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# GENERALIZED $\mathcal{L}$-CONTRACTIVE MAPPING THEOREMS IN PARTIALLY ORDERED SETS WITH b-METRIC SPACES

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ABSTRACT. In this paper, the notion of generalized $\mathcal{L}$-contractions is introduced in partially ordered sets with b-metric spaces and a new fixed point theorem for such contractions is established. An example and an application to differential equation are given to support the validity of the main theorem.

## 1. INTRODUCTION AND PRELIMINARIES

Banach's contraction principle, which plays a very important role in nonlinear analysis, has been generalized and expanded by many researchers.

In particular, Chatterjea [3] gave a generalization of Banach contraction principle as follows.

**Theorem 1.1.** *[3] Let $(X, d)$ be a complete metric space, and $T : X \to X$ be a C-contraction, i.e.*

$$\exists\alpha \in (0, \frac{1}{2}) : \ \forall x, y \in X, \ d(Tx, Ty) \leq \alpha[d(x, Ty) + d(y, Tx)].$$

*Then $T$ has a unique fixed point.*

Choudhury [5] introduced a generalization of the notion of C-contraction and he obtained the following result which is a generalization of Theorem 1.1.

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**Theorem 1.2.** *[5] Let $(X, d)$ be a complete metric space, and $T : X \to X$ be a weakly C-contraction, i.e.*

$$\forall x, y \in X,\ d(Tx, Ty) \leq \frac{1}{2}[d(x, Ty) + d(y, Tx)] - \varphi(d(x, Ty), d(y, Tx))$$

*where $\varphi : [0, \infty) \times [0, \infty) \to [0, \infty)$ is continuous with $\varphi(x, y) = 0 \Leftrightarrow x = y = 0$. Then $T$ has a unique fixed point.*

Harjani *et al.* [8] extended the result of [5] to partially ordered sets with metric spaces.

**Theorem 1.3.** *[8] Let $(X, \preceq)$ be a partially ordered set. Suppose that there exists a metric $d$ on $X$ such that $(X, d)$ is a complete metric space.*

*Let $T : X \to X$ be a non-decreasing mapping, i.e. $Tx \preceq Ty$ whenever $x \preceq y$, such that*

$$\forall x, y \in X : y \preceq x,\ d(Tx, Ty) \leq \frac{1}{2}[d(x, Ty) + d(y, Tx)] - \varphi(d(x, Ty), d(y, Tx))$$

*where $\varphi : [0, \infty) \times [0, \infty) \to [0, \infty)$ is continuous with $\varphi(x, y) = 0 \Leftrightarrow x = y = 0$. Assume that there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$.*

*If either $T$ is continuous or $x_n \preceq x$ for any non-decreasing sequence $\{x_n\} \subset X$ with*

$$\lim_{n\to\infty} d(x, x_n) = 0,$$

*then $T$ has a fixed point. Further if for $x, y \in X$, there exists $z \in X$ such that either $z \preceq x$ or $z \preceq y$, then $T$ has a unique fixed point.*

Let $\theta : (0, \infty) \to (1, \infty)$ be a function. Consider the following conditions:

($\theta 1$) $\theta$ is non-decreasing;

($\theta 2$) $\forall \{t_n\} \subset (0, \infty)$,

$$\lim_{n\to\infty} \theta(t_n) = 1 \Leftrightarrow \lim_{n\to\infty} t_n = 0;$$

($\theta 3$) $\exists r \in (0, 1) \wedge l \in (0, \infty)$:

$$\lim_{t\to 0^+} \frac{\theta(t) - 1}{t^r} = l;$$

($\theta 4$) $\theta$ is continuous on $(0, \infty)$.

Denote $\Theta_{123}$ by the family of all functions $\theta : (0, \infty) \to (1, \infty)$ satisfying conditions ($\theta 1$), ($\theta 2$) and ($\theta 3$), and $\Theta_{124}$ by the class of all functions $\theta : (0, \infty) \to (1, \infty)$ such that ($\theta 1$), ($\theta 2$) and ($\theta 4$) holds.

Recently, Jleli and Samet [10] introduced the notion of $\theta$-contractions and gave a generalization of the Banach contraction principle in generalized metric spaces, where $\theta \in \Theta_{123}$. Also, Ahmad *et al.* [1] extended the result of Jleli and Samet [10] to metric spaces by using $\theta \in \Theta_{124}$.

Very recently, Cho [4] introduced the notion of $\mathcal{L}$-contractions by introducing the concept of $\mathcal{L}$-simulation function and obtained fixed point result for such contractions in the setting of generalized metric spaces, which is a generalization of result [10].

In the paper, we introduce the concept of a new type of contraction maps which is generalization of C-contractions and weak C-contractions, and we establish a new fixed point theorem for such contraction maps in the setting of partially ordered sets with b-metric spaces.

Recall the concept of $\mathcal{L}$-simulation functions.

A function $\xi : [1, \infty) \times [1, \infty) \to \mathbb{R}$ is called *$\mathcal{L}$-simulation* [4] if and only if it satisfies the following conditions:

($\xi$1) $\xi(1, 1) = 1$;
($\xi$2) $\xi(t, s) < \frac{s}{t} \ \ \forall s, t > 1$;
($\xi$3) for any sequence $\{t_n\}, \{s_n\} \subset (1, \infty)$ with $t_n \leq s_n \ \ \forall n = 1, 2, 3, \cdots$

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 1 \Rightarrow \lim_{n\to\infty} \sup \xi(t_n, s_n) < 1.$$

Denote $\mathcal{L}$ by the family of all $\mathcal{L}$-simulation functions.

Note that $\xi(t, t) < 1 \ \forall t > 1$.

**Example 1.** *[4] Let $\xi_b, \xi_w, \xi, \xi_{wc} : [1, \infty) \times [1, \infty) \to \mathbb{R}$ be functions defined as follows, respectively:*

(1) $\xi_b(t, s) = \frac{s^k}{t} \ \ \forall t, s \geq 1$ *where* $k \in (0, 1)$;
(2) $\xi_w(t, s) = \frac{s}{t\phi(s)} \ \ \forall t, s \geq 1$, *where* $\phi : [1, \infty) \to [1, \infty)$ *is non-decreasing and lower semicontinuous such that* $\phi^{-1}(\{1\}) = 1$;
(3) $\xi(t, s) = \begin{cases} 1 & \text{if } (s, t) = (1, 1), \\ \frac{s}{2t} & \text{if } s < t, \\ \frac{s^\lambda}{t} & \text{otherwise,} \end{cases}$
$\forall s, t \geq 1$, *where* $\lambda \in (0, 1)$;

(4) $\xi_{wc}(t, s_1 s_2) = \frac{s_1 s_2}{t\psi(s_1,s_2)}$ $\forall t, s_1, s_2 \geq 1$, *where* $\psi : [1,\infty) \times [1,\infty) \to [1,\infty)$ *is continuous such that* $\psi(\mu,\nu) = 1$ *if and only if* $\mu = \nu = 1$.

