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PERIOD DOUBLING BIFURCATION ANALYSIS OF A SIMPLE NONLINEAR DIFFERENTIAL EQUATION USING NUMERICAL SCHEME

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ABSTRACT. It is shown taking an example of fundamental approach which leads to the chaos in the order of the differential system and the same has been verified by that of a numerical tool to numerically simulate in the defined initial and boundary condition which is leading to the bifurcation of the nature of period doubling in the very essential nature of ordinary differential equation thus formed. The yielding reliable bifurcation diagrams when applied to the pseudo spectral approximation of a one-parameter family of nonlinear differential equations it is perfectly getting to be verified and the same has been verified by the proposed lemmas. The example shows a resemblance to one of the fundamental type of population equations and exhibits period doubling bifurcations. The reliability has been demonstrated by comparing the results obtained by a reduction to a differential dynamical system and to those obtained by direct application of differential system and latter modeling it to numerically estimate for positive Lyapunov exponents in the chaotic regime. We conclude that the methodology described here works well for a class of delay equations for which currently no tailor-made tools and solved using latest python solver available in the cloud repository.

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1. INTRODUCTION

The pendulum is one of the most interesting objects to be acted upon in the sort of executing mathematical aspects to the direct physical interpretation in majority of the cases. It describes both by a velocity dependent force which we call to be a damping force, and a periodic driving force, which we can think of displaying its nature of variations both ordered and chaotic behaviors, for certain ranges of parameters.

Nature is the source of most of the chaotic system; it is as common as any symmetry in the working of the basic functioning of nature. The dynamics and the damped driven pendulum are examined and first best example to figure out the nature of the pendulum and of chaotic dynamics have something in common and also mentioning how these two sub-fields of physics coincide at a fundamental field.

The pendulum equation as a matter of fact is governed by Newton's Second Law, which demonstrates the pendulum equation admits the potential for chaotic dynamics by equivalence to a system of three first order differential equations. It is explained by way of an example, that numerical bifurcation tools for ODE yield reliable bifurcation diagrams when applied to the pseudo spectral approximation of a one-parameter family of nonlinear differential equations. A cascade is a path of regular periodic orbits that has infinitely many period-doubling bifurcations with the periods going to infinity at the end of the path. Solid lines denote regular orbits, and dashed lines denote flip orbits. Each branch of hyperbolic periodic orbits can be parameterized by μ and is easy to find numerically by solving a differential equation.

The aim of the present paper is to contribute to substantiating this claim by way of the analysis of a simple and an interesting problem. It derives from a demographic equation that was introduced in the literature. A simplified version of that equation appears. The stability of slowly oscillating solutions of an even more stylized equation was established. It is conjectured many at times that a branch of symmetric periodic solutions of fixed period would lose stability by a symmetry breaking period doubling bifurcation. The conjecture has to be verified with the existence of the solution and as a very special feature of the example is that the branch of symmetric periodic solutions can be characterized by simple Hamiltonian systems in the plane, as first demonstrated by Kaplan et.al for delay differential equations.

2. MATHEMATICAL DEFINITION

The equation and the numerical results could be evaluated in accordance with the established modeled equation of the process of the physics.

Let $h : R \to R$ be a C^1 function with $h(0) = 0$ and $h(z) \geq 0$ for $z \geq 0$. We are interested in the equation

$$z(t) = \int_0^{-\infty} A(\tau)\, h(z(t-\tau))\, d\tau,$$

where $A :\to R$ takes only positive values. We choose the midpoint to be $\tau = 2$. To be more precise we assume that A is a bounded, nonnegative and measurable function to satisfy $A(\tau) = 0$ for $\tau > 4$,.

$$A(\tau) = A(4 \,\breve{}\, \tau), 0 \ \leq\ \tau \ \leq\ 4$$

and further, $\int_0^{-\infty} A(\tau)\, d\tau = 0$ and

$$A(\tau) = \begin{cases} \frac{1}{2}, 1 \ \leq\ \tau \ \leq\ 3, \\ 0, 0 \ \leq\ \tau < 1 \text{ or } \tau > 3 \end{cases}.$$

The quadratic linearity and non linearity will be decided with the constraints mentioned here with depicted equation

$$h(z) = \gamma z(1 - \ z).$$

The exponential Ricker-type nonlinearity

$$h(z) = \gamma z e^{-z}.$$

The main advantage of the pseudo spectral method considered is that a generic nonlinear delay equation is reduced to a low-dimensional system of ODE, which can be written in a convenient way using the right-hand side of the original equation and an additional block of linear equations that depends only on the discretization mesh and on the delay and therefore it is independent of the specific problem under a suitable scaling of time.

