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A NUMERICAL METHOD BY FINITE ELEMENT METHOD (FEM) OF AN EULER-BERNOULLI BEAM TO VARIABLE COEFFICIENTS

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ABSTRACT. We consider the Euler-Bernoulli beam equation with variable coefficients $\rho(x)w_{tt}(x,t) + (EI(x)w_{xx}(x,t))_{xx} = 0$ in the domain $(0,1)$ clamped at one end and controlled at the other end, in force and in moment, as well as its corresponding Cauchy problem $\frac{d}{dt}y(t) = Ay(t);\ y(0) = y_0$ in the appropriate state space. For this boundary value problem, we establish existence and uniqueness results adapting the standard strategy presented in [7, 9]. Indeed, using its weak formulation, we adapt the techniques of Faedo-Galerkin raised in [7, 9] to show the existence and uniqueness of weak solution for the case of the linear boundary conditions. Finally, we develop a stable and convergent numerical method by the finite element method. This method is validated by numerical simulations.

1. INTRODUCTION

In this paper, we establish existence and uniqueness theorems and develop a numerical method for an Euler-Bernoulli beam with variable coefficients. This beam is clamped at one end and controlled at the other end, in force and in moment by a linear combination of the velocity, velocity of rotation and the position term (respectively the angle of rotation) in the control of force (respectively moment). The vibrating beam system can be described as follows:

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(1.1) $$\rho(x)w_{tt}(x,t) + (EI(x)w_{xx}(x,t))_{xx} = 0, \quad 0 < x < 1, t > 0$$

(1.2) $$w(0,t) = w_x(0,t) = 0, \quad t > 0$$

(1.3) $$-EI(1)w_{xx}(1,t) = (2\mu_{11}w_t + \mu_{12}w_{xt} + \alpha w_x)(1,t), \quad t > 0$$

(1.4) $$(EI(.)w_{xx})_x (1,t) = (\mu_{21}w_t + 2\mu_{22}w_{xt} + \beta w)(1,t), \quad t > 0$$

(1.5) $$w(x,0) = w_0(x), w_t(x,0) = v_0(x), \quad 0 < x < 1$$

where $w(x,t)$ is the transversal deviation at position x and time t. We suppose that $\alpha \geq 0$, $\beta \geq 0$ and $\mu_{ij} \geq 0$, $1 \leq i,j \leq 2$ are constants such that

(1.6) $$\mu_{12} > 0 \quad \text{and} \quad \mu_{12}\mu_{21} \geq (\mu_{11} + \mu_{22})^2 .$$

In the system (1.1)–(1.5), we assume that the length of the beam is equal unity. Also, the function $\rho(x)$ is the mass density of the beam and the function $EI(x)$ is its flexural rigidity of the beam satisfying the following conditions:

$$(\rho(x), EI(x)) \in \left[C^4(0,1)\right]^2, \text{with } \rho(x) > 0, \ EI(x) > 0,$$

for all $x \in (0,1)$.

This system has been studied by My Driss Aouragh and Naji Yebari [1] where it is shown that a set of generalized eigenfunctions of problem (1.1)–(1.4) forms a Riesz basis for the appropriate Hilbert space and that the spectrum-determined growth condition holds. In addition, to study numerically the spectrum of system, they used the finite difference scheme with the QZ method [8, 12].

A study of existence and uniqueness of weak solution was made by Bomisso et al [5] who analyzed a flexible Euler-Bernoulli beam with a force control in rotation and velocity rotation (a study with constant coefficients). The same study of this type was made for an Euler-Bernoulli beam with tip body and passivity-based boundary control (a study with variable coefficients) by Miletic et al [11]. Their studies were based on the Faedo-Galerkin's method and the work of Lions et al in [9]. We adapt the same procedure to study the existence and uniqueness of the problem (1.1) under the conditions (1.2)–(1.4). The main difficulty lies in the appearance of the time terms which appear in the definition of the weak solution of boundary value problem. The presence of this time terms can, greatly, complicate the application of standard techniques stated in [7, 9]. We overcome this difficulty by adapting these standard techniques. In addition, we develop a numerical method which preserves unconditionally the results obtained in the continuous case. We derive an error estimate for a semi-discrete approximation and establish convergence.

The paper consists of five sections. In section 2, after proving that the problem is well posed in the sense C_0-semigroup of contractions, we show that this system is Lyapunov stable. In section 3, from the weak formulation, we show the existence, uniqueness and higher regularity of the weak solution. In section 4, a semi-discrete numerical method is constructed using the finite element method and in section 5, the numerical simulations are presented in order to illustrate the results obtained.

2. PRELIMINARIES

2.1. **Semigroup formulation.** Let us introduce the following spaces:

$$H_E^2(0,1) = \left\{ w \in H^2(0,1) : w(0) = w_x(0) = 0 \right\}.$$

The Hilbert space is defined by:

$$\mathcal{H} = H_E^2(0,1) \times L^2(0,1)$$

with the inner product

$$\langle w, v \rangle_{\mathcal{H}} = \int_0^1 \rho g_1 \overline{g_2} dx + \int_0^1 EI (f_1)_{xx} \overline{(f_2)}_{xx} dx + \alpha (f_1)_x(1) \overline{(f_2)}_x(1) + \beta f_1(1) \overline{f_2}(1)$$

where $w = (f_1, g_1)^T \in \mathcal{H},\ v = w_t = (f_2, g_2)^T \in \mathcal{H}$ and we denote by $\|.\|_{\mathcal{H}}$ the corresponding norm. The superscript T stands for the transpose.
Let $\mathbb{H} = (H_E^2(0,1) \cap H^4(0,1)) \times H_E^2(0,1)$ and we consider $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ an linear operator with the domain

$$D(A) = \Big\{ (f,g) \in \mathbb{H} : -EI(1) f_{xx}(1) = 2\mu_{11} g(1) + \mu_{12} g_x(1) + \alpha f_x(1),$$
$$\left(EI(.) f_{xx}(.)\right)_x (1) = \mu_{21} g(1) + 2\mu_{22} g_x(1) + \beta f(1) \Big\} \tag{2.1}$$

defined by

$$A \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} g(.) \\ -\frac{1}{\rho(.)} \left(EI(.) f_{xx}(.)\right)_{xx} \end{pmatrix}. \tag{2.2}$$

Now (1.1)–(1.5) can be written as a following Cauchy problem:

$$\begin{cases} \frac{d}{dt}y(t) = Ay(t) \\ y(0) = y_0 \in \mathcal{H} \end{cases} \tag{2.3}$$

where $y(t) = (w(.,t), w_t(.,t))^T$, $y(0) = (w_0, v_0)^T$ for all $t > 0$.
We give the proof on the well-posedness result of the system (1.1)–(1.5):

Theorem 2.1. *The operator A defined by* (2.1)–(2.2) *is closed, densely defined and dissipative. Furthermore, A is invertible with A^{-1} being compact and A generates a C_0-semigroup of contractions on $\mathcal{H}$ denoted by $\{S(t)\}_{t\geq 0}$.*

Proof. We start by showing that A is the dissipative operator.
For all $w = (f,g)^T \in D(A)$,

$$\begin{aligned}\langle Aw, w\rangle_{\mathcal{H}} &= \left\langle \left(g(x), -\frac{1}{\rho(x)}\left(EI(x)f_{xx}(x)\right)_{xx}\right)^T, (f,g)^T \right\rangle_{\mathcal{H}} \\ &= \int_0^1 -\left(EI(x)f_{xx}\right)_{xx}\overline{g}(x)dx + \int_0^1 EI(x)g_{xx}(x)\overline{f}_{xx}(x)dx \\ &\quad + \alpha g_x(1)\overline{f}_x(1) + \beta g(1)\overline{f}(1).\end{aligned}$$

Integrating twice by parts, using (1.2)–(1.4) and taking real part, we obtain:

$$Re\langle Aw, w\rangle_{\mathcal{H}} = -\Big(\mu_{21}|g(1)|^2 + \mu_{12}|g_x(1)|^2 + 2\mu_{22}|g_x(1)g(1)| + 2\mu_{11}|g(1)g_x(1)|\Big) \leq 0.$$

Thus, A is a dissipative operator. Now let us show that the operator A is maximal i.e. prove that for all $(f,g)^T \in \mathcal{H}$, we can find $(w,v)^T \in D(A)$ such that $(I - A)(w,v) = (f,g)$. This leads us in an equivalent way to seek (w,v) solution of the following system:

$$v = w - f$$

$$\rho(x)w + \left(EI(x)w_{xx}(x,t)\right)_{xx} = \rho(x)(g+f) \tag{2.4}$$

$$-EI(1)w_{xx}(1,t) = \left(2\mu_{11}v + \mu_{12}v_x + \alpha w_x\right)(1,t) \tag{2.5}$$

$$\left(EI(.)w_{xx}\right)_x(1,t) = \left(\mu_{21}v + 2\mu_{22}v_x + \beta w\right)(1,t) \tag{2.6}$$

with $w \in H^4(0,1) \cap H^2_E(0,1)$, $v \in H^2_E(0,1)$ and $g \in L^2(0,1)$.
Now, multiply (2.4) by $\varphi \in H^2_E(0,1)$ and after integrating twice by parts and using the conditions (2.5)-(2.6), we obtain:

$$a(w,\varphi) = L(\varphi) \tag{2.7}$$

where

$$\begin{aligned} a(w,\varphi) &= \int_0^1 \rho(x)w\varphi dx + \int_0^1 \left(EI(x)w_{xx}\right)\varphi_{xx}dx + \alpha w_x(1)\varphi_x(1) + \beta w(1)\varphi(1) \\ L(\varphi) &= \int_0^1 \rho(x)(g+f)\varphi dx + 2\mu_{11}v(1)\varphi_x(1) + \mu_{12}v_x(1)\varphi_x(1) + \mu_{21}v(1)\varphi(1) \\ &+ 2\mu_{22}v_x(1)\varphi(1). \end{aligned}$$

We easily show that, for all $\varphi \in H_E^2(0,1)$, the bilinear form $a(.,.)$ is continuous, coercive and the linear form $L(.)$ is continuous on $H_E^2(0,1)$.