Czerwik [6] introduced the concept of a b-metric.

A function $d : X \times X \to [0,\infty)$, where $X$ is a non-empty set, is called *b-metric* [6] on $X$ if and only if it satisfies the following conditions:

for all $x, y, z \in X$

(d1) $d(x,y) = 0$ if and only if $x = y$;
(d2) $d(x,y) = d(y,x)$;
(d3) $d(x,y) \leq 2[d(x,z) + d(z,y)]$.

In this case, the pair $(X, d)$ is called a b-metric space.

Also, Czerwik [7] gave a generalization of this concept by replacing constant $2$ in condition (d3) with constant $s \geq 1$ as follows:

Let $X$ be a non-empty set, and $d : X \times X \to [0,\infty)$ be a function such that for all $x, y, z \in X$

(d1) $d(x,y) = 0$ if and only if $x = y$;
(d2) $d(x,y) = d(y,x)$;
(d3) $d(x,y) \leq s[d(x,z) + d(z,y)]$ where $s \geq 1$.

Then $d$ is also called a b-metric and $(X, d)$ is called a b-metric space.

Note that if $s$=1, then a b-metric reduce to a metric.

Let $(X, d)$ be a b-metric space, $\{x_n\} \subset X$ be a sequence and $x \in X$. Then we say that

(1) $\{x_n\}$ is *convergent* to $x$ (denoted by $\lim_{n\to\infty} x_n = x$) if and only if for all $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0,\ d(x_n, x) < \epsilon, \quad i.e. \lim_{n\to\infty} d(x, x_n) = 0;$$

(2) $\{x_n\}$ is *Cauchy* if and only if for all $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n, m \geq n_0,\ d(x_n, x_m) < \epsilon, \quad i.e. \lim_{m,n\to\infty} d(x_n, x_m) = 0;$$

(3) The b-metric space $(X, d)$ is *complete* if and only if every Cauchy sequence in $X$ is convergent to some point in $X$.

Note that every convergent sequence in a b-metric space has a unique limit. In fact, if $\lim_{n\to\infty} d(x,x_n) = 0$ and $\lim_{n\to\infty} d(y,x_n) = 0$, then

$$d(x,y) \le s[d(x,x_n) + d(x_n,y)]$$

which yields $d(x,y) = 0$, and $x = y$.

Also, note that every convergent sequence in a b-metric space is a Caucy sequence.

Khamsi and Hussein [11] defined a toplogy $\sigma_d$ on b-metric space $(X,d)$ by

$$U \in \sigma_d \Longleftrightarrow \forall x \in U,\ \exists \epsilon > 0 : B(x,\epsilon) = \{y : d(x,y) < \epsilon\} \subset U.$$

Let $(X,d)$ be a b-metric space.

A map $T : X \to X$ is called *continuous* at $x \in X$ if for any $V \in \sigma_d$ containg $Tx$, there exists $U \in \sigma_d$ containg $x$ such that $TU \subset V$.

We say that a map $T : X \to X$ is *continuous* whenever it is continuous at each point in $X$.

**Proposition 1.1.** *Let $(X,d)$ be a b-metric space, and let $T : X \to X$ be a map. Then the followings are equivalent.*

(1) *$T$ is continuous at $x \in X$;*
(2) *$\forall \epsilon > 0$, $\exists \delta > 0$:*

$$d(x,y) < \delta \Longrightarrow d(Tx,Ty) < \epsilon;$$

(3) *$T$ is sequentially continuous at $x$, i.e. $\lim_{n\to\infty} d(Tx_n,Tx) = 0$ for any sequence $\{x_n\} \subset X$ with $\lim_{n\to\infty} d(x_n,x) = 0$.*

**Remark 1.1.** *[9] If $d$ is a b-metric on $X$, then d is not generally continuous in each coordinates.*

**Proposition 1.2.** *[2] Let $(X,d)$ be a b-metric space. If $d$ is continuous in one variable, then $d$ is continuous in other variable. Moreover, we have that $\forall x \in X, \forall r > 0$*

(1) *$B(x,r) \in \sigma_d$;*
(2) *$X \backslash B[x,r] \in \sigma_d$.*

## 2. Fixed point theorems

Let $(X, \preceq)$ be a partially ordered set.

A mapping $T : X \to X$ is called *non-decreasing* if and only if for $x, y \in X$,

$$x \preceq y \text{ implies } Tx \preceq Ty.$$

Now, we prove our main result.

**Theorem 2.1.** *Let $(X, \preceq)$ be a partially ordered set. Suppose that there exists a b-metric $d$ on $X$ such that $(X, d)$ is complete. Let $T : X \to X$ be a non-decreasing mapping such that for all $x, y \in X$ with $y \preceq x$*

$$d(Tx, Ty) > 0 \Rightarrow \xi(\theta(sd(Tx, Ty)), \theta(\frac{1}{1+s}[d(x, Ty) + d(y, Tx)])) \geq 1. \tag{2.1}$$

*Assume that there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$. If $T$ is continuous, then $T$ has a fixed point.*

*Proof.* Suppose that $x_0 \preceq Tx_0$. Since $T$ is non-decreasing,

$$x_0 \preceq Tx_0 \preceq T^2x_0.$$

Inductively, we have

$$x_0 \preceq Tx_0 \preceq T^2x_0 \preceq T^3x_0 \cdots \preceq T^nx_0 \preceq T^{n+1}x_0 \preceq \cdots .$$

Let $\{x_n\} \subset X$ be a sequence defined by

$$x_n = Tx_{n-1} = T^nx_0, \forall n = 1, 2, 3, \cdots .$$

Then

$$x_n \preceq x_{n+1}, \forall n = 1, 2, 3, \cdots .$$

If $x_n = x_{n+1}$ for some $n \geq 1$, then $x_n = Tx_n$ and the proof is finished. Thus assume that

$$x_n \preceq x_{n+1} \text{ and } x_n \neq x_{n+1}, \forall n = 1, 2, 3, \cdots .$$

