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ON SOME SANDWICH THEOREMS OF ANALYTIC FUNCTIONS INVOLVING NOOR-SALAGEAN OPERATOR

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ABSTRACT. The main object of this paper is to find sufficient conditions for certain normalized analytic functions to satisfy sandwich conditions. We obtain some subordination and superordination results involving by Noor-Salagean operator $DI^{m}_{\lambda,n}$.

1. INTRODUCTION

Let $H = H(U)$ be the class of analytic functions in the open unit disk $U = \{z \in \mathrm{C} : |z| < 1\}$. For n a positive integer and $a \in \mathrm{C}$. Let $H[a, n]$ be the subclass of H consisting of functions of the form:

$$f(z) = a + a_n\, z^n + a_{n+1}\, z^{n+1} + ... \qquad (a \in \mathbb{C}).$$

Also, let S be the subclass of H consisting of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k\, z^k. \tag{1.1}$$

Let $f,\ g \in H$. The function f is said to be subordinate to g, or g is said to be subordinate to f, if there exists a Schwarz function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $z \in U$, such that $f(z) = g(w(z))$, In such a case we write $f \prec g$

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or $f(z) \prec g(z) \ (z \in U)$. If g is univalent function in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let $p, \ h \in H$ and $\psi(r, \delta, t, z) : \mathbb{C}^3 \times U \to \mathbb{C}$. If p and $\psi(p(z), zp^{'}(z), \ z^2 p^{''}(z); z)$ are univalent functions in U and if p satisfies the second-order differential superordination.

$$h(z) \prec \psi(p(z), zp^{'}(z), z^2 p^{''}(z); z), \tag{1.2}$$

then p is called a solution of the differential superordination (1.2).

(If f is subordinate to g , then g is superordinate to f.) An analytic function q is called a subordinant of (1.2), if $q \prec p$ for all the functions p satisfying (1.2).

An univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all the subordinants q of (1.2) is called the best subordinant. Miller and Mocanu [11] have obtained conditions on the functions h, q and ψ for which the following implication holds:

$$h(z) \prec \psi(p(z), zp^{'}(z), z^2 p^{''}(z); z) \to q(z) \prec p(z).$$

Let f given by (1.1) and g is defined by

$$g(z) = z + \sum_{k=2}^{\infty} b_k \ z^k \qquad z \in U.$$

Then the Hadamard product (or convolution) $f * g$ of the functions f and g is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k \ z^k \qquad z \in U.$$

Definition 1.1. *[4] For $f \in S$, $\lambda \geq 0$ and $m \in N$, the operator D_λ^m is defined by:*

$D_\lambda^m : S \to S,$

$D_\lambda^0 f(z) = f(z),$

$D_\lambda^1 f(z) = (1-\lambda) f(z) - \lambda z f^{'}(z) = D_\lambda f(z),$

$\vdots$

$D_\lambda^m f(z) = (1-\lambda) \ D_\lambda^{m-1} f(z) + \lambda z \left(D_\lambda^{m-1} f(z)\right)^{'} = D_\lambda \left(D_\lambda^{m-1} f(z)\right), (z \in U).$

$\quad = z + \sum_{k=2}^{\infty} \left[1 + (k-1)\lambda\right]^m a_k z^k.$

The operator D_λ^m is called the generalised Salagean operator.

Definition 1.2. *[13] Let $f_n(z) = \frac{z}{(1-z)^{n+1}}$ and $f_n^{-1}(z)$ be defined as*

$$f_n(z) * f_n^{-1}(z) = \frac{z}{(1-z)}, \tag{1.3}$$

then the operator $I_n : S \to S$ *is defined as*

$$I_n f(z) = f_n^{-1}(z) * f(z) = \left[\frac{z}{(1-z)^{n+1}}\right]^{-1} * f(z),$$

which is called Noor integral operator.

