping satisfying $f(0)=0$ and

$$\begin{aligned}\Bigg\|2a^2f\left(\frac{s+t}{2}\right)+2a^2f\left(\frac{s-t}{2}\right)+2a^2f\left(\frac{s}{2}+t\right)+2a^2f\left(\frac{s}{2}-t\right)&\\ -2f(as)-5f(at)\Bigg\|\leq\varphi(s,t)&\end{aligned} \tag{2.8}$$

for all $a \in F$ and $s, t \in X$. If for each $s \in X$, the mapping $f(\lambda s)$ is continuous in $\lambda \in \mathbb{R}$, then there exists a unique Jensen F-quadratic mapping $Q : X \longrightarrow Y$ such that for all $s \in X$,

$$\|f(s) - Q(s)\| \leq \frac{\alpha}{1-\alpha}\varphi(s,0).$$

Proof. Putting $t = 0$ and $a = 1$ (the multiplicative identity in F) in (2.8), we get

$$\left\|8f\left(\frac{s}{2}\right) - 2f(s)\right\|_Y \leq \varphi(s,0)$$

for all $s \in X$. So, we have, for all $s \in X$,

$$\left\|f(s) - \frac{1}{4}f(2s)\right\|_Y \leq \frac{1}{8}\varphi(2s,0) \leq \alpha\varphi(s,0).$$

Consider the set $S := \Big\{g : X \longrightarrow Y,\ g(0) = 0\Big\}$ and introduce the generalized metric on S by

$$d(g,h) = d_\varphi(g,h) = \inf S_\varphi(g,h), \qquad \forall\, g,h \in S,$$

$$\text{where} \qquad S_\varphi(g,h) = \Big\{\mu \in (0,\infty) : \|g(t) - h(t)\| \leq \mu\varphi(t,0),\ \forall\, t \in X\Big\}.$$

Then as in the proof of Theorem 2.2 in [19], it is seen that (S,d) is complete. Now, considering the mapping $J : S \longrightarrow S$ such that

$$Jg(t) = \frac{1}{4}g(2t), \qquad \forall\, t \in X,$$

we can show that J is a contraction mapping. Using Theorem 1.1, we can arrive at the required results as in Theorem 2.2. □

2.3. **An orthogonally quadratic equation.** In this section, we discuss the stability in the sense of Rassias for the functional equation

$$f(s+t) + 2f\left(\frac{s}{2}+t\right) + 2f\left(\frac{s}{2}-t\right) = 2f(s) + 5f(t), \tag{2.9}$$

for all $s, t \in X$, which is a combination of Cauchy's additive functional equation

$$f(s+t) = f(s) + f(t)$$

and the quadratic functional equation (2.3).

We note that $f(s) = c\|s\|^2,\ c \in \mathbb{R}$ is a solution of (2.9) if X is an inner product space. For, substituting $f(s) = c\|s\|^2$ in (2.9), we get

$$\begin{aligned}
&c\|s+t\|^2 + 2c\left\|\frac{s}{2}+t\right\|^2 + 2c\left\|\frac{s}{2}-t\right\|^2 = 2c\|s\|^2 + 5c\|t\|^2\\
\text{i.e.,}\quad &c\|s\|^2 + 2c(s,t) + c\|t\|^2 + 2c\left\{2\left\|\frac{s}{2}\right\|^2 + 2\|t\|^2\right\} = 2c\|s\|^2 + 5c\|t\|^2\\
\text{i.e.,}\quad &2c\|s\|^2 + 2c(s,t) + 5c\|t\|^2 = 2\|s\|^2 + 5\|t\|^2,
\end{aligned}$$

which holds true if the inner product of s and t, $(s,t) = 0$.

In light of this, we call the functional equation (2.9), an orthogonally quadratic equation and its solution, an orthogonally quadratic function.

We now obtain a stability result for the functional equation (2.9) using the direct method.

