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VAGUE GROUPS REDEFINED WITH RESPECT TO t-NORM

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ABSTRACT. The basic aim of the present research paper is to the studies of vague sets and vague groups. Some properties of vague groups based on vague sets are redefined with respect to t-norm and consequently some results in the form of vague groups with respect to t-norm have been proved. We have also given some counter examples redefined vague groups with respect to our results.

1. INTRODUCTION

First introduction of fuzzy subset was given by L.A. Zadeh. In his first observation on fuzzy set in 1965, L.A. Zadeh [1] opened new avenues and wide range for the researchers in many challenging areas in many scientific fields for new research. The real-life situations are very often not crisp and cannot be answered in just yes or no. Prof. L.A. Zadeh describes this vagueness mathematically, by giving some degree of membership to each element of the given set. A very first paper on fuzzy groups was published in 1971 by A. Rosenfield [4] in which the concept of fuzzy sub groupoid and fuzzy groups was introduced. Fuzzy logic was further extended into the generalization of another logic which was first introduced by Gau and Buehrer [2] is known as vague logic. Vague logic deals not only the membership grade but also with the non-membership grade, the

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basic platform on which the vague logic plays depends upon the two membership functions i.e., truth membership and the false membership. R. Biswas [3] defined the vague group. Rosenfeld presented a similar application to the elementary theory of groups. Later, Anthony and Sherwood [5] used fuzzy logic to redefined the groups into fuzzy groups and consequently gave some results on fuzzy sub-groupoid and fuzzy subgroups with some beautiful counter examples. Khan et al., [6] have also given an article on study of vague sets and vague groups. Demirci [7] first introduced fuzzy logic in place of crisp logic to define some properties of groups and then extended these properties by using vague logic based on two membership functions. Kaufmann [8] defined fuzzy subset as a mapping on the elements of a universal set into the closed interval [0, 1]. This mapping, named as a generalization of the characteristic function called the membership function. Later on, Zadeh [9] gave his valuable comments in the form of linguistic variables and defining it on approximate reasoning and makes an increment on the differences in the judgment about the degree of membership. Kiran et al. [10] discussed about the conversion of vague groups into a group induced from the structure of vague group. In 2011, Solairaju et al. [11] gave wonderful results on vague group and vague normal subgroups with the help of (T, S) norms. They extended their results from R. Biswas (2006) by characterising homologous groups with supporting results. Chaun [12] in 2005 presented his ideas on vague groups and presented his research ideas in the form of generalization fuzzy groups into vague groups and provides the vague homomorphism and Sylow theorem based on vague group known as vague Sylow theorem. Xu et al. [13] presented a novel approach for the distance measurements in vague sets. The main objective of our present paper is to focus on the novelty of vague results related to vague group based on t-norm and its related properties.

The whole work done in this present article is constituted into five segments. Second segment of the research paper deals with some elementary definitions related to vague sets and their types. The third segment of the research paper describes some pivotal results interconnected to vague sets and vague groups. In segment four of the paper, we have conveyed with the argument results for vague groups directly fastened t-norms. The last segment of the research paper represents the conclusions of exertion of the work.

2. Vague Sets And Their Types

Definition 2.1. *A Vague set in the universe of discourse U is distinguished by two integrating (membership) functions known as:*
A truth membership function