Using (1.3), we have the following recursive relation for Noor integral operator.

$$z\left(I_{n+1} f(z)\right)' = (n+1)\ I_n f(z) - n\ I_{n+1} f(z).$$

By taking convolution of the Noor integral operator and generalized Salagean operator, Fayyaz and Noor defined a new operator named Noor-Salagean operator as follows:

Definition 1.3. *[10] For* $f \in S$ *,* $\lambda \geq 0$ *and* $m \in N$*, the operator*
$DI_{\lambda,n}^m : S \to S$ *is defined as*
$DI_{\lambda,n}^m f(z) = (D_\lambda^m * I_n)\ f(z),$
for $z \in U$ *and each nonnegative integer* m, n*.*

By simple computations, we can have the following relations. For $m, n \in$ Nand $\lambda \geq 0$, we have

$$DI_{\lambda,n}^{m+1} f(z) = (1-\lambda)\ DI_{\lambda,n}^m f(z) + \lambda z\left(DI_{\lambda,n}^m f(z)\right)',$$

$$z\left(DI_{\lambda,n+1}^m f(z)\right)' = (n+1)\ DI_{\lambda,n}^m f(z) - n\ DI_{\lambda,n+1}^m f(z). \tag{1.4}$$

Ali et al. [2] obtained sufficient conditions for certain normalized analytic functions to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$. Also, Al-Ziadi and Atshan [3] obtained differential sandwich theorems for analytic functions associated with convolution structure. Atshan et. al [3–8] studied sandwich theorems for another conditions. Al-Ameedee et. al [1] obtained sandwich results for certain classes of analytic functions.

The main object of the present paper is to find sufficient conditions for certain normalized analytic functions f to satisfy:

$$q_1(z) \prec \left(\frac{DI^m_{\lambda,n+1}f(z)}{z}\right)^{\delta} \prec q_2(z),$$

and

$$q_1(z) \prec \left(\frac{tDI^m_{\lambda,n+1}f(z) + (1-t)\, DI^m_{\lambda,n}f(z)}{z}\right)^{\delta} \prec q_2(z)$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$.

2. Preliminaries

In order to prove our subordination and superordination results, we need the following definition and lemmas.

Definition 2.1. *[11] Denote by Q the set of all functions f that are analytic and injective on $\overline{U}\backslash E(f)$, where*

$$E(f) = \{\xi \in \partial U : \lim_{z\to\xi} f(z) = \infty\}$$

and are such that $f'(\xi) \neq 0$ for $\xi \in \partial U \backslash E(f)$.

Lemma 2.1. *[11] Let q be univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\phi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that*

(i) $Q(z)$ is starlike univalent in U,

(ii) $Re\{\frac{zh'(z)}{Q(z)}\} > 0$ for $z \in U$.

If p is analytic in U with $p(0) = q(0)$, $p(U) \subset D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{2.1}$$

then $p \prec q$ and q is the best dominant of (2.1).

Lemma 2.2. *[12] Let q be convex univalent in function in U and let $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}/\{0\}$ with*

$$Re\{1 + \frac{zq''(z)}{q'(z)}\} > max\{0, -Re\left(\frac{\alpha}{\beta}\right)\}.$$

If p is analytic in U, and

$$\alpha p(z) + \beta zp'(z) \prec \alpha q(z) + \beta zq'(z), \tag{2.2}$$

then $p \prec q$ *and* q *is the best dominant of (2.2).*

Lemma 2.3. *[12] Let* q *be convex univalent in* U *and let* $\beta \in \mathbb{C}$*, further assume that* $Re\,(\beta) > 0$ *. If* $P \in H\,[q\,(0)] \cap Q$ *and* $p\,(z) + \beta z p'\,(z)$ *is univalent in* U*, then*

$$q\,(z) + \beta z q^{'}\,(z) \prec p\,(z) + \beta z p^{'}\,(z)\,, \tag{2.3}$$

which implies that $q \prec p$ *and* q *is the best subordinant of (2.3).*

Lemma 2.4. *[9] Let* q *be convex univalent in the unit disk* U *and let* θ *and* ϕ *be analytic in domain* D *containing* $q\,(U)$*. Suppose that*

(i) $Re\left\{\frac{\theta^{'}(q(z))}{\phi(q(z))}\right\} > 0$ *for* $z \in U$*,*

(ii) $Q\,(z) = zq^{'}\,(z)\,\phi\,(q\,(z))$ *is starlike univalent in* U.