It follows from (2.1) that

$$\begin{aligned}
1 \leq &\xi(\theta(sd(Tx_{n-1}, Tx_n)), \theta(\frac{1}{1+s}[d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})])) \\
= &\xi(\theta(sd(x_n, x_{n+1})), \theta(\frac{1}{1+s}[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)])) \\
= &\xi(\theta(sd(x_n, x_{n+1})), \theta(\frac{1}{1+s}d(x_{n-1}, x_{n+1})))
\end{aligned}$$

$$<\frac{\theta(\frac{1}{1+s}d(x_{n-1},x_{n+1}))}{\theta(sd(x_n,x_{n+1}))}$$

which implies

$$\theta(sd(x_n,x_{n+1}))<\theta(\frac{1}{1+s}d(x_{n-1},x_{n+1})) \tag{2.2}$$

and so

$$\begin{aligned}&\theta(sd(x_n,x_{n+1}))<\theta(\frac{1}{1+s}d(x_{n-1},x_{n+1}))\\ \leq&\theta(\frac{s}{1+s}[d(x_{n-1},x_n)+d(x_n,x_{n+1})]).\end{aligned} \tag{2.3}$$

Since $\theta$ is non-decreasing,

$$\begin{aligned}&sd(x_n,x_{n+1})\\ <&\frac{s}{1+s}[d(x_{n-1},x_n)+d(x_n,x_{n+1})]\end{aligned}$$

which implies

$$\frac{s^2}{1+s}d(x_n,x_{n+1})<\frac{s}{1+s}d(x_{n-1},x_n).$$

Thus we have

$$\frac{s}{1+s}d(x_n,x_{n+1})<\frac{s^2}{1+s}d(x_n,x_{n+1})<\frac{s}{1+s}d(x_{n-1},x_n).$$

Hence

$$d(x_n,x_{n+1})<d(x_{n-1},x_n),\forall n=1,2,3,\cdots.$$

Since $\{d(x_{n-1},x_n)\}\subset[0,\infty)$ is a decreasing sequence, there exists $r\geq 0$ such that

$$\lim_{n\to\infty}d(x_{n-1},x_n)=r.$$

We now show that $r=0$. Suppose that $r\neq 0$. Let $t_n=\theta(sd(x_n,x_{n+1}))$ and $s_n=\theta(\frac{1}{1+s}d(x_{n-1},x_{n+1}))\forall n=1,2,3,\cdots$. Then it follows from (2.2) that

$$t_n<s_n\forall n=1,2,3,\cdots.$$

We have

$$\lim_{n\to\infty}t_n=\theta(sr).$$

It follows from (2.3) that

$$
\begin{aligned}
&\theta(sd(x_n, x_{n+1}))\\
&<\theta(\frac{1}{1+s}d(x_{n-1}, x_{n+1}))\\
&<\theta(\frac{s}{1+s}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})])\\
&<\theta(\frac{s}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]).
\end{aligned} \tag{2.4}
$$

Letting $n \to \infty$ in (2.4), we have

$$\lim_{n\to\infty} s_n = \theta(sr).$$

Hence

$$1 \leq \lim_{n\to\infty} \sup \xi(\theta(t_n, s_n)) < 1$$

which is a contradiction. Thus we have

$$\lim_{n\to\infty} d(x_{n-1}, x_n) = 0$$

and so

$$\lim_{n\to\infty} \theta(d(x_{n-1}, x_n)) = 1.$$

We show that $\{x_n\}$ is a Cauchy sequence. On the contrary, assume that $\{x_n\}$ is not a Cauchy sequence. Then there exists an $\epsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $m(k)$ is the smallest index for which $m(k) > n(k) > k \; \forall k = 1, 2, 3, \cdots$

$$d(x_{m(k)}, x_{n(k)}) \geq \epsilon \text{ and } d(x_{m(k)-1}, x_{n(k)}) < \epsilon.$$

Since $m(k) > n(k) > k \; \forall k = 1, 2, 3, \cdots$,

$$x_{n(k)} \preceq x_{m(k)} \; \forall k = 1, 2, 3, \cdots .$$

It follows from (2.1) that

$$
\begin{aligned}
1 \leq &\xi(\theta(sd(x_{n(k)}, x_{m(k)})), \theta(\frac{1}{1+s}[d(x_{n(k)-1}, x_{m(k)}) + d(x_{m(k)-1}, x_{n(k)})])\\
< &\frac{\theta(\frac{1}{1+s}[d(x_{n(k)-1}, x_{m(k)}) + d(x_{m(k)-1}, x_{n(k)})])}{\theta(sd(x_{n(k)}, x_{m(k)}))}
\end{aligned}
$$

which implies

$$\theta(sd(x_{n(k)}, x_{m(k)})) < \theta(\frac{1}{1+s}[d(x_{n(k)-1}, x_{m(k)}) + d(x_{m(k)-1}, x_{n(k)})]).$$

Hence we have

$$
\begin{aligned}
&s\epsilon\\
&\leq sd(x_{n(k)}, x_{m(k)})\\
&< \frac{1}{1+s}[d(x_{n(k)-1}, x_{m(k)}) + d(x_{m(k)-1}, x_{n(k)})]\\
&< \frac{1}{1+s}[\epsilon + d(x_{n(k)-1}, x_{m(k)})].
\end{aligned} \tag{2.5}
$$

We infer that

$$
\begin{aligned}
&d(x_{n(k)-1}, x_{m(k)})\\
\leq& sd(x_{m(k)}, x_{n(k)}) + sd(x_{n(k)}, x_{n(k)-1})\\
\leq& s^2 d(x_{m(k)}, x_{m(k)-1}) + s^2 d(x_{m(k)-1}, x_{n(k)}) + sd(x_{n(k)}, x_{n(k)-1})\\
<& s^2 d(x_{m(k)}, x_{m(k)-1}) + s^2\epsilon + sd(x_{n(k)}, x_{n(k)-1}).
\end{aligned} \tag{2.6}
$$

Letting $n \to \infty$ in (2.6), we obtain

$$\lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)}) \leq s^2\epsilon. \tag{2.7}$$

