Advances in Mathematics: Scientific Journal **9** (2020), no.10, 8469–8474
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.10.75

NOTE ON COEFFICIENTS FOR THE CLASS $\mathcal{U}(\lambda)$

EL MOCTAR OULD BEIBA

ABSTRACT. Let $\mathcal{A}$ be the class of analytic function in the unit disc $\mathbb{D} = \{z \in \mathbb{C},\ |z| < 1\}$, normalized by $f^{'}(0) - 1 = f(0) = 0$. For $0 < \lambda \leq 1$, let $\mathcal{U}(\lambda) = \{f \in \mathcal{A},\ \left|\left(\frac{z}{f(z)}\right)^2 f'(z) - 1\right| < \lambda,\ z \in \mathbb{D}\}$. The aim of the present article is to present some coefficients results for the class $\mathcal{U}(\lambda)$. We give a new proof for a known results on the second coefficient a_2 (Corollary 2.1) and for the Koebe domain (Corollary 2.3) for the class $\mathcal{U}(\lambda)$. At the end of the note a problem.is proposed.

1. INTRODUCTION

Let $\mathcal{A}$ be the class of analytic function in the unit disc $\mathbb{D} = \{z \in \mathbb{C}\ |z| < 1\}$, normalized by $f^{'}(0) - 1 = f(0) = 0$. Let $\mathcal{S}$ denote the set of functions in $\mathcal{A}$ which are univalent. For general theory of univalent function see [2] and [5].

For $0 < \lambda \leq 1$, let

$$\mathcal{U}(\lambda) = \{f \in \mathcal{A},\ \left|U_f(z)\right| < \lambda,\ z \in \mathbb{D}\},$$

where $U_f(z) = \left(\frac{z}{f(z)}\right)^2 f'(z) - 1$, for $f \in \mathcal{A}$ with $\frac{f(z)}{z} \neq 0$ for $z \in \mathbb{D}$. In [6], Nunokawa and Ozaki proved the inclusion $\mathcal{U} := \mathcal{U}(1) \subset \mathcal{S}$. The classe $\mathcal{U}(\lambda)$ has been extensively studied in the recent years. For more details concerning this classe, see [4, 7, 8] and references therein. In [8], the authors proved, among other results, that the class $\mathcal{U}(\lambda)$ is preserved under some transformations such

2020 *Mathematics Subject Classification.* 30C45.
Key words and phrases. Univalent Functions, Koebe domain, Coefficient Problems.

as rotations, dilation, conjugation and omitted-value transformations. They also proposed the following conjecture:

Conjecture 1. *Suppose that* $f = z + \sum_{n=1}^{\infty} a_n z^n \in \mathcal{U}(\lambda)$ *for some* $0 < \lambda < 1$*, then*

$$|a_n| \leq \sum_{k=0}^{n-1} \lambda^k, \quad n \geq 2.$$

The conjecture 1 has been proved in the case $n = 2$ by Vasudevarao and Yanagihara ([10], Theorem 2.6). In [7], the authors proved (Theorem 3) the Conjecture 1, by using subordination techniques, for $n = 2, 3$ and 4 .

The following theorem is due to Rogosinski ([2], p.192):

Theorem 1.1 (Rogosinski's Theorem)**.** *Let* $f(z) = \sum_{n=1}^{\infty} a_n z^n$ *and* $g(z) = \sum_{n=1}^{\infty} b_n z^n$ *be analytic in* $\mathbb{D}$*, and suppose* $g \prec f$*. Then*

$$\sum_{k=1}^{n} |b_k|^2 \leq \sum_{k=1}^{n} |a_k|^2, \;\; n \geq 1.$$

Here, "$\prec$" is the symbol of subordination: $g \prec f$ means that

$$g(z) = f(w(z)), \;\; z \in \mathbb{D},$$

for some analytic function function w with $|w(z)| \leq |z|$ for $z \in \mathbb{D}$. For $f \in \mathcal{U}(\lambda)$, we have the following relation of subordination ([8], Theorem 4):

$$\frac{f(z)}{z} = 1 + \sum_{n=2}^{\infty} a_n z^{n-1} \prec \frac{1}{(1-z)(1-\lambda z)} = 1 + \sum_{n=1}^{\infty} \frac{1-\lambda^{n+1}}{1-\lambda} z^n, \; 0 < \lambda < 1. \tag{1.1}$$

2. RESULTS

Theorem 2.1. *Let* $f = z + \sum_{n=1}^{\infty} a_n z^n \in \mathcal{U}(\lambda)$ *for some* $0 < \lambda \leq 1$*. Then*

$$\sqrt{\sum_{k=2}^{n} |a_k|^2} \leq \sqrt{n-1} \sum_{k=0}^{n-1} \lambda^k. \tag{2.1}$$

Proof. For $\lambda = 1$, the theorem is a consequence, according to the fact that $\mathcal{U} \subset \mathcal{S}$, of the de Brange's theorem.