If $p \in H\,[q\,(0)\,, 1] \cap Q$*, with* $p\,(\cup) \subset D, \theta\,(p\,(z)) + zp^{'}\,(z)\,\phi p\,(z)$ *is univalent in* U *and*

$$\theta\,(q\,(z)) + zq^{'}\,(z)\,\phi\,(q\,(z)) \prec \theta\,(p\,(z)) + zp^{'}\,(z)\,\phi\,(p\,(z))\,, \tag{2.4}$$

then $q \prec p$ *and* q *is the best subordination of (2.4).*

3. Subordination Results

Theorem 3.1. *Let* q *be convex univalent function in* U *with* $q\,(0) = 1,\ 0 \neq \Psi \in \mathbb{C}, \delta > 0$ *and suppose that* q *satisfies:*

$$Re\left\{1 + \frac{zq''\,(z)}{q'\,(z)}\right\} \geq \max\left\{0, -Re\left(\frac{\delta}{\Psi}\right)\right\}. \tag{3.1}$$

If $f \in S$ *satisfies the subordination*

$$(1 - \Psi\,(n+1)) \quad \left(\frac{DI^m_{\lambda,n+1}f(z)}{z}\right)^{\delta} + \Psi\,(n+1)\left(\frac{DI^m_{\lambda,n+1}f(z)}{z}\right)^{\delta}\left(\frac{DI^m_{\lambda,n}f(z)}{DI^m_{\lambda,n+1}f(z)}\right) \prec q\,(z) + \tfrac{\Psi}{\delta}zq^{'}\,(z)\,, \tag{3.2}$$

then

$$\left(\frac{DI^m_{\lambda,n+1}f\,(z)}{z}\right)^{\delta} \prec q\,(z)\,, \tag{3.3}$$

and q *is the best dominant of (3.2).*

Proof. Define the function p by

$$p\,(z) = \left(\frac{DI^m_{\lambda,n+1}f\,(z)}{z}\right)^{\delta}. \tag{3.4}$$

Differentiating (3.4) with respect to z logarithmically, we get

$$\frac{zp'(z)}{p(z)}=\delta\left(\frac{z\left(DI^m_{\lambda,n+1}f(z)\right)'}{DI^m_{\lambda,n+1}f(z)}-1\right).$$

Now, in view of (1.4), we obtain the following subordination

$$\frac{zp'(z)}{p(z)}=\delta\left(n\left(\frac{DI^m_{\lambda,n}f(z)}{DI^m_{\lambda,n+1}f(z)}-1\right)+\left(\frac{DI^m_{\lambda,n}f(z)}{DI^m_{\lambda,n+1}f(z)}-1\right)\right),$$

therefore,

$$\frac{zp'(z)}{\delta}=\left(\frac{DI^m_{\lambda,n+1}f(z)}{z}\right)^{\delta}\left(n\left(\frac{DI^m_{\lambda,n}f(z)}{DI^m_{\lambda,n+1}f(z)}-1\right)+\left(\frac{DI^m_{\lambda,n}f(z)}{DI^m_{\lambda,n+1}f(z)}-1\right)\right).$$

The subordination (3.2) from the hypothesis becomes

$$p(z)+\frac{\Psi}{\delta}zp'(z)\prec q(z)+\frac{\Psi}{\delta}zq'(z).$$

An application of Lemma 2.2 with $\beta=\frac{\Psi}{\delta}$ and $\alpha=1$, we obtain (3.3). □