Theorem 2.4. *Let X be an inner product space and Y be a real Banach space. Let $f : X \longrightarrow Y$ be a function such that there exists $\Omega \geq 0$ and $0 \leq \gamma < 1$ with*

$$\begin{aligned}
&\left\|f(s+t) + 2f\left(\frac{s}{2}+t\right) + 2f\left(\frac{s}{2}-t\right) - 2f(s) - 5f(t)\right\|\\
&\qquad\leq \Omega\left(\|s\|^\gamma + \|t\|^\gamma\right)
\end{aligned}\tag{2.10}$$

for all $s, t \in X$. Then there exists a unique orthogonally quadratic even function $M : X \longrightarrow Y$ with

$$\|f(s) - M(s)\| \leq \frac{\Omega}{1-2^{\gamma-2}}\|s\|^\gamma, \qquad \forall\ s \in X. \tag{2.11}$$

Lemma 2.1. *With the above given conditions in Theorem 2.4, we have*

$$\left\|\frac{1}{2^{2n}}f(2^n s) - f(s)\right\| \leq \frac{\Omega}{1-2^{\gamma-2}}\|s\|^\gamma \sum_{k=1}^{n} 2^{k(\gamma-2)} \tag{2.12}$$

for all $s \in X$.

Putting $t = 0$ in (2.10), we get

$$\left\|\frac{1}{2^2}f(2s) - f(s)\right\| \leq \Omega\|s\|^\gamma 2^{\gamma-2} \qquad \forall\ s \in X.$$

Thus (2.12) is true for $n = 1$, and assume that it is true for some $n \in \mathbb{N}$. Then

$$\begin{aligned}\left\|\frac{1}{2^{2(n+1)}}f(2^{(n+1)}s) - f(s)\right\| &\leq \frac{1}{2^{2n}}\left\|\frac{1}{2^2}f(2(2^n s)) - f(2^n s)\right\| + \left\|\frac{1}{2^{2n}}f(2^n s) - f(s)\right\| \\ &\leq \frac{1}{2^{2n}}\Omega\|s\|^\gamma 2^{\gamma-2} + \frac{\Omega}{1-2^{\gamma-2}}\|s\|^\gamma \sum_{k=1}^{n} 2^{k(\gamma-2)} \\ &= \frac{\Omega}{1-2^{\gamma-2}}\|s\|^\gamma \sum_{k=1}^{n} 2^{k(\gamma-2)}.\end{aligned}$$

Thus (2.12) is valid for all $n \in \mathbb{N}$. $\square$

Proof. (**Proof of Theorem 2.4**) First, we observe that

$$\sum_{k=1}^{n} 2^{k(\gamma-2)} = \frac{1-2^{(n+1)(\gamma-2)}}{1-2^{\gamma-2}} \leq \frac{1}{2^{\gamma-2}}, \qquad \text{since } 2^{\gamma-2} < 1.$$

Hence from Lemma 2.1, we have

$$\left\|\frac{1}{2^{2n}}f(2^n s) - f(s)\right\| \leq \frac{\Omega}{2^{\gamma-2}}\|s\|^\gamma \qquad \forall\, s \in X. \tag{2.13}$$

Now, for $m > n > 0$, we have from (2.13)

$$\begin{aligned}\left\|\frac{1}{2^{2m}}f(2^m s) - \frac{1}{2^{2n}}f(2^n s)\right\| &= \frac{1}{2^{2n}}\left\|\frac{1}{2^{2(m-n)}}f(2^{m-n}2^n s) - f(2^n s)\right\| \\ &\leq \frac{1}{2^{2n}}\frac{\Omega}{2^{\gamma-2}}\left\|2^n s\right\|^\gamma = \Omega\frac{2^{n(\gamma-2)}}{1-2^{\gamma-2}}\left\|s\right\|^\gamma.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\left\|\frac{1}{2^{2m}}f(2^m s) - \frac{1}{2^{2n}}f(2^n s)\right\| = \lim_{n\to\infty}\frac{2^{n(\gamma-2)}}{1-2^{(\gamma-2)}}\Omega\|s\|^\gamma = 0.$$