By taking $k \to \infty$ in (2.5) and applying (2.7), we have

$$
\begin{aligned}
&s\epsilon\\
&\leq \lim_{k\to\infty} sd(x_{m(k)}, x_{n(k)})\\
&\leq \lim_{k\to\infty} \frac{1}{1+s}[d(x_{n(k)-1}, x_{m(k)}) + d(x_{m(k)-1}, x_{n(k)})]\\
&\leq \lim_{k\to\infty} \frac{1}{1+s}[d(x_{n(k)-1}, x_{m(k)}) + \epsilon]\\
&\leq \frac{1}{1+s}[\epsilon + s^2\epsilon]\\
&= \frac{(1+s^2)}{1+s}\epsilon\\
&\leq \frac{(1+s)s}{1+s}\epsilon\\
&= s\epsilon
\end{aligned}
$$

which implies

$$\lim_{k\to\infty} sd(x_{m(k)}, x_{n(k)}) = \lim_{k\to\infty} \frac{1}{1+s}[d(x_{n(k)-1}, x_{m(k)}) + d(x_{m(k)-1}, x_{n(k)})] = s\epsilon.$$

Let $t_k=\theta(sd(x_{m(k)},x_{n(k)}))$ and $s_k=\theta(\frac{1}{1+s}[d(x_{n(k)-1},x_{m(k)})+d(x_{m(k)-1},x_{n(k)})])$. Then

$$t_k<s_k\ \forall k=1,2,3,\cdots, \text{ and } \lim_{k\to\infty}t_k=\lim_{k\to\infty}s_k>1.$$

Hence

$$1\leq\lim_{k\to\infty}\sup\xi(t_k,s_k)<1$$

which is a contradiction.

Thus $\{x_n\}$ is a Cauchy sequence, and so there exists $x_*\in X$ such that

$$\lim_{n\to\infty}d(x_*,x_n)=0.$$

Since $T$ is continuous,

$$\lim_{n\to\infty}d(Tx_*,x_{n+1})=\lim_{n\to\infty}d(Tx_*,Tx_n)=0.$$

Thus we have

$$d(x_*,Tx_*)\leq s\lim_{n\to\infty}[d(x_*,x_{n+1})+d(x_{n+1},Tx_*)]=0.$$

Hence $x_*=Tx_*$. □

**Theorem 2.2.** *Let $(X,\preceq)$ be a partially ordered set. Suppose that there exists a b-metric $d$ on $X$ such that $(X,d)$ is complete. Let $T:X\to X$ be a non-decreasing mapping such that (2.1) holds. Assume that there exists $x_0\in X$ such that $x_0\preceq Tx_0$. If, for any non-decreasing sequence $\{x_n\}$ with $\lim_{n\to\infty}d(x,x_n)=0$,*

(2.8) $$x_n\preceq x$$

*then $T$ has a fixed point.*

*Proof.* Following proof of Theorem 2.1, we have a sequence $\{x_n=T^nx_0\}\subset X, x_0\in X$ such that for all $n=1,2,3,\cdots$,

$$x_n\preceq x_{n+1},\ x_n\neq x_{n+1},\ \lim_{n\to\infty}d(x_*,x_n)=0 \text{ and } \lim_{n\to\infty}d(x_{n-1},x_n)=0.$$

It follows from (2.1) and (2.8) that

$$\begin{aligned}
1\leq&\,\xi(\theta(sd(Tx_*,Tx_n)),\theta(\frac{1}{1+s}[d(x_*,Tx_n)+d(x_n,Tx_*)]))\\
=&\,\xi(\theta(sd(Tx_*,Tx_n)),\theta(\frac{1}{1+s}[d(x_*,x_{n+1})+d(x_n,Tx_*)]))\\
<&\,\frac{\theta(\frac{1}{1+s}[d(x_*,x_{n+1})+d(x_n,Tx_*)])}{\theta(sd(Tx_*,Tx_n))}
\end{aligned}$$

which implies

$$\theta(sd(Tx_*, Tx_n)) < \theta(\frac{1}{1+s}[d(x_*, x_{n+1}) + d(x_n, Tx_*)]). \tag{2.9}$$

Hence

$$\begin{aligned} sd(Tx_*, x_{n+1})) &< \frac{1}{1+s}[d(x_*, x_{n+1}) + d(x_n, Tx_*)] \\ &< \frac{1}{1+s}[d(x_*, x_{n+1}) + sd(x_n, x_*) + sd(x_*, Tx_*)] \\ &< \frac{1}{1+s}d(x_*, x_{n+1}) + d(x_n, x_*) + d(x_*, Tx_*)]. \end{aligned}$$

Thus

$$\lim_{n\to\infty} sd(Tx_*, x_{n+1}) \leq d(x_*, Tx_*).$$

Hence

$$d(x_*, Tx_*) \leq \lim_{n\to\infty}[sd(x_*, x_{n+1}) + sd(x_{n+1}, Tx_*)] \leq d(x_*, Tx_*)$$

which implies

$$d(x_*, Tx_*) \leq \lim_{n\to\infty} sd(x_{n+1}, Tx_*) \leq d(x_*, Tx_*)$$

and so

$$\lim_{n\to\infty} sd(x_{n+1}, Tx_*) = d(x_*, Tx_*).$$

It follows from (2.9) that

$$sd(Tx_*, Tx_n) < \frac{1}{1+s}[d(x_*, x_{n+1}) + d(x_n, Tx_*)] < \frac{s}{1+s}[d(x_*, x_{n+1}) + d(x_n, Tx_*)].$$

By letting $n \to \infty$ in above we have

$$d(x_*, Tx_*) \leq \frac{1}{1+s}d(x_*, Tx_*)$$

which is a contradiction if $d(x_*, Tx_*) > 0$.

Hence $d(x_*, Tx_*) = 0,$ and hence $x_* = Tx_*$. $\square$

**Theorem 2.3.** *Let $(X, \preceq)$ be a partially ordered set. Suppose that there exists a b-metric $d$ on $X$ such that $(X, d)$ is complete. Let $T : X \to X$ be a non-decreasing mapping such that (2.1) holds. Suppose that there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, and assume that either $T$ is continuous or (2.8) is satisfied. If for $x, y \in X$, there exists $z \in X$ such that either $z \preceq x$ or $z \preceq y$, then $T$ has a unique fixed point.*

*Proof.* By Theorem 2.1 or Theorem 2.2, $T$ has a fixed point.