Putting $q(z)=\left(\frac{1+z}{1-z}\right)$ in Theorem 3.1, we obtain the following:

Corollary 3.1. *Let* $0\neq\Psi\in\mathbb{C},\delta>0$ *and*

$$Re\left\{1+\frac{2z}{1-z}\right\}>\max\left\{0,-Re\left(\frac{\delta}{\Psi}\right)\right\}.$$

If $f\in S$ *satisfies the subordination*

$$(1-\Psi(n+1))\left(\frac{DI^m_{\lambda,n+1}f(z)}{z}\right)^{\delta}+\Psi(n+1)\left(\frac{DI^m_{\lambda,n+1}f(z)}{z}\right)^{\delta}\left(\frac{DI^m_{\lambda,n}f(z)}{DI^m_{\lambda,n+1}f(z)}\right)\prec\left(\frac{1-z^2+2\frac{\Psi}{\delta}z}{(1-z)^2}\right),$$

then

$$\left(\frac{DI^m_{\lambda,n+1}f(z)}{z}\right)^{\delta}\prec\left(\frac{1+z}{1-z}\right),$$

and $q(z)=\left(\frac{1+z}{1-z}\right)$ *is the best dominant.*

Theorem 3.2. *Let* q *be convex univalent function in* U *with* $q(0)=1,q(z)\neq 0\,(z\in U)$ *and assume that* q *satisfies*

$$Re\left\{1-\frac{\delta}{\Psi}+\frac{zq''(z)}{q'(z)}\right\}>0, \tag{3.5}$$

where $\Psi\in\mathbb{C}/\{0\},\delta>0$ *and* $z\in U$.

Suppose that $-\Psi zq^{'}(z)$ *is starlike univalent in* U*, if* $f \in S$ *satisfies:*

$$\phi(\delta, n, \lambda, m, \Psi; z) \prec \delta q(z) - \Psi zq^{'}(z), \tag{3.6}$$

where

$$\phi(\delta, m, n, \lambda, \Psi; z) = \delta\left(\frac{tDI_{\lambda,n+1}^{m}f(z) + (1-t)DI_{\lambda,n}^{m}f(z)}{z}\right)^{\delta} \tag{3.7}$$

$$- \quad \delta\Psi\left(\frac{tDI_{\lambda,n+1}^{m}f(z) + (1-t)DI_{\lambda,n}^{m}f(z)}{z}\right)^{\delta} \left(\frac{tDI_{\lambda,n}^{m}f(z) + (1-t)DI_{\lambda,n-1}^{m}f(z)}{tDI_{\lambda,n+1}^{m}f(z) + (1-t)DI_{\lambda,n}^{m}f(z)} - 1\right), \tag{3.8}$$

then

$$\left(\frac{tDI_{\lambda,n+1}^{m}f(z) + (1-t)DI_{\lambda,n}^{m}f(z)}{z}\right)^{\delta} \prec q(z),$$

and $q(z)$ *is the best dominant of (3.6).*

Proof. Define the function p by

$$p(z) = \left(\frac{tDI_{\lambda,n+1}^{m}f(z) + (1-t)DI_{\lambda,n}^{m}f(z)}{z}\right)^{\delta}, \tag{3.9}$$

by setting :

$$\theta(w) = \delta w \text{ and } \phi(w) = -\Psi, w \neq 0.$$

We see that $\theta(w)$ is analytic in $\mathbb{C}$, $\phi(w)$ is analytic in $\mathbb{C}/\{0\}$ and that $\phi(w) \neq 0, w \in \mathbb{C}/\{0\}$.