Thus, according to the Lax-Milgram theorem, there exists a unique solution $w \in H^4(0,1) \cap H_E^2(0,1)$ for (2.7). Then the operator A is maximal.
Next, the domain $D(A)$ is clearly dense in $\mathcal{H}$ and the operator is closed.

Finally, we prove that A^{-1} exists. For any $\Psi = (g_1, g_2)^T$, we need to find a unique $\Phi = (f_1, f_2)^T \in D(A)$ such that $A\Phi = \Psi$ which yields

$$\begin{aligned} f_2(x) &= g_1(x), \quad g_1 \in H_E^2(0,1) \\ (EI(x)(f_1)_{xx})_{xx} &= -\rho(x)g_2(x), \quad g_2 \in L^2(0,1) \\ f_1(0) &= (f_1)_x(0) = 0 \\ -EI(1)(f_1)_{xx}(1) &= 2\mu_{11}g_1(1) + \mu_{12}(g_1)_x(1) + \alpha(f_1)_x(1) \\ \big(EI(.)(f_1)_{xx}(.)\big)_x(1) &= \mu_{21}g_1(1) + 2\mu_{22}(g_1)_x(1) + \beta(f_1)(1). \end{aligned}$$

A direct computation shows that the above solution is given by:

$$\begin{cases} f_2(x) &= g_1(x), \\ f_1(x) &= \int_0^x \int_0^s \Big[-\beta f_1(1)\frac{1-\xi}{EI(\xi)} - \alpha(f_1)_x(1)\frac{1}{EI(\xi)} - \frac{1}{EI(\xi)}\int_\xi^1\int_\eta^1 \rho(x)g_2(x)drd\eta \\ & \quad -\big(\mu_{21}g_1(1) + 2\mu_{11}(g_1)_x(1)\big)\frac{1-\xi}{EI(\xi)} - \big(2\mu_{11}g_1(1) + \mu_{12}(g_1)_x(1)\big)\frac{1}{EI(\xi)}\Big] d\xi ds. \end{cases}$$

Thus, A^{-1} exists and is bounded on $\mathcal{H}$. Furthermore, the Sobolev embedding theorem implies that A^{-1} is compact on $\mathcal{H}$. □

Now according to Lumer-Phillips theorem (see [13]), the statement of the theorem follows.

Theorem 2.2. (2.3) *has a unique mild solution* $y(t) = S(t)y_0 \in C([0;\infty);\mathcal{H})$ *for all* $y_0 \in \mathcal{H}$.

2.2. **Stability in the sense of Lyapunov of** (2.3)**.** The idea of this part is that if there exists a functional $\mathcal{V} \geq 0$ on Hilbert space $\mathcal{H}$ such that $\frac{d}{dt}\mathcal{V} \leq 0$ (the generalized time derivative of $\mathcal{V}$) and the set $\{x : \mathcal{V}(x) = 0\}$ does not contain any complete negative orbit except the trivial one $x = 0$ then the system is Lyapunov stable.

Notice that the total mechanical energy of the system (1.1)–(1.5) is given by :

$$E(t,w) = \frac{1}{2}\left(\int_0^1 \rho(x)w_t^2 dx + \int_0^1 EI(x)w_{xx}^2 dx + \alpha w_x^2(1) + \beta w^2(1)\right). \tag{2.8}$$

The time derivative of the energy function $E(t)$ along the classical solutions of (1.1)–(1.5) is

$$\frac{d}{dt}E(t,w) = -2\left(\mu_{22} + \mu_{11}\right) w_{tx}(1,t)w_t(1,t) - \mu_{21}w_t^2(1,t) - \mu_{12}w_{tx}^2(1,t) \leq 0. \tag{2.9}$$

The property of semigroup contractions in Theorem 2.1 implies that $\|.\|_{\mathcal{H}}$ is a good candidate for the Lyapunov functional for (2.3).

Let the functional $\mathcal{V} : \mathcal{H} \to \mathbb{R}$ defined as follows

$$\mathcal{V}(y) = \|y\|_{\mathcal{H}}^2 = \frac{1}{2}\int_0^1 EI(x)w_{xx}^2 dx + \frac{1}{2}\int_0^1 \rho(x)w_t^2 dx + \frac{1}{2}\alpha w_x^2(1,t) + \frac{1}{2}\beta w^2(1,t). \tag{2.10}$$

Analogously as in (2.9), for all classical solutions y it follows that:

$$\frac{d}{dt}\mathcal{V}(y) = \frac{d}{dt}\|y\|_{\mathcal{H}}^2 = -2\left(\mu_{22} + \mu_{11}\right) w_{tx}(1,t)w_t(1,t) - \mu_{21}w_t^2(1,t) - \mu_{12}w_{tx}^2(1,t) \leq 0. \tag{2.11}$$

So, the time evolution of the Lyapunov functional along the classical solutions is non-increasing. Furthermore, from Theorem 2.1, the decay of energy along the classical solutions can be extended to mild solutions. Hence, $\mathcal{V}$ is the Lyapunov functional for (2.3). The system (2.3) is stable in the sense of Lyapunov. Now, applying LaSalle's invariance principle [10], the stability result is enunciated as follows:

Theorem 2.3. *Assume that $y(t)$ is the mild solution of* (2.3) *for all $y_0 \in \mathcal{H}$. Then $y(t) \to 0 \in \mathcal{H}$ when $t \to \infty$.*

3. Existence, uniqueness and higher regularity of the weak Solution

Before defining the weak solution of the boundary problem, let's define what is called "intermediate spaces" (see [9]), which spaces will be useful in the following.

Let X and Y two hilbert spaces. Let $<,>_X$ and $<,>_Y$ be the inner products in X and Y respectively. X is densely embedded in Y and suppose that the canonical injection of X into Y is continuous. The space X may be defined as the domain of an operator Λ which is self-adjoint, positive and unbounded in Y. The norm in X being equivalent to the norm of the graph

$$\left(\|u\|_Y^2 + \|\Lambda u\|_Y^2\right)^{\frac{1}{2}}, \quad u \in D(\Lambda) = X.$$

We denote by $D(S)$ the set of elements u such that the antilinear form

$$v \mapsto < u, v >_X, \quad v \in X$$

is continuous in the induced topology by Y. Then

$$< u, v >_X = < Su, v >_Y, \tag{3.1}$$

which defines S as an unbounded operator in Y, with domain $D(S)$, dense in Y. S is a self-adjoint and strictly positive operator. Indeed, there exists a constant M such that

$$< Sv, v >_Y = \|v\|_X^2 \geq M\|v\|_Y^2.$$

Using the spectral decomposition of self-adjoint operators, for θ real or complex, the powers S^θ of S, may be defined. In particular, we take $\theta = \frac{1}{2}$ and we shall use the fractional space

$$\Lambda = S^{\frac{1}{2}}. \tag{3.2}$$

The operator Λ is self-adjoint and positive in Y, with domain X. From (3.1)–(3.2), we deduce

$$< u, v >_X = < \Lambda u, \Lambda v >_Y, \quad \forall u, v \in X.$$

We give the following definition of the intermediate spaces $[X, Y]_\theta$:

Definition 3.1. *Let X and Y be two Hilbert spaces that we assume to be separable with $X \subset Y$, X being dense in Y with continuous injection and Λ defined by* (3.2). *The intermediate space $[X, Y]_\theta$ is defined by*

$$[X, Y]_\theta = D(\Lambda^{1-\theta}), \quad 0 \leq \theta \leq 1$$

with $D(\Lambda^{1-\theta})$ the domain of $\Lambda^{1-\theta}$ and the norm of $[X,Y]_\theta$ equal norm of the graph of $\Lambda^{1-\theta}$ i.e. $\left(\|u\|_Y^2 + \|\Lambda^{1-\theta}u\|_Y^2\right)^{\frac{1}{2}}$.

From the properties of the spectral decomposition, we immediately have that X is dense in $[X,Y]_\theta$. Furthermore, according to the duality theorem, we have for all $\theta \in]0,1[$, $[X,Y]_\theta^{'} = [Y',X']_{1-\theta}$. In the next paragraph, we arbitrarily choose $\theta = \frac{1}{2}$.