Since Y is a Banach space, the Cauchy sequence $\left\{\frac{1}{2^{2n}}f(2^n s)\right\}$ converges for every $s \in X$. We define the function $M : X \longrightarrow Y$ by

$$M(s) = \lim_{n\to\infty}\frac{1}{2^{2n}}f(2^n s), \qquad s \in X.$$

Substituting s and t by $2^n s$ and $2^n t$, respectively in (2.10), we have

$$\begin{aligned}&\left\|f(2^n(s+t)) + 2f\left(2^n(\frac{s}{2}+t)\right) + 2f\left(2^n(\frac{s}{2}-t)\right) - 2f(2^n s) - 5f(2^n t)\right\| \\ &\leq \Omega\left(\|2^n s\|^\gamma + \|2^n t\|^\gamma\right) = \Omega 2^{n\gamma}\left(\|s\|^\gamma + \|t\|^\gamma\right),\end{aligned}$$

or,

$$\lim_{n\to\infty}\frac{1}{2^{2n}}\left\|f(2^n(s+t))+2f\left(2^n(\frac{s}{2}+t)\right)+2f\left(2^n(\frac{s}{2}-t)\right)-2f(2^ns)-5f(2^nt)\right\|$$
$$\leq \lim_{n\to\infty}\Omega 2^{n(\gamma-2)}\left(\|s\|^\gamma+\|t\|^\gamma\right)=0.$$

By the continuity of the norm function, we have

$$M(s+t)+2M\left(\frac{s}{2}+t\right)+2M\left(\frac{s}{2}-t\right)=2M(s)+5M(t), \tag{2.14}$$

showing that M is an orthogonally quadratic function.

Putting $s=t=0$ in (2.14), we get $M(0)=0$. And the substitution of $s=0$ in (2.14) gives $M(-t)=M(t)$ for all $t\in X$, showing that M is an even function.

Also, from (2.13) it follows that

$$\lim_{n\to\infty}\left\|\frac{1}{2^{2n}}f(2^ns)-f(s)\right\|\leq \lim_{n\to\infty}\frac{\Omega}{2^{\gamma-2}}\|s\|^\gamma=\frac{\Omega}{2^{\gamma-2}}\|s\|^\gamma,$$

or,

$$\left\|M(s)-f(s)\right\|\leq \varepsilon\|s\|^\gamma, \qquad \text{where } \varepsilon=\frac{\Omega}{1-2^{\gamma-2}},$$

which proves (2.11).

To show the uniqueness of M, suppose that there exist $\xi : X\longrightarrow Y$ with $\xi(s)\neq M(s)$ for all $s\in X$ such that for some $\varepsilon_1\geq 0$ and $0\leq\gamma'<1$,

$$\|\xi(s)-f(s)\|\leq\varepsilon_1\|s\|^{\gamma'}.$$

But then

$$\|M(s)-\xi(s)\|\leq\|M(s)-f(s)\|+\|f(s)-\xi(s)\|\leq\varepsilon\|s\|^\gamma+\varepsilon_1\|s\|^{\gamma'}.$$

Therefore,

$$\|M(s)-\xi(s)\|=\frac{1}{2^{2n}}\|M(2^ns)-\xi(2^ns)\|\leq\frac{1}{2^{2n}}\left(\|2^ns\|^\gamma+\|2^ns\|^{\gamma'}\right)$$
$$\leq 2^{\gamma-2}\left(\|s\|^\gamma+\|s\|^{\gamma'}\right).$$

We note that if f is a solution of (2.9), then $f(2^ns)=\frac{1}{2^{2n}}f(s)$ for all $n\in\mathbb{N}$. To see this, we put $t=0$ in (2.9) to get $f(2s)=2^2f(s)$, for all $s\in X$ and the result follows by induction.

Thus,

$$\lim_{n\to\infty}\|M(s)-\xi(s)\|=0 \qquad \forall\, s\in X,$$

and hence $M(s)=\xi(s)$ for all $s\in X$, as required. □

Conclusion

In this paper, the Hyers-Ulam stability of some functional equations, which are combinations of different known functional equations, are discussed for mappings defined from a normed space to a Banach space. Similar problems can also be investigated in case of other type of functional equations.

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