Also, we get

$$Q(z) = zq^{'}(z)\phi q(z) = -\Psi zq^{'}(z),$$

and

$$h(z) = \theta q(z) + Q(z) = \delta q(z) - \Psi zq^{'}(z).$$

It is clear that $Q(z)$ is starlike univalent in U,

$$Re\{\frac{zh^{'}(z)}{Q(z)}\} = Re\{1 - \frac{\delta}{\Psi} + \frac{zq^{''}(z)}{q^{'}(z)}\} > 0.$$

By a straightforword computation, we obtain

$$\delta p(z) - \Psi zp^{'}(z) = \phi(\delta, n, \lambda, m, \Psi; z), \tag{3.10}$$

where $\phi(\delta, n, \lambda, m, \Psi; z)$ is given by (3.7). From (3.6) and (3.10), we have

$$\delta p(z) - \Psi z\, p^{'}(z) \prec \delta q(z) - \Psi z\, q^{'}(z).$$

Therefore, by Lemma 2.1, we get $p(z) \prec q(z)$. By using (3.9), we obtain the result. □

Putting $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem 3.1, we obtain the following corollary:

Corollary 3.2. *Let* $-1 \leq B < A \leq 1$ *and*

$$Re\{1 - \frac{\delta}{\Psi} + \frac{z2B}{(1+Bz)}\} > 0,$$

where $\Psi \in \mathbb{C}/\{0\}$ *and* $z \in U$, *if* $f \in S$ *satisfies*

$$\phi(\delta, n, \lambda, m, \Psi; z) \prec \left(\delta\left(\frac{1+Az}{1+Bz}\right) - \Psi z \frac{A-B}{(1+Bz)^2}\right),$$

and $\phi(\delta, n, \lambda, m, \Psi; z)$ *is given by (3.7),*

$$\left(\frac{tDI^m_{\lambda,n+1}f(z) + (1-t)\,DI^m_{\lambda,n}f(z)}{z}\right)^{\delta} \prec \frac{1+Az}{1+Bz}$$

and $q(z) = \frac{1+Az}{1+Bz}$ *is the best dominant.*

4. Superordination results

Theorem 4.1. *Let* q *be convex univalent function in* U *with* $q(0) = 1, \delta > 0$ *and* $Re\{\Psi\} > 0$. *Let* $f \in S$ *satisfies:*

$$\left(\frac{DI^m_{\lambda,n+1}f(z)}{z}\right)^{\delta} \in H[q(0), 1] \cap Q,$$

and

$$(1 - \Psi(n+1))\left(\frac{DI^m_{\lambda,n+1}f(z)}{z}\right)^{\delta} + \Psi(n+1)\left(\frac{DI^m_{\lambda,n+1}f(z)}{z}\right)^{\delta}\left(\frac{DI^m_{\lambda,n}f(z)}{DI^m_{\lambda,n+1}f(z)}\right),$$

be univalent in U. *If*

$$q(z) + \frac{\Psi}{\delta} zq'(z) \prec (1 - \Psi(n+1))\left(\frac{DI^m_{\lambda,n+1}f(z)}{z}\right)^{\delta} + \Psi(n+1)\left(\frac{DI^m_{\lambda,n+1}f(z)}{z}\right)^{\delta}\left(\frac{DI^m_{\lambda,n}f(z)}{DI^m_{\lambda,n+1}f(z)}\right), \tag{4.1}$$

then

$$q(z) \prec \left(\frac{DI^m_{\lambda,n+1}f(z)}{z}\right)^{\delta},$$

and q is the best subordinant of (4.1).

Proof. Define the function p by

$$p(z)=\left(\frac{DI^{m}_{\lambda,n+1}f(z)}{z}\right)^{\delta}. \tag{4.2}$$

Differentiating (4.2) with respect to z logarithmically, we get

$$\frac{zp'(z)}{p(z)}=\delta\left(\frac{z\left(DI^{m}_{\lambda,n+1}f(z)\right)'}{DI^{m}_{\lambda,n+1}f(z)}-1\right). \tag{4.3}$$