3.1. **Definition of weak solution.** Let $\phi \in H_E^2(0,1)$. Multiplying (1.1) by ϕ, integrating twice by parts over $(0,1)$ and using (1.2)–(1.4), we have the following identity:

$$\begin{aligned}(3.3)\quad &\int_0^1 \rho(x) w_{tt}\phi dx + \int_0^1 EI(x) w_{xx}\phi_{xx} dx + \alpha w_x(1)\phi_x(1) + \beta w(1)\phi(1) \\ &+ 2\mu_{11} w_t(1)\phi_x(1) + \mu_{12} w_{xt}(1)\phi_x(1) + \mu_{21} w_t(1)\phi(1) + 2\mu_{22} w_{xt}(1)\phi(1) = 0.\end{aligned}$$

In order to give the definition of the weak solution, we will adapt the ideas of the authors H.T. Banks et al [2] and Evans [7] to the problem (1.1)–(1.5).
We build the following functional spaces:

$$X = \mathbb{R}^2 \times H_E^2(0,1) = \left\{\widehat{\phi} = (\phi(1), \phi_x(1), \phi), \phi \in H_E^2(0,1)\right\}$$

and

$$Y = \mathbb{R}^2 \times L^2(0,1) = \left\{\widehat{z} = (z(1), z_x(1), z), z \in L^2(0,1)\right\}$$

with the following respect inner product

$$\langle \widehat{\phi}_1, \widehat{\phi}_2 \rangle_X = \langle (\phi_1)_{xx}, (\phi_2)_{xx} \rangle_{L^2(0,1)} \quad \text{and} \quad \langle \widehat{z}_1, \widehat{z}_2 \rangle_Y = \langle \rho z_1, z_2 \rangle_{L^2(0,1)}.$$

We notice that X is densely embedded in Y and suppose that the canonical injection of X into Y is continuous. Therefore, we obtain a Gelfand triple : $X \subset Y \equiv Y' \subset X'$ where X' is the dual of X and Y' the dual of Y. Consider the bilinear forms $a^{(1)} : X \times X \to \mathbb{R}$ and $a^{(2)} : Y \times Y \to \mathbb{R}$ defined as follows:

$$a^{(1)}\left(\widehat{\phi}_1, \widehat{\phi}_2\right) = \langle EI\widehat{\phi}_1, \widehat{\phi}_2 \rangle_X + \alpha(\phi_1)_x(1)(\phi_2)_x(1) + \beta\phi_1(1)\phi_2(1)$$

$$a^{(2)}\left(\widehat{z}_1, \widehat{z}_2\right) = 2\mu_{11} z_1(1)(z_2)_x(1) + \mu_{12}(z_1)_x(1)(z_2)_x(1) + \mu_{21} z_1(1) z_2(1) + 2\mu_{22}(z_1)_x(1) z_2(1).$$

We can now define the weak solution. In it, note that $\langle .,. \rangle_{X,X'}$ represents the duality pairing between X and X'.

Definition 3.2. *Let $T > 0$ be fixed. We say that $\hat{w} = (w(1), w_x(1), w)$ is a weak solution of problem* (1.1) – (1.5) *on* $(0,1)$ *if*

$$\hat{w} \in L^2(0,T;X) \quad \textit{with} \quad \hat{w}_t \in L^2(0,T;Y), \quad \hat{w}_{tt} \in L^2(0,T;X')$$

and satisfies

$$\langle \hat{w}_{tt}, \hat{\phi} \rangle_{X,X'} + a^{(1)}(\hat{w}, \hat{\phi}) + a^{(2)}(\hat{w}_t, \hat{\phi}) = 0 \tag{3.4}$$

almost everywhere on $t \in (0,T)$ and for all $\hat{\phi} \in H^2_E(0,1)$, with the following initial conditions

$$\hat{w}(0) = \hat{w}_0 = (w_0(1), (w_0)_x(1), w_0) \in X, \tag{3.5}$$

$$\hat{w}_t(0) = \hat{v}_0 = (v_0(1), (v_0)_x(1), v_0) \in Y. \tag{3.6}$$

Remark 3.1. *In the Definition 3.2, notice that the assumptions $\hat{w} \in L^2(0,T;X)$ and $\hat{w}_t \in L^2(0,T;Y)$ implies that $\hat{w} \in C\left([0,T], [X,Y]_{\frac{1}{2}}\right)$ after, possibly, a modification on a set of measure zero (see [9]). This gives meaning to the initial conditions* (3.5)–(3.6).

For the proof of the existence of the weak solution, we will need the following lemma whose proof appears in [5, 11]:

Lemma 3.1. *Let $H^2_E(0,1)$ be a subspace of $H^2(0,1)$. Then there exists a infinite sequence of functions $\{\phi_i\}_{i=1}^{\infty}$ such that $\{\phi_i\}_{i=1}^{\infty}$ is an orthogonal basis of $H^2_E(0,1)$ and $\{\phi_i\}_{i=1}^{\infty}$ is an orthonormal basis of $L^2(0,1)$.*

3.2. **Existence of the weak solution.**

Theorem 3.1. *There exists a weak solution $\hat{w}$ of weak formulation* (3.4)–(3.6) *such that*

$$\hat{w} \in L^{\infty}(0,T;X) \tag{3.7}$$

$$\hat{w}_t \in L^{\infty}(0,T;Y) \tag{3.8}$$

$$\hat{w} \in C\left([0,T], [X,Y]_{\frac{1}{2}}\right) \tag{3.9}$$

$$\hat{w}_t \in C\left([0,T], [X,Y]'_{\frac{1}{2}}\right). \tag{3.10}$$

Proof. This proof is based on the Faedo-Galerkin's method. According to Lemma 3.1, there exists by extension an infinite sequence of functions $\{\phi_i\}_{i=1}^{\infty}$ that is an orthogonal basis for X and an orthonormal basis for Y. Consider such a sequence. Introduce the following finite dimensional spaces spanned by $\{\phi_i\}_{i=1}^{m}$ defined as

$$\forall m \in \mathbb{N}, \quad \hat{V}_m := \text{span}\left\{\hat{\phi}_1, ..., \hat{\phi}_m\right\} = \left\{\sum_{j=1}^{m} \alpha_j \hat{\phi}_j, \quad \text{with} \quad \alpha_j \in \mathbb{R}\right\}.$$

Step 1 : Construction of approximate solutions

We seek $\hat{w} = \hat{w}_m(t) \in \hat{V}_m$ the approximate solution of the problem in the form:

$$\hat{w}_m = \sum_{i=1}^{m} g_{im}(t)\hat{\phi}_i$$

where $g_{im}(t) \in \mathbb{R}$ $(0 \le t \le T, \quad i = 1, ..., m)$ are the solutions of the weak formulation (3.3) on $\hat{V}_m$. So for a fixed integer m, we have:

$$\langle(\hat{w}_m)_{tt}, \hat{\phi}\rangle_Y + a^{(1)}(\hat{w}_m, \hat{\phi}) + a^{(2)}((\hat{w}_m)_t, \hat{\phi}) = 0, \quad \forall\hat{\phi} \in \hat{V}_m \tag{3.11}$$

with the initial conditions:

$$\hat{w}_m(0) = \hat{w}_{m0}, \ \hat{w}_{m0} = \sum_{i=1}^{m} \alpha_{im}\hat{\phi}_i \longrightarrow \hat{w}_0 \ \text{ in } X \text{ when } m \longrightarrow \infty, \tag{3.12}$$

(3.13)

$$(\hat{w}_m)_t(0) = \hat{v}_m(0), \ \ \hat{v}_m(0) = \hat{v}_{m0}, \ \ \hat{v}_{m0} = \sum_{i=1}^{m} \beta_{im}\hat{\phi}_i \longrightarrow \hat{v}_0 \ \text{ in } Y \text{ when } m \longrightarrow \infty$$

with $\alpha_{im} = g_{im}(0)$ and $\beta_{im} = (g_{im})_t(0)$.

Thus, according to the general results on the systems of the second order differential equations , one is assured of the existence of a solution $\hat{w}_m \in C^2\left([0;T], X\right)$ of (3.11) – (3.13), for $0 \le t \le T$.

Step 2 : A-priori estimates on approximate solutions

Let $\hat{E} : \mathbb{R} \times X \to \mathbb{R}$ the energy functional for the trajectory $\hat{w}$ analogous to that defined by the expression of the Lyapunov functional (2.10).

$$\hat{E}(t, \hat{w}) = \frac{1}{2}\int_0^1 EI(x)\hat{w}_{xx}^2 dx + \frac{1}{2}\int_0^1 \rho(x)\hat{w}_t^2 dx + \frac{1}{2}\alpha\hat{w}_x^2(1, t) + \frac{1}{2}\beta\hat{w}^2(1, t)$$

$$\hat{E}(t, \hat{w}) = \|(\hat{w}, v)\|_{\mathcal{H}}.$$

Assuming that there exists a solution $\hat{w}_m \in C^2\left([0;\tau], \hat{V}_m\right)$ to (3.11) on some interval $[0;\tau]$. Let's take $(\hat{w}_m)_t = \hat{\phi}$ in (3.11), we obtain

$$\langle(\hat{w}_m)_{tt}, (\hat{w}_m)_t\rangle_Y + a^{(1)}(\hat{w}_m, (\hat{w}_m)_t) + a^{(2)}((\hat{w}_m)_t, (\hat{w}_m)_t) = 0$$

and

$$\frac{d}{dt}\hat{E}(t,\hat{w}_m) \le 0$$

for all $t \in [0,\tau]$. So we have uniform boundedness of the solution on $[0,\tau]$:

$$\hat{E}(t,\hat{w}_m) \le \hat{E}(0,\hat{w}_{m0}), \quad t \ge 0$$

which implies that :