$$t_A : U \to [0, 1],$$

and a false membership function

$$f_A : U \to [0, 1].$$

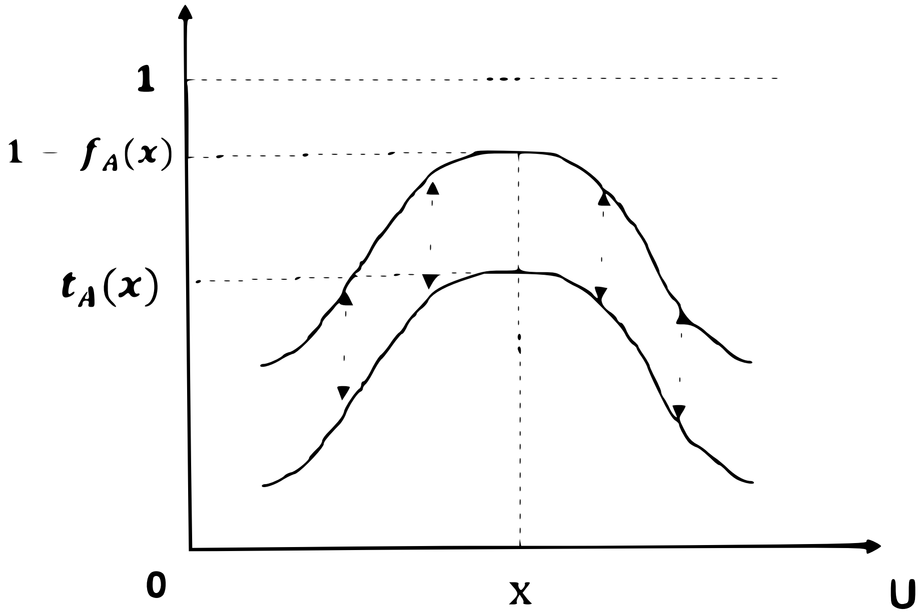

Figure 1. Vague Set

One can easily observe from the figure that, the lower membership grade be represented by $t_A(x)$ and it will be acquired from the "evidence of x" and the upper bound will be represented by $f_A(x)$ and it will be procured from the negation of x for the "evidences against x" with the condition that $0 \leq t_A(x) + f_A(x) \leq 1$ From this condition one can observe that the membership grade of any element in the universe of discourse U will be represented by a closed subinterval $[t_A(x), 1 - f_A(x)]$ for any value lie in [0,1], as shown in figure 1. Mathematically, the vague set can be represented as $A = [(x, t_A(x), f_A(x))/x \in U]$. f_A can also be represented as $f_A = 1 - t_A$. So, the interval $[t_A(x), 1 - f_A(x)]$ is called the value of x in A.

Definition 2.2. *A vague set possessing the value of its truth- membership and false-membership function identically zero is known as an empty vague set.*

Definition 2.3. *Vague set A of a universal set U is known as zero vague set if its truth membership value $t_A(x) = 0$ and and false membership value $f_A(x) = 1$, $\forall x \in U$.*

Definition 2.4. *Vague set A of a universal set U is known as zero vague set if its truth membership value $t_A(x) = 1$ and and and false membership value $f_A(x) = 0$, $\forall x \in U$.*

3. Vague Group And Some Propositions

Definition 3.1. *Let G be any group and let V_A be any vague set defined on G. then V_A is known as Vague Group (VG) of G if it is satisfying the following properties defined as:*
Sub property 1: $V_A(xy) \geq \min(V_A(x), V_A(y))$, i.e., $t_A(xy) \geq \min(t_A(x), t_A(y))$ and $1 - f_A(xy) \geq \min(1 - f_A(x), 1 - f_A(y))$
Sub property 2: $V_A(x^{-1}) >= V_A(x)$, i.e., $t_A(x^{-1}) >= t_A(x)$ and $1 - f_A(x^{-1}) >= 1 - f_A(x)$
Any element (x y) stands for $(x * y)$ where '' is binary operation defined on G.*

Example 1. *Let $G = Z_3 = 0, 1, 2$ be any group under addition modulo 3. Consider, $V_A(G) = [(0, 1, 0.2), (1, 0.5, 0.3), (2, 0.5, 0.8)]$ be the vague set of G. Then we can say that $V_A(G)$ is VG.*

Proposition 3.1. *Let G be any group and A be any vague group defined on G then we have the condition that $V_A(x^{-1}) = V_A(x), \forall x \in G$.*

Proposition 3.2. *V_A is a vague group defined on a group G if and only if $V_A(xy^{-1}) \geq \min(V_A(x), V_A(y))$.*

Proposition 3.3. *Intersection of any sets of vague groups is a vague group.*

Proposition 3.4. *The union of any set of vague groups is not necessarily a vague group.*

Example 2. *Let G be a Klien four K_4 group defined in the following manner, $G = e, a, b, c$ with $a^2 = b^2 = e$ and $ab = ba$. Let V_A and V_B be any two vague groups of G defined as; $V_A = [(e, 0.6, 0.2), (a, 0.4, 0.3), (b, .1, 0.7), (ab, 0.1, 0.7)]$ and $V_B = [(e, 0.9, 0.1), (a, 0.5, 0.4), (b, .5, 0.4), (ab, 0.9, 0.1)]$.*

Now, $V_C = V_A \cup V_B = [(e, 0.9, 0.1), (a, 0.5, 0.3), (b, .5, 0.4), (ab, 0.9, 0.1)]$.

Noted that, $1 - f_C(a.ab) = 1 - f_C(a^2b) = f_C(e.b) = 1 - f_C(b) = 1 - 0.4 = 0.6$ and $\min(1 - f_C(a), 1 - f_C(ab)) = \min(1 - 0.3, 1 - 0.1) = 0.7$, i.e., $1 - f_C(a.ab) < \min(1 - f_C(a), 1 - f_C(b))$ Which contradict the proposition 3.1, so we can easily observe that V_C is not a VG. So, union of vague groups is not necessarily a VG.