After some computations and using (1.4) , from (4.3), we obtain

$$\begin{aligned}(1-\Psi(n+1))\left(\tfrac{DI^{m}_{\lambda,n+1}f(z)}{z}\right)^{\delta}+\Psi(n+1)\left(\tfrac{DI^{m}_{\lambda,n+1}f(z)}{z}\right)^{\delta}\left(\tfrac{DI^{m}_{\lambda,n}f(z)}{DI^{m}_{\lambda,n+1}f(z)}\right)\\ =p(z)+\tfrac{\Psi}{\delta}zp'(z),\end{aligned}$$

and now , by using Lemma 2.3, we get the desired result. □

Putting $q(z)=\frac{1+z}{1-z}$ in Theorem 4.1, we obtain the following corollary:

Corollary 4.1. *Let $\delta>0$ and $Re\{\Psi\}>0$. If $f\in S$ satisfies:*

$$\left(\frac{DI^{m}_{\lambda,n+1}f(z)}{z}\right)^{\delta}\in H[q(0),1]\cap Q,$$

and

$$(1-\Psi(n+1))\left(\frac{DI^{m}_{\lambda,n+1}f(z)}{z}\right)^{\delta}+\Psi(n+1)\left(\frac{DI^{m}_{\lambda,n+1}f(z)}{z}\right)^{\delta}\left(\frac{DI^{m}_{\lambda,n}f(z)}{DI^{m}_{\lambda,n+1}f(z)}\right),$$

be univalent in U. If

$$\begin{aligned}\left(\frac{1-z^{2}+2\frac{\Psi}{\delta}z}{(1-z)^{2}}\right) \prec\ & (1-\Psi(n+1))\left(\frac{DI^{m}_{\lambda,n+1}f(z)}{z}\right)^{\delta}\\ & +\Psi(n+1)\left(\frac{DI^{m}_{\lambda,n+1}f(z)}{z}\right)^{\delta}\left(\frac{DI^{m}_{\lambda,n}f(z)}{DI^{m}_{\lambda,n+1}f(z)}\right),\end{aligned}$$

then

$$\left(\frac{1+z}{1-z}\right)\prec\left(\frac{DI^{m}_{\lambda,n+1}f(z)}{z}\right)^{\delta},$$

and

$$q(z)=\frac{1+z}{1-z}$$

is the best subordinant.

Theorem 4.2. *Let q be convex univalent function in U with $q(0)=1$, and assume that q satisfies:*

$$Re\{\frac{-\delta q^{'}(z)}{\Psi}\}>0, \tag{4.4}$$

where $\eta \in \mathbb{C}/\{0\}$ and $z \in U$. Suppose that $-\Psi zq^{'}(z)$ is starlike univalent function in U, let $f \in S$ satisfies:

$$\left(\frac{tDI^{m}_{\lambda,n+1}f(z)+(1-t)DI^{m}_{\lambda,n}f(z)}{z}\right) \in H[q(0),1] \cap Q,$$

and $\phi(\delta,m,n,\lambda,\Psi;z)$ is univalent function in U, where $\phi(\delta,m,n,\lambda,\Psi;z)$ is given by (3.7). If

$$\delta q(z)-\Psi zq^{'}(z) \prec \phi(\delta,n,\lambda,m,\Psi;z), \tag{4.5}$$

then

$$q(z) \prec \left(\frac{tDI^{m}_{\lambda,n+1}f(z)+(1-t)DI^{m}_{\lambda,n}f(z)}{z}\right)^{\delta},$$

and q is the best subordinant of (4.5).