(3.14) $$\{\hat{w}_m\}_{m\in\mathbb{N}} \quad \text{is bounded in} \quad C\left([0,T];X\right)$$

(3.15) $$\{(\hat{w}_m)_t\}_{m\in\mathbb{N}} \quad \text{is bounded in} \quad C\left([0,T];Y\right).$$

We further to show that $\{(\hat{w}_m)_{tt}\}_{m\in\mathbb{N}}$ is bounded.
Let m be fixed. We Consider $\hat{\phi} \in X$ and $\hat{\phi} = \hat{\varphi}_1 + \hat{\varphi}_2$ such that $\hat{\varphi}_1 \in \hat{V}_m$ and $\hat{\varphi}_2$ orthogonal to $\hat{V}_m$ in Y. Hence $\langle(\hat{w}_m)_{tt},\hat{\varphi}_1\rangle_Y = 0$. From (3.11), we have:

$$\langle(\hat{w}_m)_{tt},\hat{\phi}\rangle_Y = -a^{(1)}(\hat{w}_m,\hat{\phi}) - a^{(2)}((\hat{w}_m)_t,\hat{\phi}) \le M\|\hat{\varphi}_1\|_X \le M\|\hat{\phi}\|_X,$$

where M is a positive constant which does not depend on m. This implies that

$$\{(\hat{w}_m)_{tt}\}_{m\in\mathbb{N}} \quad \text{is bounded in} \quad C\left([0,T];X'\right).$$

Step 3 : Passage to the limit
According to the Eberlein-Smulian Theorem [6], we can extract weakly convergent subsequences $\{\hat{w}_{m_l}\}_{l\in\mathbb{N}}$, $\{(\hat{w}_{m_l})_t\}_{l\in\mathbb{N}}$ and $\{(\hat{w}_{m_l})_{tt}\}_{l\in\mathbb{N}}$ and $\hat{w} \in L^2(0,T;X)$, $\hat{w}_t \in L^2(0,T;Y)$, $\hat{w}_{tt} \in L^2\left(0,T;X'\right)$ such that:

(3.16) $$\{\hat{w}_{m_l}\} \rightharpoonup \hat{w} \quad \text{in} \quad L^2(0,T;X)$$

(3.17) $$\{(\hat{w}_{m_l})_t\} \rightharpoonup \hat{w}_t \quad \text{in} \quad L^2(0,T;Y)$$

(3.18) $$\{(\hat{w}_{m_l})_{tt}\} \rightharpoonup \hat{w}_{tt} \quad \text{in} \quad L^2\left(0,T;X'\right).$$

Let $m_0 \in \mathbb{N}$. For all functions $\hat{\varphi} \in L^2\left(0,T;\hat{V}_{m0}\right)$ of the form

(3.19) $$\hat{\varphi}(x,t) = \sum_{j=1}^{m_0} \alpha_j(t)\phi_j(x)$$

where $\alpha_j \in L^2(0,T;\mathbb{R})$ and for all $m_l \ge m_0$, the formulation (3.11) becomes:

(3.20) $$\int_0^T \left[\langle(\hat{w}_{m_l})_{tt},\hat{\varphi}\rangle_Y + a^{(1)}(\hat{w}_{m_l},\hat{\varphi}) + a^{(2)}((\hat{w}_{m_l})_t,\hat{\varphi})\right] dt = 0.$$

Therefore, passing on to the limit in (3.20) for $m = m_l$ when $l \to \infty$ and using the convergence results (3.16)–(3.18), one obtains:

$$\int_0^T \left[\langle \hat{w}_{tt}, \hat{\varphi}\rangle_{X,X'} + a^{(1)}(\hat{w}, \hat{\varphi}) + a^{(2)}(\hat{w}_t, \hat{\varphi})\right] dt = 0. \tag{3.21}$$

Then $\langle(\hat{w})_{tt}, \hat{\varphi}\rangle + a^{(1)}(\hat{w}, \hat{\varphi}) + a^{(2)}(\hat{w}_t, \hat{\varphi}) = 0$ a.e on $[0, T]$ for all $\hat{\varphi} \in L^2(0, T; X)$. The functions $\hat{\varphi}$ of the form (3.19) are dense in $L^2(0, T; X)$. So (3.21) is well defined for all $\hat{\varphi} \in L^2(0, T; X)$. We thus obtain the expression of the weak formulation almost everywhere on $[0, T]$. $\hat{w}$ is well the solution of (3.3).

Regarding additional regularities, we have that $\hat{w}$ follows the regularity (3.7)–(3.8), by definition of weak solution and the important results (3.14)–(3.15). □

In addition, note that (3.9) follows directly from Remark 3.1 after possible modification on a set of measures zero and the regularity of (3.10) is deduced from Remark 3.1 and from the previous duality theorem.

3.3. **Uniqueness of the Weak Solution.** Let us state the following theorem on the uniqueness of the solution for the weak formulation.

Theorem 3.2. *The weak formulation* (3.4) *with the conditions* (3.5) – (3.6) *has a unique solution* $\hat{w}$.

Proof. This proof is an adaptation of proof of Theorem 8.1 pp. $290 - 291$ in [9]. First let us show that the solution $\hat{w}$ satisfies the conditions (3.5)–(3.6). Let $\hat{\phi} \in C^2([0, T], X)$ such that $\hat{\phi}(T) = 0$ and $\hat{\phi}_t(T) = 0$. Integrating the identity (3.4) over $[0, T]$, we get:

$$\int_0^T \left[\langle \hat{w}_{tt}, \hat{\phi}\rangle_{X,X'} + a^{(1)}(\hat{w}, \hat{\phi}) + a^{(2)}(\hat{w}_t, \hat{\phi})\right] d\tau = 0.$$

By integrating by parts over $[0, T]$ under the duality pairing, we have:

$$\int_0^T \left[\langle \hat{w}, \hat{\phi}_{tt}\rangle_Y + a^{(1)}(\hat{w}, \hat{\phi}) + a^{(2)}(\hat{w}_t, \hat{\phi})\right] d\tau = \langle \hat{w}_t(0), \hat{\phi}(0)\rangle_{X,X'} - \langle \hat{w}(0), \hat{\phi}_t(0)\rangle_Y. \tag{3.22}$$

Similarly, for a fixed m, it follows by integrating twice by parts (3.11):

$$\int_0^T \left[\langle \hat{w}_m, \hat{\phi}_{tt}\rangle_Y + a^{(1)}(\hat{w}_m, \hat{\phi}) + a^{(2)}((\hat{w}_m)_t, \hat{\phi})\right] d\tau = \langle \hat{v}_{m0}, \hat{\phi}(0)\rangle_Y - \langle \hat{w}_{m0}, \hat{\phi}_t(0)\rangle_Y. \tag{3.23}$$

Using (3.12)–(3.13) with (3.16)–(3.18), and passing to the limit in (3.23) along the convergent subsequence $\{\hat{w}_{m_l}\}$, this gives:

(3.24)
$$\int_0^T \left[\langle \hat{w}, \hat{\phi}_{tt}\rangle_Y + a^{(1)}(\hat{w}, \hat{\phi}) + a^{(2)}((\hat{w})_t, \hat{\phi})\right] d\tau = \langle \hat{v}_0, \hat{\phi}(0)\rangle_Y - \langle \hat{w}_0, \hat{\phi}_t(0)\rangle_Y.$$

Comparing (3.24) with (3.22), we have $\hat{w}(0) = \hat{w}_0$ and $\hat{v}_0 = \hat{w}_t(0)$. Thus the initial conditions are satisfied.

Let us now show the uniqueness of the weak solution of (3.4).

Let $0 < s < T$ be fixed and introduce an auxiliary function:

$$\hat{\psi} :]0, T[\to \mathbb{R}$$
$$t \mapsto \hat{\psi}(t) = \begin{cases} \int_t^s \hat{w}(\tau)d\tau, & 0 < t < s \\ 0, & t \geq s. \end{cases}$$

By replacing $\hat{\phi}$ by $\hat{\psi}$ in (3.4) and by integrating by parts on $]0, T[$, we obtain

$$\int_0^s \left[\langle \hat{w}_t(\tau), \hat{w}(\tau)\rangle_Y - a^{(1)}(\hat{\psi}_t(\tau), \hat{\psi}(\tau)) + a^{(2)}(\hat{w}(\tau), \hat{w}(\tau))\right] d\tau = 0$$

$$\int_0^s \frac{d}{dt}\left[\frac{1}{2}\langle \hat{w}(\tau), \hat{w}(\tau)\rangle_Y - \frac{1}{2}a^{(1)}(\hat{\psi}(\tau), \hat{\psi}(\tau))\right] d\tau = -\int_0^s a^{(2)}(\hat{w}(\tau), \hat{w}(\tau))d\tau.$$

This is equivalent to

$$\left[\frac{1}{2}\langle \hat{w}(\tau), \hat{w}(\tau)\rangle_Y - \frac{1}{2}a^{(1)}(\hat{\psi}(\tau), \hat{\psi}(\tau))\right]_0^s = -\int_0^s a^{(2)}(\hat{w}(\tau), \hat{w}(\tau))d\tau.$$

Therefore,

$$\frac{1}{2}\|\hat{w}(s)\|_Y + \frac{1}{2}a^{(1)}(\hat{\psi}(0), \hat{\psi}(0)) \leq 0.$$

The bilinear form $a^{(1)}(.,.)$ is coercive. Hence $\hat{w}(s) \equiv 0$ et $\hat{\psi}(0) = 0$. Since $s \in]0, T[$ was arbitrary then $\hat{w} \equiv 0$. □

3.4. **Higher regularity results.** Before showing the theorem of higher regularity (Theorem 3.3) of the weak solution $\hat{w}$ solving (3.4)-(3.6), let's state a following lemma which is proved in [9].