We show that the fixed point of $T$ is unique. Let $u = Tu$ and $v = Tv$. We consider the following two cases.

*Case 1.* Let $v \preceq u$.
Suppose that $u \neq v$. Then $d(u, v) > 0$, and form (2.1) we have

$$\begin{aligned} 1 \leq &\xi(\theta(sd(Tu, Tv)), \theta(\frac{1}{1+s}[d(u, Tv) + d(v, Tu)])) \\ = &\xi(\theta(sd(u, v)), \theta(\frac{1}{1+s}[d(u, v) + d(v, u)])) \\ < &\frac{\theta(\frac{1}{1+s}[d(u, v) + d(v, u)])}{\theta(sd(u, v))} \end{aligned}$$

which implies

$$\theta(sd(u, v)) < \theta(\frac{1}{1+s}[d(u, v) + d(v, u)]) = \theta(\frac{2}{1+s}[d(u, v)]) \leq \theta(d(u, v))$$

which is a contradiction. Thus $u = v$, and $T$ has a unique fixed point.

*Case 2.* If $u \npreceq v$, then there exists $z \in X$ such that $z \preceq u$ or $z \preceq v$.
Suppose that $z \preceq u$. Since $T$ is non-decreasing,

$$T^{n-1}z \preceq T^{n-1}u \;\forall n = 1, 2, 3, \cdots .$$

It follows from (2.1) that $\forall n = 1, 2, 3, \cdots$

$$\begin{aligned} 1 \leq &\xi(\theta(sd(T^n u, T^n z)), \theta(\frac{1}{1+s}[d(u, T^n z) + d(T^{n-1}z, T^n u)])) \\ = &\xi(\theta(sd(u, T^n z)), \theta(\frac{1}{1+s}[d(u, T^n z) + d(T^{n-1}z, u)])) \\ < &\frac{\theta(\frac{1}{1+s}[d(u, T^n z) + d(T^{n-1}z, u)])}{\theta(sd(u, T^n z))} \end{aligned}$$

which implies

$$\theta(sd(u, T^n z)) < \theta(\frac{1}{1+s}[d(u, T^n z) + d(T^{n-1}z, u)])$$

and so

$$sd(u, T^n z) < \frac{1}{1+s}[d(u, T^n z) + d(T^{n-1}z, u)]. \tag{2.10}$$

Hence

$$\frac{1}{1+s}d(u, T^n z) \leq (s - \frac{1}{1+s})d(u, T^n z) < \frac{1}{1+s}d(u, T^{n-1}z)$$

and hence

$$d(u, T^n z) < d(u, T^{n-1}z) \ \forall n = 1, 2, 3, \cdots .$$

Thus there exists $l \geq 0$ such that $\lim_{n\to\infty} d(u, T^{n-1}z) = l$. By letting $n \to \infty$ in (2.10), we have

$$sl \leq \frac{2}{1+s}l$$

which is a contradiction if $l \neq 0$. Hence $l = 0$ and hence $\lim_{n\to\infty} d(u, T^n z) = 0$.

Similary, we can prove $\lim_{n\to\infty} d(v, T^n z) = 0$. Thus we have

$$d(u, v) \leq \lim_{n\to\infty}[sd(u, T^n z) + sd(T^n z, v)] = 0.$$

Hence $u = v$, and $T$ has a unique fixed point. □

By taking $\xi_c(t, q) = \frac{q^k}{t}, k \in (0, 1)$ in Theorem 2.3, we have the following result.

**Corollary 2.1.** *Let $(X, \preceq)$ be a partially ordered set. Suppose that there exists a b-metric $d$ on $X$ such that $(X, d)$ is complete. Let $T : X \to X$ be a non-decreasing mapping such that for all $x, y \in X$ with $y \preceq x$*

$$d(Tx, Ty) > 0 \Rightarrow \theta(sd(Tx, Ty)) \leq [\theta(\frac{1}{1+s}\{d(x, Ty) + d(y, Tx)\})]^k$$

*where $k \in (0, 1)$.*

*Suppose that there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, and assume that either $T$ is continuous or (2.8) is satisfied.*

*Then $T$ has a fixed point. Further if for $x, y \in X$, there exists $z \in X$ such that $z \preceq x$ or $z \preceq y$, then $T$ has a unique fixed point.*

**Corollary 2.2.** *Let $(X, \preceq)$ be a partially ordered set. Suppose that there exists a b-metric $d$ on $X$ such that $(X, d)$ is complete. Let $T : X \to X$ be a non-decreasing mapping such that for all $x, y \in X$ with $y \preceq x$*

$$sd(Tx, Ty) \leq \frac{k}{1+s}[d(x, Ty) + d(y, Tx)]$$

*where $k \in (0, 1)$.*

*Suppose that there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, and assume that either $T$ is continuous or (2.8) is satisfied.*

*Then $T$ has a fixed point. Further if for $x, y \in X$, there exists $z \in X$ such that $z \preceq x$ or $z \preceq y$, then $T$ has a unique fixed point.*

By taking $\xi_{wc}$ in Theorem 2.3, we have the following result.

**Corollary 2.3.** *Let $(X, \preceq)$ be a partially ordered set. Suppose that there exists a b-metric $d$ on $X$ such that $(X, d)$ is complete. Let $T : X \to X$ be a non-decreasing mapping such that for all $x, y \in X$ with $y \preceq x$*

$$d(Tx, Ty) > 0 \Rightarrow \theta(sd(Tx, Ty)) \leq \frac{\theta(\frac{1}{1+s}[d(x, Ty) + d(y, Tx)])}{\psi(\theta(\frac{1}{1+s}d(x, Ty)), \theta(\frac{1}{1+s}d(y, Tx)))}$$

*where $\theta \in \Theta$ with $\theta(p_1 + p_2) = \theta(p_1)\theta(p_2)$.*

*Suppose that there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, and assume that either $T$ is continuous or (2.8) is satisfied.*

*Then $T$ has a fixed point. Further if for $x, y \in X$, there exists $z \in X$ such that $z \preceq x$ or $z \preceq y$, then $T$ has a unique fixed point.*