Proposition 3.5. *Let G be a group and let T be any subset of G i.e., $T \subseteq G$ and ψ_T is the characteristic function defined on a set T then ψ_T is a vague group if and only if T is a subgroup of G.*

Proof. Let ψ_T be a VG, then we have to show that-T is a subgroup of G. First, let $x, y \in T$ that is $\psi_T(x) = 1 = \psi_T(y)$ then $\psi_T(xy) >= 1$ (since ψ_T is a VG)implies that $\psi_T(xy) = 1$ (since $\psi_T(xy)$ not greater than 1) thus, $x \in T$. Now if $x \in T$ that is $\psi_T(x) = 1$ then $\psi_T(x^{-1}) >= 1$ (since ψ_T is a VG) $\psi_T(x^{-1}) = 1$ thus $x^{-1} \in T$. Hence, T is a subgroup of G.

Conversely, let T be a subgroup of G then we have to show that ψ_T is a VG. Let $x, y \in T$ then $\psi_T(x) = 1 = \psi_T(y)$, on the basis of this we have $\psi_T(xy) = 1$ (since $xy \in T$). Therefore,$\psi_T(xy) \geq \min(\psi_T(x), \psi_T(y))$. Also, noted that for any $(x \in T)$we have $(x^{-1} \in T)$that is for $\psi_T(x) = 1$, we have $\psi_T(x^{-1}) = 1$ Which conclude that $\psi_T(x^{-1}) \geq \psi_T(x)$.

Thus, ψ_T is a VG. □

Proposition 3.6. *Show that set $S = (x : V_A(x) = V_A(e))$ is a subgroup of G, where e is the identity element of group G and V_A is a VG of G.*

Example 3. *Let $x, y \in S$, then $V_A(x) = V_A(e) = V_A(y))$.*

Now, consider $V_A(xy) \geq \min(V_A(x), V_A(y)) = V_A(e) \Rightarrow V_A(xy) \geq V_A(e)$ (since $V_A$is a vague group and we have given that $V_A(x) = V_A(e)$). Again, $V_A(e) = V_A(xyx^{-1}y^{-1}) \geq \min(V_A(xy), V_A(y^{-1}x^{-1})) = \min(V_A(xy), V_A(xy)^{-1}) = V_A(xy)$ (since, $V_A(x^{-1}) = V_A(x)$). Therefore, $V_A(xy) = V_A(e)$ this implies that $xy \in S$. Now, let $x \in S$, we have to show that $x^{-1} \in S$, since $V_A(x^{-1}) = V_A(x) = V_A(e)$, so $x^{-1} \in S$.

Thus, S is a subgroup of G.

4. Vague Groups And T-Norm

Definition 4.1. *t-Norm was first used J.M. Anthony and H. Sherwood [5] in redefined fuzzy group in 1979. T-norm was used in semi group in the form of metric and its general probabilistic metric spaces. A function* $T : [0,1] \times [0,1]$ *is said to form a t-norm if it satisfies the following properties for each* $\alpha, \beta, \gamma, \delta$ *in [0,1]:*

a. $T(0,0) = 0, T(\gamma, 1) = \gamma = T(1, \gamma)$;
b. $T(\alpha, \beta) \leq T(\gamma, \delta)$, *if* $\alpha \leq \gamma$ *and* $\beta \leq \delta$;
c. $T(\alpha, \beta) = T(\delta, \alpha)$;
d. $T(\alpha, T(\delta, \gamma)) = T(T(\alpha, \delta), \gamma)$.

Definition 4.2. *t-Norm Vague group (tVG) can be formulated in the following manner. Let G be any group and* V_A *be any vague set defined on G. then any vague group is said to be tVG if it follows the following conditions:*

a. $V_A(xy) \geq T((V_A(x), V_A(y))$, *i.e.,* $t_A(xy) \geq T((t_A(x), t_A(y))$ *and* $1 - f_A(xy)$ $\geq T((1 - f_A(x), 1 - f_A(y))$.
b. $V_A(x^{-1}) = V_A(x)$ *i.e.,*
$t_A(x^{-1}) = t_A(x)$ *and* $1 - f_A(x^{-1}) = 1 - f_A(x)$, $\forall x, y \in G$.

Example 4. *Let G be a Klien four* K_4 *group defined in the following manner,* $G = e, a, b, c$ *with* $a^2 = b^2 = e$ *and* $ab = ba$. *Let* V_A *be vague set of G defined as;* $V_A = [(e, 1, 0), (a, 0.2, 0.5), (b, .3, 0.4), (ab, 0.4, 0.5)]$. *From this we may observe that* $T(a, b) = ab, a, b \in [0, 1]$. *Then* V_A *is vague group with t-norm 'T' and is known as tVG.*

Proposition 4.1. *If* V_A *is a vague group with norm t of any group. Then,* $H = [x \in G : V_A(x) = 1 i.e., t_A(x) = 1 or 1 - f_A(x) = 1]$ *is either empty or is a subgroup of G.*