Lemma 3.2. *Let X and Y two Banach spaces, $X \subset Y$ with continuous injection, the space X being reflexive. We set:*

$$C_w([0, T]; Y) = \left\{w \in L^\infty(0, T; Y) : t \longrightarrow \langle f, w(t)\rangle \text{ is continuous on } [0, T], \forall f \in Y'\right\}$$

which denotes the space of weakly continuous functions with values in Y. Thus we get

$$L^{\infty}\left(0,T;X\right)\cap C_w\left(\left[0,T\right];Y\right)=C_w\left(\left[0,T\right];X\right).$$

Theorem 3.3. *After, possibly, a modification on a set of measure zero, the weak solution $\hat{w}$ of* (3.4)-(3.6) *satisfies*

$$\begin{aligned} \hat{w} &\in C\left(\left[0,T\right];X\right) \\ \hat{w}_t &\in C\left(\left[0,T\right];Y\right). \end{aligned}$$

Proof. This proof is an adaption of standard strategies in section 8.4 of [9] pp. $297-301$ and in section 2.4 of [14].

Considering Lemma 3.2, it follows from (3.7) and (3.9) that $\hat{w}\in C_w\left(\left[0,T\right];X\right)$. Similarly (3.8) and (3.10) imply $\hat{w}_t\in C_w\left(\left[0,T\right];Y\right)$.

Let $\xi\in C^{\infty}(\mathbb{R})$ a scalar cutoff function such that $\xi(x)=1$ if $x\in J\subset\subset[0,T]$ and $\xi(x)=0$ else. Then the function $\xi\hat{w}$ is then compactly supported. Let η^{ε} be a standard mollifier in time. Using convolutional regulation of distributions, we define

$$\hat{w}^{\varepsilon}=\eta^{\varepsilon}*\xi\hat{w}\in C_c^{\infty}(\mathbb{R},X).$$

$\hat{w}^{\varepsilon}$ converges to $\hat{w}$ in X and $\hat{w}_t^{\varepsilon}$ converges to $\hat{w}_t$ in X almost everywhere on J. Hence, $\hat{E}(t,\hat{w}^{\varepsilon})$ converges to $\hat{E}(t,\hat{w})$ almost everywhere on J. Since $\hat{w}^{\varepsilon}$ is smooth, we obtain by a straightforward calculation on J:

(3.25)
$$\frac{d}{dt}\hat{E}(t,\hat{w}^{\varepsilon})=-2\left(\mu_{11}+\mu_{22}\right)\left[\frac{d}{dt}\hat{w}_x^{\varepsilon}(t).\frac{d}{dt}\hat{w}^{\varepsilon}(t)\right]-\mu_{12}\left[\frac{d}{dt}\hat{w}_x^{\varepsilon}(t)\right]^2-\mu_{21}\left[\frac{d}{dt}\hat{w}^{\varepsilon}(t)\right]^2.$$

Passing to the limit in (3.25) as $\varepsilon\to 0$, we obtain:

(3.26)
$$\frac{d}{dt}\hat{E}(t,\hat{w})=-2\left(\mu_{11}+\mu_{22}\right)\left[\frac{d}{dt}\hat{w}_x(t).\frac{d}{dt}\hat{w}(t)\right]-\mu_{12}\left[\frac{d}{dt}\hat{w}_x(t)\right]^2-\mu_{21}\left[\frac{d}{dt}\hat{w}(t)\right]^2$$

in the sense of distributions on J. (3.26) holds on all compact subintervals of $[0,T]$, since J was arbitrary.

Let $t\in[0,\infty[$ be fixed and $(t_n)_{n\in\mathbb{N}}$ a sequence such that $\lim_{n\longrightarrow\infty}t_n=t$. Let $(\nu_n)_{n\in\mathbb{N}}$ be defined by

$$\nu_n=\frac{1}{2}\|\hat{w}_t(t)-\hat{w}_t(t_n)\|_Y^2+\frac{1}{2}\|\hat{w}(t)-\hat{w}(t_n)\|_X^2+\frac{\alpha}{2}\left[\hat{w}_x(t)-\hat{w}_x(t_n)\right]^2+\frac{\beta}{2}\left[\hat{w}(t)-\hat{w}(t_n)\right]^2.$$

Then

$$\nu_n = \hat{E}(t,\hat{w}) + \hat{E}(t_n,\hat{w}) - \langle \hat{w}_t(t), \hat{w}_t(t_n)\rangle_Y - \langle \hat{w}(t), \hat{w}(t_n)\rangle_X - \alpha\hat{w}_x(t)\hat{w}_x(t_n) - \beta\hat{w}(t)\hat{w}(t_n). \tag{3.27}$$

Since $\hat{w}$, $\hat{w}_t$ are weakly continuous and $\hat{E}$ is continuous in t, we have, passing to the limit in (3.27):

$$\lim_{n\to\infty} \nu_n = 0.$$

Therefore, this implies that

$$\lim_{n\to\infty} \|\hat{w}_t(t) - \hat{w}_t(t_n)\|_Y^2 = 0 \text{ and } \lim_{n\to\infty} \|\hat{w}(t) - \hat{w}(t_n)\|_X^2 = 0.$$

Finally, $\hat{w} \in C([0,T];X)$ and $\hat{w}_t \in C([0,T];Y)$. □

4. SEMI-DISCRETE SCHEME BY THE FINITE ELEMENT METHOD (FEM)

4.1. **Piecewise Cubic Hermite Polynomials.** In this subsection, we recall the construction of these functions as for example in [5,15] and another references.

We construct an appropriate piecewise space of C^1-functions on $\bar{\Omega} = [0,1]$. Let us start with the reference domain $\bar{\Omega} = [0,1]$ where $\Omega = (0,1)$. We first introduce the local nodal basis functions on the reference domain (see Figure 1) defined by

$$N_1(\xi) = 2\xi^3 - 3\xi^2 + 1, \quad N_2(\xi) = \xi^3 - 2\xi^2 + \xi, \tag{4.1}$$

$$N_3(\xi) = -2\xi^3 + 3\xi^2, \quad N_4(\xi) = \xi^3 - \xi^2. \tag{4.2}$$

We need to use the Hermite cubic nodal basis functions in an arbitrary interval rather than the reference interval $[0,1]$. This can be done by transforming our formulas in (4.1)-(4.2) to an arbitrary interval, for example $[x_l; x_r]$ with the function $\zeta : [x_l, x_r] \longrightarrow [0,1]$ as $\zeta(x) = \frac{x-x_l}{x_r-x_l}$. This leads to functions $\psi_i : [x_l, x_r] \to \mathbb{R}$, $i = 1,2,3,4$, defined as

$$\psi_1(x) = N_1(\zeta(x)), \quad \psi_2(x) = (x_r - x_l)\left(N_2(\zeta(x))\right)$$
$$\psi_3(x) = N_3(\zeta(x)), \quad \psi_4(x) = (x_r - x_l)\left(N_4(\zeta(x))\right).$$

Notice that the nodal value of $\psi_i(x)$ is equal the nodal value of $N_i(x)$ for $i = 1,2,3,4$, and $\psi_i(x)$ preserves the properties of the Hermite cubic nodal basis functions.

Let n be a fixed integer and let us introduce a partition of $\bar{\Omega} = [0,1]$ defined by $\Omega_h = \cup_{m=1}^{n-1} Z_m$ where $Z_m = [x_m; x_{m+1}]$ for $m = 1,2,...,n-1$, is called an element of this partition, and we assume that these elements are formed by node points $0 = x_1 < x_2 < ... < x_{n-1} < x_n = 1$, with $n-1$ being the number of elements. Notice that, if the subdivision is uniform, let us denote the step length by $h = 1/n$ and $Z_m = [mh; (m+1)h] = [x_m; x_{m+1}]$.