**Corollary 2.4.** *Let $(X, \preceq)$ be a partially ordered set. Suppose that there exists a b-metric $d$ on $X$ such that $(X, d)$ is complete. Let $T : X \to X$ be a non-decreasing mapping such that for all $x, y \in X$ with $y \preceq x$*

$$sd(Tx, Ty) \leq \frac{1}{1+s}[d(x, Ty) + d(y, Tx)]) - \varphi(d(x, Ty), d(y, Tx)) \tag{2.11}$$

*where $\varphi : [0, \infty) \times [0, \infty) \to [0, \infty)$ is continuous and $\varphi(u, v) = 0$ if and only if $u = v = 0$.*

*Suppose that there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, and assume that either $T$ is continuous or (2.8) is satisfied.*

*Then $T$ has a fixed point. Further if for $x, y \in X$, there exists $z \in X$ such that $z \preceq x$ or $z \preceq y$, then $T$ has a unique fixed point.*

*Proof.* Let $\theta(t) = e^t, \forall t > 0$, and let $\varphi(u, v) = \ln(\psi(\theta(\frac{1}{1+s}u), \theta(\frac{1}{1+s}v)))$, $\forall u, v > 0$ such that $\psi : [1, \infty) \times [1, \infty) \to [1, \infty)$ is continuous and $\psi(\mu, \nu) = 1$ if and only if $\mu = \nu = 1$.

Then we have

$$\begin{aligned}
&\varphi(u, v) = 0 \\
\Leftrightarrow &\ln(\psi(\theta(\frac{1}{1+s}u), \theta(\frac{1}{1+s}v)) = 0 \\
\Leftrightarrow &\psi(\theta(\frac{1}{1+s}u), \theta(\frac{1}{1+s}v)) = 1 \\
\Leftrightarrow &\theta(\frac{1}{1+s}u) = \theta(\frac{1}{1+s}v) = 1 \\
\Leftrightarrow &u = v = 0.
\end{aligned}$$

It follows from (2.11) that for all $x, y \in X$ with $y \preceq x$ and $d(Tx, Ty) > 0$

$$\begin{aligned}
&\theta(sd(Tx, Ty)) = e^{sd(Tx,Ty)} \\
\leq &e^{\frac{1}{1+s}[d(x,Ty)+d(y,Tx)] - \varphi(d(x,Ty),d(y,Tx))} \\
= &\frac{e^{\frac{1}{1+s}[d(x,Ty)+d(y,Tx)]}}{e^{\varphi(d(x,Ty),d(y,Tx))}} \\
= &\frac{\theta(\frac{1}{1+s}[d(x,Ty) + d(y,Tx)])}{\psi(\theta(\frac{1}{1+s}d(x,Ty)), \theta(\frac{1}{1+s}d(y,Tx)))}.
\end{aligned}$$

By Corollary 2.3, $T$ has a unique fixed point. □

**Remark 2.1.** *Corollary 2.4 reduces to Theorem 3.2 of [8] by taking $s = 1$. Also, by taking $s = 1$ in Corollary 2.2, we have an extension of Theorem 1.1 to partially ordered sets with metric spaces.*

We give an example to illustrate Theorem 2.1.

**Example 2.** *Let $X = \{\frac{1}{n} : n = 1, 2, 3, \cdots\} \cup \{0\}$ and $\rho(x, y) =\mid x - y \mid$ and $d(x, y) =\mid x - y \mid^2$ .*

*Define*

$$y \preceq x \Leftrightarrow x \leq y.$$

*Then* $(X, \preceq)$ *is partially ordered set, and* $(X, \rho)$ *is a complete metric space and* $(X, d)$ *is a complete b-metric space with* $s = 2$.

*Obviously, we have that for any non-decreasing sequence* $\{x_n\} \subset X$ *with* $\lim_{n\to\infty} x_n = x \in X$,

$$x_n \preceq x, \forall n = 1, 2, 3, \cdots .$$

*Thus condition (2.8) holds.*

*Define a map* $T : X \to X$ *by*

$$Tx = \begin{cases} \frac{1}{n+1} & (x = \frac{1}{n}, n = 1, 2, 3, \cdots), \\ 0 & (x = 0). \end{cases}$$

*and a function* $\theta : (0, \infty) \to (1, \infty)$ *by*

$$\theta(t) = e^t.$$

*For* $x_0 = 1$, $Tx_0 = T1 = \frac{1}{2}$, *and so* $Tx_0 \leq x_0$, *which yields*

$$x_0 \preceq Tx_0.$$

*Let* $\psi(u, v) = \frac{u}{v^6}$, $\forall u, v \geq 1$. *We now show that (2.1) hold with respect to* $\xi_{wc}$. *Consider the following two case.*

*Case 1. Let* $x = 0, y = \frac{1}{n}$. *Then*

$$\begin{aligned}
&\xi(\theta(\frac{1}{3}d(0, T\frac{1}{n})), \theta(\frac{1}{3}[d(0, T\frac{1}{n}) + d(\frac{1}{n}, T0)]))\\
=&\frac{\theta(\frac{1}{3}[d(0, T\frac{1}{n}) + d(\frac{1}{n}, T0)])}{\theta(2d(T0, T\frac{1}{n}))\psi(\theta(\frac{1}{3}d(0, T\frac{1}{n})), \theta(\frac{1}{3}d(\frac{1}{n}, T0)))}\\
=&\frac{\theta(\frac{1}{3}[\frac{1}{(n+1)^2} + \frac{1}{n^2}])}{\theta(\frac{2}{n^2})\psi(\theta(\frac{1}{3}\frac{1}{(n+1)^2})), \theta(\frac{1}{3}\frac{1}{n^2})}\\
=&\frac{e^{\frac{1}{3}[\frac{1}{(n+1)^2} + \frac{1}{n^2}]}}{e^{\frac{2}{n^2}} e^{\frac{1}{3}\frac{1}{(n+1)^2}} e^{\frac{-2}{n^2}}}\\
=&e^{\frac{1}{3n^2}} \geq 1 \ \forall n = 1, 2, 3, \cdots .
\end{aligned}$$

*Thus we have*

$$\frac{\theta(\frac{1}{3}[d(0, T\frac{1}{n}) + d(\frac{1}{n}, T0)])}{\psi(\theta(\frac{1}{3}d(0, T\frac{1}{n})), \theta(\frac{1}{3}d(\frac{1}{n}, T0)))} \geq \theta(2d(T0, T\frac{1}{n})) \ \forall n = 1, 2, 3, \cdots .$$