For $0 < \lambda < 1$, we have from (1.1), by applying the Rogosinskis's theorem,

$$\sum_{k=2}^{n} |a_k|^2 \leq \sum_{k=1}^{n-1} \left(\frac{1-\lambda^{k+1}}{1-\lambda}\right)^2. \tag{2.2}$$

Since $0 < \lambda < 1$, we have

$$1-\lambda^{k+1} \leq 1-\lambda^n, \quad \text{for } 1 \leq k \leq n-1. \tag{2.3}$$

Taking (2.3) in (2.2), we obtain

$$\sum_{k=2}^{n} |a_k|^2 \leq (n-1)\Big(\frac{1-\lambda^n}{1-\lambda}\Big)^2$$

which yields

$$\sqrt{\sum_{k=2}^{n} |a_k|^2} \leq \sqrt{n-1}\frac{1-\lambda^n}{1-\lambda} = \sqrt{n-1}\sum_{k=0}^{n-1}\lambda^k.$$

□

Remark 2.1. *The theorem 2.1 is a necessary condition for the Conjecture 1 to be true.*

Corollary 2.1. *Let f as in the Theorem 2.1. Then*

$$|a_n| \leq \sqrt{n-1}\sum_{k=0}^{n-1}\lambda^k, \quad \textit{for } n \geq 2. \tag{2.4}$$

In particular, we have

$$|a_2| \leq 1+\lambda. \tag{2.5}$$

Proof. (2.4) is an immediate consequence of (2.1). (2.5) follows from (2.4) applying to $n = 2$. □

As a second consequence of the Theorem 2.1, we have the following asymptotic behavior of the coefficient a_n:

Corollary 2.2. *Let $f = z + \sum_{n=1}^{\infty} a_n z^n \in \mathcal{U}(\lambda)$ for some $0 < \lambda < 1$. Then*

$$\lim_{n\to\infty} \frac{|a_n|}{n} = 0.$$

Proof. From (2.4), we have

$$\frac{|a_n|}{n} \leq \sqrt{\frac{n-1}{n^2}}\sum_{k=0}^{n-1}\lambda^k, \quad \text{for } n \geq 2 \tag{2.6}$$

Since $0 < \lambda < 1$, we have $\lim_{n\to\infty}\sum_{k=0}^{n-1}\lambda^k = \frac{1}{1-\lambda}$. Thus the limit of the right side of (2.6) is 0. This gives the desired result. □

Remark 2.2. *The Corollary 2.2 shows that all functions in $\mathcal{U}(\lambda)$ have the same Hayman number which is zero.*

In the following corollary we give, based on the upper bound 2.5 for the second coefficient, a new proof of a result due to Vasudevarao and Yanagihara ([10], Theorem 3.4) concerning the Koebe domain for the the class $\mathcal{U}(\lambda)$.

Corollary 2.3. *Let f be a function in $\mathcal{U}(\lambda)$. Then the range of f contains the open disc of center the origin and of radius $\frac{1}{2(1+\lambda)}$. The result is sharp.*

Proof. Let w be an omitted-value for f. Since $\mathcal{U}(\lambda)$ is preserved by omitted-value transformation, then the function

$$F(z) = \frac{wf(z)}{w - f(z)}, \quad z \in \mathbb{D}$$

belongs to $\mathcal{U}(\lambda)$. we have

$$(w - f(z))F(z) = wf(z), \quad z \in \mathbb{D} \tag{2.7}$$

Deriving twice the both sides of (2.7), we get, by a little calculation,

$$F''(z)(w - f(z)) - 2f'(z)F'(z) - F(z)f''(z) = wf''(z), \quad z \in \mathbb{D},$$

which gives

$$wF''(0) - 2 = wf''(0).$$

This yields

$$1 = w(a_2(f) - a_2(F)).$$

Hence we obtain

$$1 \leq \left|w\right|(\left|a_2(f)\right| + \left|a_2(F)\right|). \tag{2.8}$$

Now, since $f,\ F \in \mathcal{U}(\lambda)$, we obtain from (2.8) and (2.5) that

$$1 \leq 2\left|w\right|(1 + \lambda).$$

Hence, $\left|w\right| \geq \frac{1}{2(1+\lambda)}$. This gives the desired result.