For each element of Z_m, we will use the notation $\psi_{m,i} = [x_m; x_{m+1}] \longrightarrow \mathbb{R}$, $i = 1,2,3,4$, to denote the Hermite cubic local nodal basis functions defined over this element. At each node x_i, $i = 1,2,...,n$, we define two global nodal basis functions as follows.
At x_1, we define

$$\phi_1(x) = \begin{cases} 0 & \text{if} \quad x \notin [x_1; x_2] \\ \psi_{1,1}(x) & \text{if} \quad x \in [x_1; x_2] \end{cases} \quad \phi_2(x) = \begin{cases} 0 & \text{if} \quad x \notin [x_1; x_2] \\ \psi_{1,2}(x) & \text{if} \quad x \in [x_1; x_2] \end{cases}.$$

At x_m, $m = 2,3,...,n-1$, we define

$$\phi_{2m-1}(x) = \begin{cases} 0 & \text{if } x \notin [x_{m-1}; x_{m+1}] \\ \psi_{m-1,3}(x) & \text{if } x \in [x_{m-1}; x_m] \\ \psi_{m,1}(x) & \text{if } x \in [x_m; x_{m+1}] \end{cases}, \phi_{2m}(x) = \begin{cases} 0 & \text{if } x \notin [x_{m-1}; x_{m+1}] \\ \psi_{m-1,4}(x) & \text{if } x \in [x_{m-1}; x_m] \\ \psi_{m,2}(x) & \text{if } x \in [x_m; x_{m+1}] \end{cases}.$$

And at x_n, we define

$$\phi_{2n-1}(x) = \begin{cases} 0 & \text{if } x \notin [x_{n-1}; x_n] \\ \psi_{n-1,3}(x) & \text{if } x \in [x_{n-1}; x_n] \end{cases}, \phi_{2n}(x) = \begin{cases} 0 & \text{if } x \notin [x_{n-1}; x_n] \\ \psi_{n-1,4}(x) & \text{if } x \in [x_{n-1}; x_n] \end{cases}.$$

Thus, the nodal value for $\phi_m(x_j)$ for $m = 1,2,...,2n$ and $j = 1,2,...,n$ is given as follows:

$$\phi_{2i-1}(x_j) = \begin{cases} 0 & \text{if} \quad x_j \neq x_i \\ 1 & \text{if} \quad x_j = x_i \end{cases} \quad \phi_{2i}(x_j) = \begin{cases} 0 & \text{if} \quad x_j \neq x_i \\ 1 & \text{if} \quad x_j = x_i \end{cases}$$

for $i = 1,2,...,n$. Notice that the nodal values for ϕ_m, $m = 1,2,...,2n-1,2n$ are carried from the properties of the Hermite cubic local nodal basis functions $\psi_{m,i}$, $i = 1,2,3,4$. Here, we plot the standard Hermite cubic global basis functions with nodes $x_1 = 0$, $x_2 = 0.2$ and $x_3 = 0.4$ on the domain $[0,1]$ to give a visualization of these global basis functions (see Figure 2).

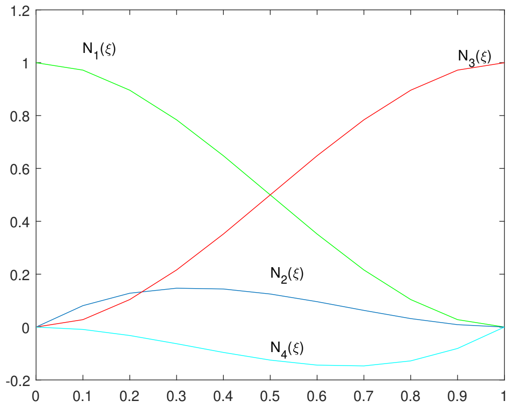

FIGURE 1. The standard Hermite cubic local basis functions $N_i(\xi)$, $i = 1, 2, 3, 4$.

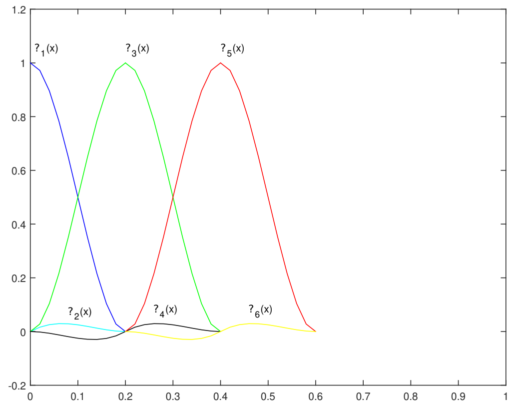

FIGURE 2. First six of the Hermite cubic global basis functions with six nodes partition on [0, 1].

Finally, our Hermite cubic finite element space is defined as:

$$S^h = \text{span}\left\{\phi_1, \phi_2, ..., \phi_{2n-1}, \phi_{2n}\right\}.$$

With the separation of variables, the approximate solution $w_h \in S^h$ which we seek can be written as follows:

$$w_h(x,t) = \sum_{i=1}^{n} \left[\bar{w}_i(t)\phi_{2i-1}(x) + (\bar{w}_i)_x(t)\phi_{2i}(x)\right]$$

for $w_h(1,t) = \bar{w}_{n-1}(t)$ and $(w_h)_x(1,t) = \bar{w}_n(t)$.

4.2. **Space discretization.** Let X^h and S^h be the respective finite dimension subspaces of X and $H^2_E(0,1)$ such that ϕ_j, $j = 1, ..., N$ (with $N = 2n$) be fixed basis for S^h. We obtain the following approximating problem :

Problem G^h: Find $\widehat{w}_h = (w_h(1), (w_h)_x(1), w_h) \in C^2([0;\infty), X^h)$ or more simply find the last component $w_h \in C^2([0;\infty), S^h)$ such that for each $t \in (0;\infty)$

(4.3) $$c((w_h)_{tt}(t), \phi_j) + a((w_h)_t(t), \phi_j) + b(w_h(t), \phi_j) = 0, \qquad \phi_j \in S^h$$

with

$$w_h(.,0) = w_0^h \in S^h \quad \text{and} \quad (w_h)_t(.,0) = w_1^h \in S^h,$$

where the inner products c and b are bilinear forms such as :

$$c((w_h)_{tt}(t),\phi_j) = \int_0^1 \rho(x)(w_h)_{tt}\phi_j dx,$$

$$b(w_h(t),\phi_j) = \int_0^1 EI(x)(w_h)_{xx}(\phi_j)_{xx}dx + \alpha(w_h)_x(1)(\phi_j)_x(1) + \beta(w_h)(1)(\phi_j)(1)$$

and a is bilinear form such as :

$$\begin{aligned} a((w_h)_t(t),\phi_j) &= 2\mu_{11}(w_h)_t(1)(\phi_j)_x(1) + \mu_{12}(w_h)_{xt}(1)(\phi_j)_x(1) \\ &+ \mu_{21}(w_h)_t(1)(\phi_j)(1) + 2\mu_{22}(w_h)_{xt}(1)(\phi_j)(1). \end{aligned}$$

By separation of variables, w_h can be written in the form: $w_h(x,t) = \sum_{i=1}^N \bar{w}_i(t)\phi_i(x)$ with $w_h(x,0) = \sum_{i=1}^N \bar{d}_i\phi_i(x)$ and $(w_h)_t(x,0) = \sum_{i=1}^N \bar{l}_i\phi_i(x)$ where W, d, l are respectively, vector representation of the functions w_h, $w_h(x,0)$ and $(w_h)_t(x,0)$ defined as follows:

$$\begin{aligned} W &= [\bar{w}_1 \quad \bar{w}_2 \quad \dots \quad \bar{w}_n \quad \bar{w}_{1x} \quad \bar{w}_{2x} \quad \dots \quad \bar{w}_{nx}]^T, \\ d &= [\bar{d}_1 \quad \bar{d}_2 \quad \dots \quad \bar{d}_n \quad \bar{d}_{1x} \quad \bar{d}_{2x} \quad \dots \quad \bar{d}_{nx}]^T, \\ l &= [\bar{l}_1 \quad \bar{l}_2 \quad \dots \quad \bar{l}_n \quad \bar{l}_{1x} \quad \bar{l}_{2x} \quad \dots \quad \bar{l}_{nx}]^T. \end{aligned}$$

Equation (4.3) is equivalent to the following equation:

$$MW_{tt} + CW_t + KW = 0, \quad \text{with} \quad W(0) = d \quad \text{and} \quad W_t(0) = l. \tag{4.4}$$

M is the mass matrix and K is the rigidity matrix. For all $i,j = 1,...,N$, the corresponding matrices M, C and K are given by:

$$M_{i,j} = c(\phi_i,\phi_j) = \int_0^1 \rho(x)\phi_i\phi_j dx,$$

$$\begin{aligned} C_{i,j} = a(\phi_i,\phi_j) &= 2\mu_{11}\phi_i(1)(\phi_j)_x(1) + \mu_{12}(\phi_i)_x(1)(\phi_j)_x(1) + \mu_{21}(\phi_i)(1)(\phi_j)(1) \\ &+ 2\mu_{22}(\phi_i)_x(1)(\phi_j)(1), \end{aligned}$$

$$K_{i,j} = b(\phi_i,\phi_j) = \int_0^1 EI(x)(\phi_i)_{xx}(\phi_j)_{xx}dx + \alpha(\phi_i)_x(1)(\phi_j)_x(1) + \beta(\phi_i)(1)(\phi_j)(1).$$

Remark 4.1. *The matrices M and K are symmetric, defined and positive therefore M and K are invertible. It follows from the theory of linear differential equations that* (4.4) *has a unique solution. This implies the existence and the uniqueness of the solution of problem G^h.*

4.3. **A-priori error estimates.** In this subsection, the a-priori error estimates for the semi-discrete solution approximation (4.3) are obtained. We will use a method used in [3, 4] to obtain error estimates.
The projection of weak solution w to S^h on $H^2_E(0,1)$ denoted by w_p is such that:

$$b(w - w_p, \phi) = 0 \quad \text{for all } \ \phi \in S^h.$$

Using the projection, we divide the error $e_h(t) = w(t) - w_h(t)$ such that $e_h(t) = e(t) + e_p(t)$ with $e(t) = w_p(t) - w_h(t)$ and $e_p(t) = w(t) - w_p(t)$.
Let us state the following proposition [3] which will be useful in the rest of our work.