Example 5. *Let* $x, y \in H$. *Now,* $t_A(xy^{-1}) \geq T(t_A(x), t_A(y^{-1})) = T(t_A(x), t_A(y))$ $= T(1, 1) = 1$. *Therefore,* $t_A(xy^{-1}) = 1$. *So,* $xy^{-1} \in H$ *and* $1 - f_A(xy^{-1}) \geq T(1 - f_A(x), 1 - f_A(y^{-1})) = T(1 - f_A(x), 1 - f_A(y))$ $T(1, 1) = 1$. *Therefore,* $1 - f_A(xy^{-1}) = 1$ *So,* $xy^{-1} \in H$. *Consequently, H is a subgroup of G.*

Proposition 4.2. *Let* V_A *be at VG and if* $V_A(xy^{-1}) = 1$ *i.e.,* $t_A(xy^{-1}) = 1$ *and* $1 - f_A(xy^{-1}) = 1$ *then* $V_A(x) = V_A(y)$ *i.e.,* $t_A(x) = t_A(y)$ *and* $1 - f_A(x) = 1 - f_A(y)$.

Example 6. *Let* $x, y \in VG$, $t_A(x) = t_A(xy^{-1})y >= T(t_A(xy^{-1}), t_A(y)) = T(1, t_A(y)) = t_A(y) = t_A(y^{-1})$ *(since,* V_A *is a Vague group).*

Now, $t_A(y^{-1}) = t_A(x^{-1}(xy^{-1})) >= T(t_A(x^{-1}), t_A(xy^{-1})) = T(t_A(x^{-1}), 1) = t_A(x^{-1}) = t_A(x)$. *(since V_A is a vague group) from this, we observe that* $t_A(x) = t_A(y)$.

Again, $1 - f_A(x) = 1 - f_A(xy^{-1})y >= T(1 - f_A(xy^{-1}), 1 - f_A(y)) = T(1, 1 - f_A(y)) = 1 - f_A(y) = 1 - f_A(y^{-1})$ *(since, V_A is a Vague group) and* $1 - f_A(y^{-1}) = 1 - f_A(x^{-1}(xy^{-1})) >= T(1 - f_A(x^{-1}), 1 - f_A(xy^{-1})) = T(1 - f_A(x^{-1}), 1) = 1 - f_A(x^{-1}) = 1 - f_A(x)$ *from this, we observe that* $1 - f_A(x) = 1 - f_A(y)$.

Hence, $V_A(x) = V_A(y)$.

Proposition 4.3. *Let V_A be a vague set on a group G and let T be a given t-norm. If $V_A(e) = 1$, i.e., $t_A(e) = 1$ and $1 - f_A(e) = 1$ and $V_A(xy^{-1}) > T(V_A(x), V_A(y))$, $\forall x, y \in G$, i.e., $t_A(xy^{-1}) > T(t_A(x), t_A(y)), \forall x, y \in G$ and $1 - f_A(xy^{-1}) > T(1 - f_A(x), 1 - f_A(y)), \forall x, y \in G$. Then, V_A is a t-norm vague group.*

Example 7. *Let $x, y \in VG$, $t_A(y^{-1}) = t_A(ey^{-1}) >= T(t_A(e), t_A(y)) = T(1, t_A(y)) = t_A(y)$ and similarly we may observe that $t_A(y) >= t_A(y^{-1})$ so that $t_A(y) = t_A(y^{-1})$. Moreover $t_A(xy) \geq t_A(xy^{-1}) \geq T(t_A(x), t_A(y^{-1})) = T(t_A(x), t_A(y))$.*

Thus,t_A is a fuzzy subgroup of G with respect to T.

Again, $1 - f_A(y^{-1}) = 1 - f_A(ey^{-1}) >= T(1 - f_A(e), 1 - f_A(y)) = T(1, 1 - f_A(y)) = 1 - f_A(y)$ *and similarly,* $1 - f_A(y) >= 1 - f_A(y^{-1})$ *so that* $1 - f_A(y) = 1 - f_A(y^{-1})$. *Moreover* $1 - f_A(xy) \geq 1 - f_A(xy^{-1}) \geq T(1 - f_A(x), 1 - f_A(y^{-1})) = T(1 - f_A(x), 1 - f_A(y))$. *Thus, V_A is a tVG.*

5. CONCLUSION

Vague set has been widely applied in decision making, medical diagnosis, logical programming, pattern recognition and it seems to have been more popular. Fuzzy group theory has very wide and efficient applications in many core (main) areas like computer science, cryptography, coding theory, digital communication, clustering, physics etc. In this paper we have studied the properties of vague group in the context of t-norm. We have also proved some propositions and gave examples in the support of our results. We have extended the result of [5] into vague group with respect to t-norm and gave some counter examples

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