Proof. Define the function p by

$$p(z)=\left(\frac{tDI^{m}_{\lambda,n+1}f(z)+(1-t)DI^{m}_{\lambda,n}f(z)}{z}\right)^{\delta}, \tag{4.6}$$

by setting $\theta(w)=\delta w$ and $\phi(w)=-\Psi$, $w \neq 0$, we see that $\theta(w)$ is analytic in $\mathbb{C}$, $\phi(w)$ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$. Also we get

$$Q(z)=zq^{'}(z)\phi q(z)=-\Psi zq^{'}(z).$$

It is clear that $Q(z)$ is starlike univalent function in U,

$$Re\left\{\frac{\theta^{'}(q(z))}{\phi(q(z))}\right\}=Re\left\{\frac{-\delta q^{'}(z)}{\Psi}\right\} >0 .$$

By a straightforword computation, we obtain

$$\phi(\delta,n,\lambda,m,\Psi;z)=\delta p(z)-\Psi zp^{'}(z), \tag{4.7}$$

where

$$\phi(\delta,n,\lambda,m,\Psi;z)$$

is given by (3.7). From (4.5) and (4.7), we have

$$\delta q(z)-\Psi zq^{'}(z) \prec \delta p(z)-\Psi p^{'}(z).$$

Therefore, by Lemma 2.4, we get $q(z) \prec p(z)$. By using (4.6), we obtain the result. □

5. Sandwich Results

Concluding the results of differential subordination and superordination, we arrive at the following "sandwich results".

Theorem 5.1. *Let q_1 be convex univalent function in U with $q_1(0) = 1$, $Re\{\Psi\} > 0$ and let q_2 be univalent function in $U, q_2(0) = 1$ and satisfies (3.1), let $f \in A$ satisfies:*

$$\left(\frac{DI^m_{\lambda,n+1} f(z)}{z} \right)^{\delta} \in H[1,1] \cap Q,$$

and

$$(1 - \Psi(n+1)) \left(\frac{DI^m_{\lambda,n+1} f(z)}{z} \right)^{\delta} + \Psi(n+1) \left(\frac{DI^m_{\lambda,n+1} f(z)}{z} \right)^{\delta} \left(\frac{DI^m_{\lambda,n} f(z)}{DI^m_{\lambda,n+1} f(z)} \right),$$

be univalent in U. If

$$q_1(z) + \frac{\Psi}{\delta} z q_1'(z) \prec (1 - \Psi(n+1)) \left(\frac{DI^m_{\lambda,n+1} f(z)}{z} \right)^{\delta}$$

$$+\Psi(c+1) \left(\frac{DI^m_{\lambda,n+1} f(z)}{z} \right)^{\delta} \left(\frac{DI^m_{\lambda,n} f(z)}{DI^m_{\lambda,n+1} f(z)} \right) \prec q_2(z) + \frac{\Psi}{\delta} z\, q_2'(z)$$

then

$$q_1(z) \prec \left(\frac{DI^m_{\lambda,n+1} f(z)}{z} \right)^{\delta} \prec q_2(z),$$

and q_1 and q_2 are respectively , the best subordinant and the best dominant.

Theorem 5.2. *Let q_1 be convex univalent function in U with $q_1(0) = 1$, and satisfies (4.4), let q_2 be univalent function in U, $q_2(0) = 1$, satisfies (3.5), let $f \in S$ satisfies*

$$\left(\frac{tDI^m_{\lambda,n+1} f(z) + (1-t)\, tDI^m_{\lambda,n} f(z)}{z} \right)^{\delta} \in H[1,1] \cap Q,$$

and

$$\phi(\delta, n, \lambda, m, \Psi; z)$$

is univalent in U *, where* $\phi(\delta, m, n, \lambda, \Psi; z)$ *is given by (3.7) . If* $\delta q_1(z) - \Psi z q_1^{'}(z) \prec \phi(\delta, n, \lambda, m, \Psi; z) \prec \delta q_2(z) - \Psi z q_2^{'}(z)$ *, then*

$$q_1(z) \prec \left(\frac{tDI_{\lambda,n+1}^{m} f(z) + (1-t) DI_{\lambda,n}^{m} f(z)}{z}\right)^{\delta} \prec q_2(z),$$

and $q_1 and\ q_2$ *are respectively , the best subordinant and the best dominant.*

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