Proposition 4.1. *Let* $w \in C^2((0,T); H^2_E(0,1))$ *and* $w_p \in C^2((0,T), S^h)$ *then*

$$c((e_h)_{tt}(t), \phi) + a((e_h)_t(t), \phi) + b(e(t), \phi) = 0, \quad \textit{for all} \quad \phi \in S^h. \tag{4.5}$$

We set $H^4_E(0,1) = \{w \in H^4(0,1) : w(0) = w_x(0) = 0\}$. From Proposition 3.2. of [4], we have the following estimations almost every in t, and where C_Π is a constant:

$$\|e_p\|_{H^2_E(0,1)} = \|w - w_p\|_{H^2_E(0,1)} \le C_\Pi h^2 \|w\|_{H^4_E(0,1)} \tag{4.6}$$

$$\|(e_p)_t\|_{H^2_E(0,1)} = \|w_t - (w_p)_t\|_{H^2_E(0,1)} \le C_\Pi h^2 \|w_t\|_{H^4_E(0,1)} \tag{4.7}$$

$$\|(e_p)_{tt}\|_{L^2(0,1)} = \|w_{tt} - (w_p)_{tt}\|_{L^2(0,1)} \le C_\Pi h^2 \|w_{tt}\|_{H^4_E(0,1)}. \tag{4.8}$$

The following lemma gives an estimate of the error $e(t) = w_p(t) - w_h(t)$.

Lemma 4.1. *Let* $w \in C^1([0,T) \,; H^2_E(0,1)) \cap C^2((0,T) \,; H^2_E(0,1))$ *then,*

$$\|e_t(t)\|_{L^2(0,1)} + \|e(t)\|_{H^2_E(0,1)} \le 4\sqrt{e^{3t}} A_T \tag{4.9}$$

with

$$\begin{aligned} A_T \;\; = \;\; & \int_0^T \|(e_p)_{tt}(.)\|_{L^2(0,1)} + 3K \max_{t\in[0,T]} \|(e_p)_t(t)\|_{H^2_E(0,1)} + 3K \int_0^T \|(e_p)_{tt}(.)\|_{H^2_E(0,1)} \\ & + \|e_t(0)\|_{L^2(0,1)} + \sqrt{1+K}\|e(0)\|_{H^2_E(0,1)} + \sqrt{K}\|(e_p)_t(0)\|_{H^2_E(0,1)} \end{aligned}$$

where K *is a constant.*

Proof. It is easy to show that the error e checks the weak formulation (3.3) for all $\phi_j \in S^h$ or (4.3). Using $\phi_j = e_t$, we obtain the energy expression for e analogously to (2.8):

$$E(t,e) = \frac{1}{2}c(e_t(t), e_t(t)) + \frac{1}{2}b(e(t), e(t)) = \|e_t(t)\|^2_{L^2(0,1)} + \|e(t)\|^2_{H^2_E(0,1)}. \tag{4.10}$$

From (4.10), and since b and c are inner products, we have

$$\frac{d}{dt}E(t,e) = c(e_{tt}(t), e_t(t)) + b(e(t), e_t(t)).$$

Taking $\phi = e_t$ in (4.5) and considering the fact that $a(e_t(t), e_t(t)) \geq 0$, we obtain

$$\frac{d}{dt}E(t,e) \leq -c((e_p)_{tt}(t), e_t(t)) - a((e_p)_t(t), e_t(t)).$$

Thus for all $t \in (0,T)$,

$$\int_0^t \frac{d}{dt}E(.,e) \leq -\int_0^t c((e_p)_{tt}(t), e_t(t)) - \int_0^t a((e_p)_t(.), e_t(.)). \tag{4.11}$$

Since $e_p \in C^1([0,T); H^2_E(0,1)) \cap C^2((0,T); H^2_E(0,1))$, then for any $t \in (0,T)$,

$$\int_0^t a((e_p)_t(.), e_t(.)) = a((e_p)_t(t), e(t)) - a((e_p)_t(0), e(0)) - \int_0^t a((e_p)_{tt}(.), e(.)). \tag{4.12}$$

Using (4.11) and (4.12) we obtain

$$\int_0^t \frac{d}{dt}E(.,e) \leq \tag{4.13}$$
$$-\int_0^t c((e_p)_{tt}(t), e_t(t)) \left(a((e_p)_t(t), e(t)) - a((e_p)_t(0), e(0)) - \int_0^t a((e_p)_{tt}(.), e(.)) \right).$$

By integration and using Cauchy Schwarz's inequality, Young's inequality and the fact that $|a(w,\phi)| \leq K\|w\|_{H^2_E(0,1)}\|\phi\|_{H^2_E(0,1)}$ with $K = k\,(\mu_{11} + \mu_{12} + \mu_{21} + \mu_{22})$, (4.13) yields

$$E(t,e) \leq 3\int_0^t E(.,e) + K_T$$

with

$$K_T = \int_0^T \|(e_p)_{tt}(.)\|^2_{L^2(0,1)} + 8K^2 \max_{t\in[0,T]} \|(e_p)_t(t)\|^2_{H^2_E(0,1)} + 8K^2 \int_0^T \|(e_p)_{tt}(.)\|^2_{H^2_E(0,1)}$$
$$+ \|e_t(0)\|^2_{L^2(0,1)} + (1+K)\|e(0)\|^2_{H^2_E(0,1)} + K\|(e_p)_t(0)\|^2_{H^2_E(0,1)}.$$

By applying the Gronwall inequality and using the fact that $(a+b)^2 \leq 2a^2 + 2b^2$, we get the result (4.9). □

Now, we give an important result for the convergence of the semi-discrete scheme:

Theorem 4.1. *Let w the solution of* (3.3) *. Assume that $w \in C^1([0,T];H^2_E(0,1)) \cap C^2([0,T];H^2_E(0,1))$, $w_{tt} \in L^2([0,T];H^4_E(0,1))$ and the expressions* (4.6)-(4.9). *Then, for all $t \in (0,T)$*

$$\begin{aligned}\|w(t)-w_h(t)\|_{H^2_E(0,1)} &+ \|w_t(t)-(w_h)_t(t)\|_{L^2(0,1)} \\ &\leq C_\Pi h^2\Big(\|w(t)\|_{H^4_E(0,1)}+\|w_t(t)\|_{H^4_E(0,1)}\Big) \\ &+ 4\sqrt{e^{3t}}C_\Pi h^2\Big[\int_0^T \|w_{tt}(.)\|_{H^4_E(0,1)} + 3K\max_{t\in[0,T]}\|w_t(t)\|_{H^4_E(0,1)} \\ &+ 3K\int_0^T \|w_{tt}(.)\|_{H^4_E(0,1)}\Big] + 4\sqrt{e^{3t}}\Big[\|w_1^P-w_1^h\|_{L^2(0,1)} \\ &+ \sqrt{1+K}\|w_0^P-w_0^h\|_{H^2_E(0,1)} + \sqrt{K}\|w_1-w_1^P\|_{H^2_E(0,1)}\Big].\end{aligned}$$

Furthermore, if w_0^h and w_1^h are respectively Hermite interpolations of w_0 and of w_1, then :

$$\begin{aligned}\|w(t)-w_h(t)\|_{H^2_E(0,1)} &+ \|w_t(t)-(w_h)_t(t)\|_{L^2(0,1)} \\ &\leq C_\Pi h^2\Bigg[\Big(\|w(t)\|_{H^4_E(0,1)}+\|w_t(t)\|_{H^4_E(0,1)}\Big) \\ &+4\sqrt{e^{3t}}\Big((T+3KT)\max_{t\in[0,T]}\|w_{tt}(t)\|_{H^4_E(0,1)} + 3K\max_{t\in[0,T]}\|w_t(t)\|_{H^4_E(0,1)} \\ &+(2+\sqrt{K})\|w_1\|_{H^4_E(0,1)} + 2\sqrt{1+K}\|w_0\|_{H^4_E(0,1)}\Big)\Bigg].\end{aligned}$$

Proof. We know that

$$\begin{aligned}\|w(t)-w_h(t)\|_{H^2_E(0,1)} + \|w_t(t)-(w_h)_t(t)\|_{L^2(0,1)} &\leq \|w(t)-(w_p)(t)\|_{H^2_E(0,1)} \\ &+ \|w_t(t)-(w_p)_t(t)\|_{L^2(0,1)} \\ &+ \|e(t)\|_{H^2_E(0,1)} + \|e_t(t)\|_{L^2(0,1)}.\end{aligned} \tag{4.14}$$

Using Lemma 4.1, (4.14) and (4.6)-(4.8), we get the Theorem 4.1. □

5. Numerical simulations

In this section, we show through simulations, the efficiency of the numerical method developed in the previous section.
In what follows, we will take : $EI = \rho = 1$.

5.1. **Order of convergence (o.o.c).** In order to verify the order of convergence proved in previous section, simulations are performed for different space discretization steps.