For the sharpness of the result, let $f_\lambda = \frac{1}{(1-z)(1-\lambda z)}$. We have $f_\lambda \in \mathcal{U}(\lambda)$ and a little calculation shows that the solutions of the equation

$$f_\lambda(z) = -\frac{1}{2(1 + \lambda)}$$

are -1 and $-\frac{1}{\lambda}$ which are out the disc $\mathbb{D}$. Thus $-\frac{1}{2(1+\lambda)}$ is an omitted value of f_λ and hence the range of f_λ does not contain a radius greater than $\frac{1}{2(1+\lambda)}$. This shows the sharpness. □

Remark 2.3. *If w is an omitted-value for a function $f \in \mathcal{U}(\lambda)$ then, by the Corollary 2.3, we have $2(1+\lambda)|w| \geq 1$. Hence a necessary condition for the Conjecture 1 to be true for the coefficient a_n of f is that*

$$|a_n| \leq 2(1+\lambda)|w| \sum_{k=0}^{n-1} \lambda^k.$$

If $\lambda = 1$, the estimation above becomes

$$|a_n| \leq 4\,|w|\,n,$$

which is a particular case of the well known Littlewood's Conjecture (see [3], p. 897) which is now a consequence of the de Branges's Theorem [1].

The Remark 2.3 leads to propose the following problem which can be considered as the analogue, for the class $\mathcal{U}(\lambda)$, of the Littlewood's Conjecture for the class $\mathcal{S}$ of univalent functions.

Problem 1. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{U}(\lambda)$, $0 < \lambda \leq 1$. If w is an omitted-value of f, then*

$$|a_n| \leq 2(1+\lambda)|w| \sum_{k=0}^{n-1} \lambda^k, \quad n \geq 2.$$

REFERENCES

[1] L. DE BRANGES: *A proof of Bieberbach Conjecture*, Acta Mathematica, **154** (1985), 137-152.

[2] P. L. DUREN: *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, 259, Springer, Niew York-Berlin-Heidelberg-Tokyo, 1983.

[3] P. L. DUREN: *Coefficients of Univalent Functions*, Bull. Amer. Math. Soc., **83**(5) (1977), 891-911.

[4] R. FOURNIER, S. PONNUSAMY: *A Class of Locally Univalent Functions defined by a differential Inequality*, Complex Variable and Elliptic Equations, **52**(1) (2007), 1-8.

[5] A. W. GOODMAN: *Univalent Functions*, Vol.1 and 2, Mariner, Florida, 1983.

[6] M. NUNOKAWA, S. OZAKI: *The Schwarzian Derivative and Univalent Functions*, Proc. Amer. Math. Soc., **33**(2) (1972), 392-394.

[7] M. OBRADOVIĆ, S. PONNUSAMY, K. J. WIRTHS: *Logarithmic Coefficients and a Coefficient Conjecture for univalent functions*, Monatsh. Math., **185** (2018), 489-501.

[8] M. OBRADOVIĆ, S. PONNUSAMY, K. J. WIRTHS: *Geometric Studies in the Class $\mathcal{U}(\lambda)$*, Bull. Malays. Math. Sci. Soc., **39** (2016), 1259–1284.

[9] S. PONNUSAMY, K. J. WIRTHS: *Coefficient Problems On The Class $\mathcal{U}(\lambda)$*, Probl. Anal. Issues Anal., **7**(25-1) (2018), 87-103.

[10] A. VASUDEVARAO, H. YANAGIHARA: *On the Growth of Analytic Functions in the Class $\mathcal{U}(\lambda)$*, Comput. Methods Funct. Theory, **13** (2013), 613-634.

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCES
FACULTY OF SCIENCES AND TECHNIQUES
UNIVERSITY OF NOUAKCHOTT AL AASRIYA
P.O. BOX 5026, NOUAKCHOTT, MAURITANIA
Email address: elbeiba@yahoo.fr