*Case 2. Let $x = \frac{1}{n}$ and $y = \frac{1}{m}(m > n)$. Then*

$$\begin{aligned}
&\xi(\theta(\frac{1}{3}d(T\frac{1}{n}, T\frac{1}{m})), \theta(\frac{1}{3}[d(\frac{1}{n}, T\frac{1}{m}) + d(\frac{1}{m}, T\frac{1}{n})]))\\
=&\frac{\theta(\frac{1}{3}[d(\frac{1}{n}, T\frac{1}{m}) + d(\frac{1}{m}, T\frac{1}{n})])}{\theta(2d(T\frac{1}{n}, T\frac{1}{m}))\psi(\theta(\frac{1}{3}d(\frac{1}{n}, T\frac{1}{m})), \theta(\frac{1}{3}d(\frac{1}{m}, T\frac{1}{n})))}\\
=&\frac{\theta(\frac{1}{3}[|\ \frac{1}{n} - \frac{1}{(m+1)}\ |^2 + |\ \frac{1}{m} - \frac{1}{n+1}\ |^2])}{\theta(2\ |\ \frac{1}{n+1} - \frac{1}{m+1}\ |^2)\psi(\theta(\frac{1}{3}[\frac{1}{n} - \frac{1}{(m+1)}\ |^2)), \theta(\frac{1}{3}\ |\ \frac{1}{m} - \frac{1}{n+1}\ |^2)}\\
=&\frac{e^{\frac{1}{3}[|\frac{1}{n} - \frac{1}{(m+1)}|^2 + |\frac{1}{m} - \frac{1}{n+1}|^2])}}{e^{2|\frac{1}{n+1} - \frac{1}{m+1}|^2}e^{\frac{1}{3}|\frac{1}{n} - \frac{1}{(m+1)}|^2}e^{-2|\frac{1}{m} - \frac{1}{n+1}|^2}}\\
=&e^{\frac{1}{3}|\frac{1}{m} - \frac{1}{(n+1)}|^2} \geq 1\ \forall m > n = 1, 2, 3, \cdots .
\end{aligned}$$

*Hence all condition of Theorem 2.2 is satisfied, $T$ has a fixed point.*

*Note that Corollary 2.2 is not applicable here. In fact, if $x = 0$ and $y = \frac{1}{n}$, then*

$$sd(T0, T\frac{1}{n}) \leq \frac{k}{1+s}[d(0, T\frac{1}{n}) + d(\frac{1}{n}, T0)]$$

*which implies*

$$\frac{2}{(n+1)^2} \leq \frac{k}{3}[\frac{1}{n^2} + \frac{1}{(n+1)^2}].$$

*Hence*

$$\frac{k}{3} \geq \frac{\frac{2}{(n+1)^2}}{\frac{1}{n^2} + \frac{1}{(n+1)^2}} = \frac{2n^4 + 4n^3 + 2n^2}{2n^4 + 6n^3 + 7n^2 + 4n + 1}\ \forall n = 1, 2, 3, \cdots .$$

*Thus $k \geq 3$, which is a contradiction. Hence Corollary 2.2 is not satisfied.*

*Also, Corollary 2.1 does not hold. In fact, let $x = 0, y = \frac{1}{n}$ and $\theta(t) = e^t, \forall t > 0$, then*

$$\theta(sd(T0, Tx)) \leq \theta(\frac{k}{1+s}[d(x, T\frac{1}{n}) + d(\frac{1}{n}, T0)]).$$

*Thus*

$$e^{\frac{2}{(n+1)^2}} \leq e^{\frac{k}{3}[\frac{1}{n^2} + \frac{1}{(n+1)^2}]}$$

*and so*

$$e^{\frac{k}{3}} \geq e^{\frac{2}{(n+1)^2}\frac{n^2(n+1)^2}{2n^2+2n+1}} = e^{\frac{2n^2}{2n^2+2n+1}}\ \forall n = 1, 2, 3, \cdots .$$

*Hence*

$$e^{\frac{k}{3}} \geq e^1,\ \text{and hence } k \geq 3$$

*which is a contradiction. Thus Corollary 2.1 is not applicable here.*

## 3. Application to differential equations

Let $\mathbb{I} = [0, T] \subset \mathbb{R}$ be a closed interval, where $T > 0$, and let $C(\mathbb{I}, \mathbb{R})$ be the class of all continuous function from $\mathbb{I}$ into $\mathbb{R}$.

Let $\rho(x, y) = \sup_{s\in\mathbb{I}} \mid x(s) - y(s) \mid \ \forall x, y \in C(\mathbb{I}, \mathbb{R})$, and $d(x, y) = [\rho(x, y)]^2 \ \forall x, y \in C(\mathbb{I}, \mathbb{R})$.

Then $(C(\mathbb{I}, \mathbb{R}), d)$ is a complete b-metric space with $s = 2$, and $(C(\mathbb{I}, \mathbb{R}), \preceq)$ is a partially ordered set with the partial order given by

$$\forall x, y \in C(\mathbb{I}, \mathbb{R}), \quad x \preceq y \iff x(s) \leq y(s) \ \forall s \in \mathbb{I}.$$

Consider the following ordinary differential equation:

$$u'(s) = f(s, u(s)), \ \forall s \in \mathbb{I}, \ u(0) = u(T) \tag{3.1}$$

where $f : \mathbb{I} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

A function $a \in C^1(\mathbb{I}, \mathbb{R})$ is a lower solution for the ordinary differential equation (3.1) if and only if

$$a'(s) \leq f(s, a(s)) \ \forall s \in \mathbb{I}, \ a(0) \leq a(T).$$

Note that if, for some $\lambda > 0$

$$G(t, s) = \begin{cases} \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1} & (0 \leq s < t \leq T), \\ \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1} & (0 \leq t < s \leq T) \end{cases}$$

then

$$\sup_{t\in\mathbb{I}} \int_0^T G(t, s) ds = \frac{1}{\lambda}.$$

In fact,

$$\begin{aligned}
&\sup_{t\in\mathbb{I}} \int_0^T G(t, s) ds = \sup_{t\in\mathbb{I}} \int_0^t \frac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1} ds + \int_t^T \frac{e^{\lambda(s-t)}}{e^{\lambda T} - 1} ds \\
=& \sup_{t\in\mathbb{I}} \frac{1}{e^{\lambda T} - 1} \Big[ (\frac{1}{\lambda} e^{\lambda(T+s-t)} \Big]_0^t + \Big[ \frac{1}{\lambda} e^{\lambda(s-t)} \Big]_t^T \\
=& \frac{1}{\lambda(e^{\lambda T} - 1)} (e^{\lambda T} - 1) \\
=& \frac{1}{\lambda}.
\end{aligned}$$