We take $\alpha = \beta = 10^{-1}$ and $\mu_{11} = 2$, $\mu_{22} = 3$, $\mu_{21} = 5$, $\mu_{12} = 10$ (according the conditions (1.6)). In following tables, the l^2-error norms of e_h are listed. The o.o.c. results are given for fixed $\Delta t = 10^{-2}$ and varying space discretization step $h = \frac{1}{2^n}$, $n = 2, 3, 4, 5, 6, 7$, on the space interval $[0, 1]$. The order is calculated using the following formula:

$$(\text{o.o.c})_i = \log\left[(e_h)_i/(e_h)_{i-1}\right]/\log\left(h_i/h_{i-1}\right), \quad i = 1, ..., 5.$$

For the initial conditions, we take two cases:

$$\text{Case a)} \quad w_0(x) = -0.6x^2 + 0.4x^3, \quad v_0 \equiv 0 \quad \text{(table on the left)}$$

$$\text{Case b)} \quad w_0(x) = \frac{1}{2}x^2, \quad v_0 \equiv 0 \quad \text{(table on the right).}$$

Δt	h	$\|e_h\|_{l^2}$	$o.o.c$
10^{-2}	$\frac{1}{4}$	$3.398 * 10^{-1}$	—
10^{-2}	$\frac{1}{8}$	$1.735 * 10^{-1}$	1.2
10^{-2}	$\frac{1}{16}$	$5.28 * 10^{-2}$	1.71
10^{-2}	$\frac{1}{32}$	$1.43 * 10^{-2}$	1.88
10^{-2}	$\frac{1}{64}$	$3.7 * 10^{-3}$	1.95
10^{-2}	$\frac{1}{128}$	$8.95 * 10^{-4}$	2.04

Δt	h	$\|e_h\|_{l^2}$	$o.o.c$
10^{-2}	$\frac{1}{4}$	$8.314 * 10^{-1}$	—
10^{-2}	$\frac{1}{8}$	$2.078 * 10^{-1}$	2.0002
10^{-2}	$\frac{1}{16}$	$5.19 * 10^{-2}$	2.0014
10^{-2}	$\frac{1}{32}$	$1.29 * 10^{-2}$	2.0042
10^{-2}	$\frac{1}{64}$	$3.2 * 10^{-3}$	2.0112
10^{-2}	$\frac{1}{128}$	$7.61 * 10^{-4}$	2.0704

We thus notice that the choice of the initial conditions have an impact on the speed of convergence.

5.2. **Energy decay and representation of solution** $\widehat{w}(x,t) = (w(1,t), w_x(1,t), w(x,t))$**.** Now, we present the numerical simulation for the case a), the energy decay, tip position $w(1,t)$, tip angle $w_x(1,t)$ and deflection $w(x,t)$ of the beam over time. Then we discuss according to the values of the different variables, α, β, μ_{ij}.

5.2.1. *Energy decay.* As for the graphical representation of energy decay, we set firstly $\alpha = \beta = 0$ and $\mu_{11} = \mu_{22} = \mu_{21} = 0$ and we vary μ_{12} (see Figure 3). Secondly, we set $\mu_{12} = 1$, $\mu_{11} = \mu_{22} = \mu_{21} = 0$ and we vary α, β (so that we always have $\alpha = \beta$) (see Figure 4). These two figures are as follows.

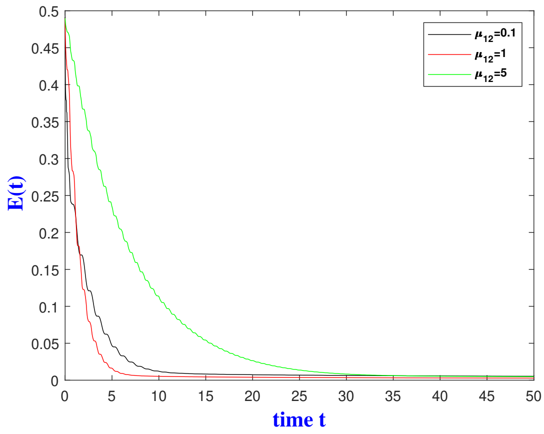

FIGURE 3. Energy for different values of μ_{12}.

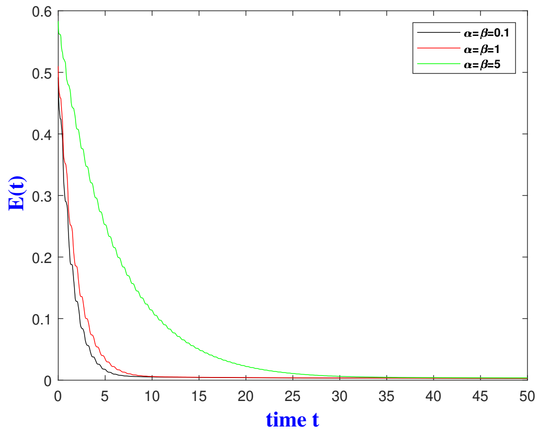

FIGURE 4. Energy for different values of α and β.

We notice that the choice of these values has an influence on the rate of energy decay.

5.2.2. *Representation of each component of the solution* $\widehat{w}(x,t)$. Firstly, we will represent the tip position and tip angle and then the deflection.

Graphical representation of tip position $w(1,t)$ **and tip angle** $w_x(1,t)$**.**

- We set $\alpha = \beta = 0$ and $\mu_{11} = \mu_{22} = \mu_{21} = 0$. Then we vary μ_{12}.

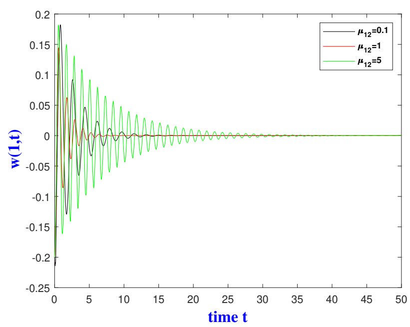

FIGURE 5. Tip position.

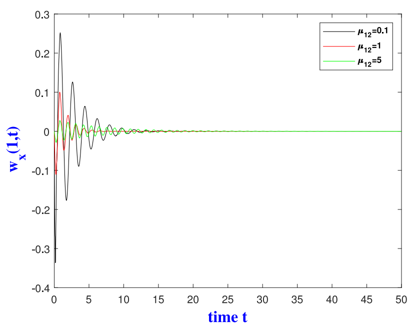

FIGURE 6. Tip angle

- We set $\mu_{12} = 1$ and $\mu_{11} = \mu_{22} = \mu_{21} = 0$. Then we vary α, β (so that we always have $\alpha = \beta$).

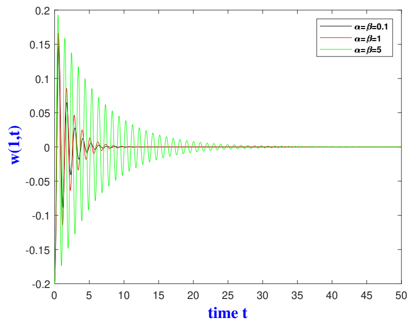

FIGURE 7. Tip position

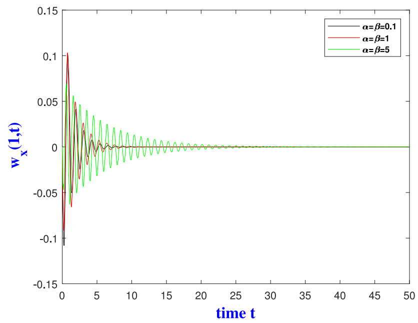

FIGURE 8. Tip angle

Graphical representation of deflection $w(x,t)$**:** First, we choose $\alpha = \beta = 0.1$, $\mu_{11} = \mu_{22} = \mu_{21} = 0$ and $\mu_{12} = 1$ (see Figure 9). Next, Figure 10 is obtained with the arbitrary values $\alpha = \beta = 0.1$ and $\mu_{11} = 2$, $\mu_{22} = 3$, $\mu_{21} = 5$, $\mu_{12} = 10$.

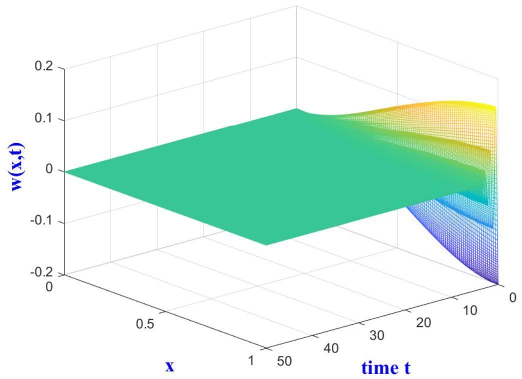

FIGURE 9. Deflection $w(x,t)$

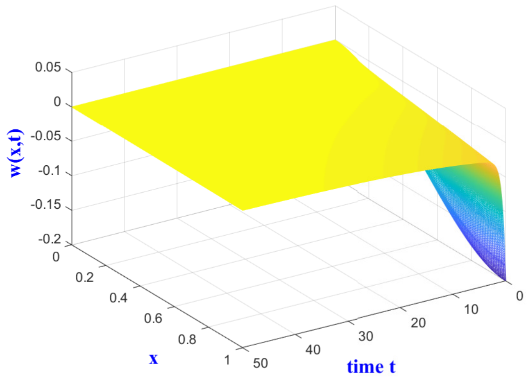

FIGURE 10. Deflection $w(x,t)$

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