**Lemma 3.1.** *[12] If $a \in C^1(\mathbb{I}, \mathbb{R})$ is a lower solution for the ordinary differential equation (3.1), then $a \preceq Fa$ where $F : C(\mathbb{I}, \mathbb{R}) \to C(\mathbb{I}, \mathbb{R})$ is a map defined by*

$$(Fu)(t) = \int_0^T G(t, s)[f(s, u(s)) + \lambda u(s)]ds. \tag{3.2}$$

**Theorem 3.1.** *Suppose that there exists $\lambda > 0$ such that for $x, y \in \mathbb{R}$ with $x \leq y$*

$$\begin{aligned} &\theta(2[f(s, y) + \lambda y - (f(s, x) + \lambda x)]) \\ \leq&[\theta(\frac{1}{3}[(f(s, x) + \lambda x) - \lambda y + (f(s, y) + \lambda y) - \lambda x])]^k \end{aligned} \tag{3.3}$$

*where $k \in (0, 1)$ and $\theta \in \Theta_{124}$.*

*Then the differential equation (3.1) has a unique solution, whenever it has a lower solution.*

*Proof.* Let $F : C(\mathbb{I}, \mathbb{R}) \to C(\mathbb{I}, \mathbb{R})$ be a map defined by (3.2). Let $u, v \in C(\mathbb{I}, \mathbb{R})$ with $v \preceq u$. Then from (3.3)

$$\theta(f(s, u(s)) + \lambda u(s) - [f(s, v(s)) + \lambda v(s)]) > 1.$$

Hence

$$f(s, u(s)) + \lambda u(s) - [f(s, v(s)) + \lambda v(s)] > 0$$

and hence

$$f(s, u(s)) + \lambda u(s) > f(s, v(s)) + \lambda v(s).$$

Thus we have that for each $t \in \mathbb{I}$

$$\begin{aligned} &(Fv)(t) \\ =&\int_0^T G(t, s)[f(s, v(s)) + \lambda v(s)]ds \\ <&\int_0^T G(t, s)[f(s, u(s)) + \lambda u(s)]ds \\ =&(Fu)(t) \end{aligned}$$

which implies

$$Fv \prec Fu.$$

Thus $F$ is non-decreasing and $0 \prec d(Fu, Fv)$.

Let $a(t) \in C'(\mathbb{I}, \mathbb{R})$ be a lower solution for (3.1). By Lemma 3.1, $a \preceq Fa$.

Let $v \preceq u$. Then we have that

$$
\begin{aligned}
1 <&\theta(2d(Fu,Fv))\\
=&\theta(2\sup_{t\in\mathbb{I}} \mid (Fu)(t)-(Fv)(t)\mid^2)\\
=&\theta(2\sup_{t\in\mathbb{I}}[\int_0^T G(t,s)[f(s,u(s))+\lambda u(s)-f(s,v(s))-\lambda v(s)]ds]^2)\\
\leq&[\theta(\frac{1}{3}\sup_{t\in\mathbb{I}}[\int_0^T G(t,s)[f(s,u(s))+\lambda u(s)]ds-\lambda v(t)]^2\\
&+\frac{1}{3}\sup_{t\in\mathbb{I}}[\int_0^T G(t,s)[f(s,v(s))+\lambda v(s)]ds-\lambda u(t)]^2)]^k\\
\leq&[\theta(\frac{1}{3}[\sup_{t\in\mathbb{I}}[(Fu)(t)-\lambda v(t)\int_0^T G(t,s)ds]^2\\
&+[(Fv)(t)-\lambda u(t)\int_0^T G(t,s)ds]^2])]^k\\
=&[\theta(\sup_{t\in\mathbb{I}}\frac{1}{3}[((Fu)(t)-v(t))^2+((Fv)(t)-u(t))^2])]^k\\
\leq&[\theta(\frac{1}{3}[d(Fu,v)+d(Fv,u)])]^k
\end{aligned}
$$

which implies

$$
\begin{aligned}
1\leq&\frac{[\theta(\frac{1}{3}[d(Fu,v)+d(Fv,u)])]^k}{\theta(2d(Fu,Fv))}\\
=&\xi_b(\theta(2d(Fu,Fv)),\theta(\frac{1}{3}[d(Fu,v)+d(Fv,u)]))\\
=&\xi_b(\theta(sd(Fu,Fv)),\theta(\frac{1}{1+s}[d(Fu,v)+d(Fv,u)])),
\end{aligned}
$$

$\forall u,v\in C(\mathbb{I},\mathbb{R})$ with $v\preceq u$. Hence (2.1) holds.

We show that (2.11) holds.

Let $\{x_n\}\subset C(\mathbb{I},\mathbb{R})$ be a non-decreasing sequence such that

$$\lim_{n\to\infty} d(x,x_n)=0 \tag{3.4}$$

where $x\in C(\mathbb{I},\mathbb{R})$. Then we have that for all $t\in\mathbb{I}$

$$x_1(t)\leq x_2(t)\leq\cdots\leq x_n(t)\leq\cdots. \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$x_n(t) \leq x(t) \; \forall t \in \mathbb{I}, n = 1, 2, 3, \cdots .$$

Thus

$$x_n \preceq x \; \forall n = 1, 2, 3, \cdots .$$

Let $u, v \in C(\mathbb{I}, \mathbb{R})$. Then $u(t), v(t) \in \mathbb{R} \; \forall t \in \mathbb{I}$, and so there exists $z \in C(\mathbb{I}, \mathbb{R})$ such that

$$\forall t \in \mathbb{I}, \quad \text{either } z(t) \leq u(t) \text{ or } z(t) \leq v(t)$$

which yields

$$\text{either } z \preceq u \text{ or } z \preceq v.$$

All conditions of Theorem 2.3 are satisfied with condition (2.8). By Theorem 2.3, $F$ has a unique fixed point, say $u_* \in C(\mathbb{I}, \mathbb{R})$. Hence $u_* \in C^1(\mathbb{I}, \mathbb{R})$ is a unique solution of differential equation (3.1). $\square$

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