The outcome of the bifurcation analysis with respect to γ of the pseudo spectral approximation with the quadratic nonlinearity which can be plotted as shown in Figure 1, we observe the nontrivial steady state bifurcating into a periodic orbit through a Hopf bifurcation, followed by the starting of a period doubling cascade.

The most important advantage of the numerical approximation through an established system of ODE is that the trajectories of the corresponding initial value problems can be simulated by using ordinary numerical solvers. The simulations produced for parameter values in the range of the period doubling cascade agree with those approximated.

In addition, the simulation of the initial value problem could be investigated with the dynamical behavior which is sometimes beyond the range of the period doubling cascade.

In particular, the system seems to show chaotic behavior for larger values of γ, see, e.g., panel D of Figure 2.1b for $\gamma = 5$.

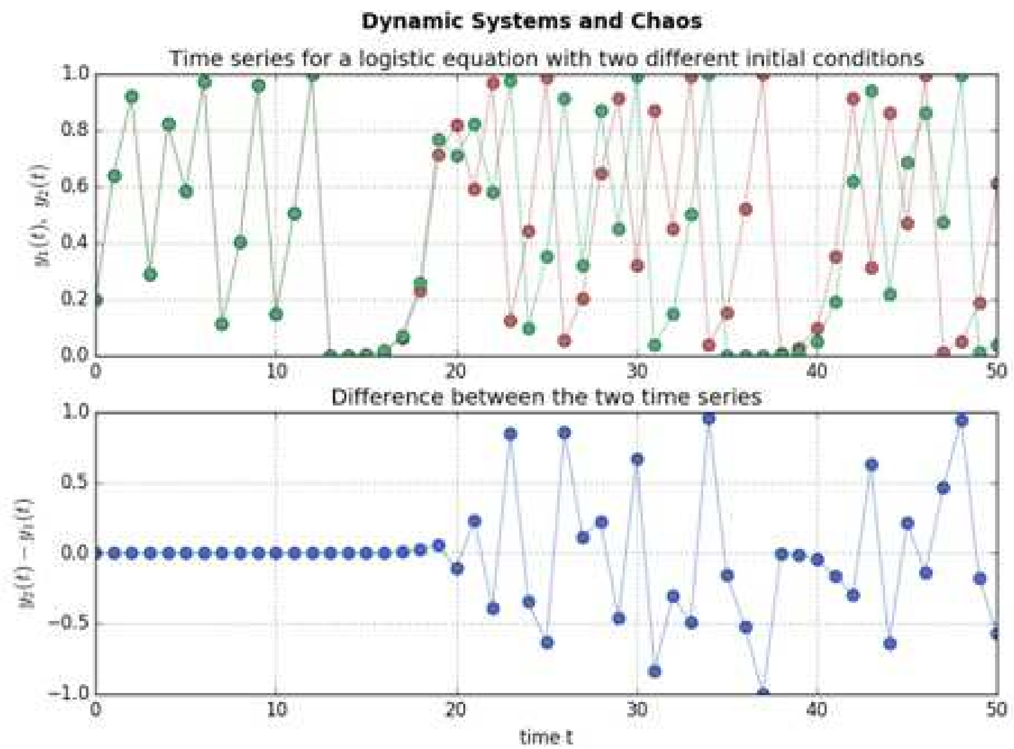

FIGURE 1. PERIOD DOUBLING IN DYNAMICAL SYSTEMS

The numerical bifurcation analysis of the exponential nonlinearity has similar features which are in quadratic case. The bifurcation diagram with respect to log

γ is plotted in Figure 2. The nontrivial equilibrium undergoes a Hopf bifurcation, followed by the starting of a period double bifurcation system.

The recorded trajectories are shown in Figure 3, for parameter values in the range of the period doubling cascade. The simulation of the initial value problem for larger values of γ shows the appearance of chaotic behavior for log (γ) = 4.3. There are, unfortunately, too few values of γ at period doubling points to check whether they display.

The results presented in this section demonstrate how the pseudo spectral approximation allows to gain valuable information about the dynamical properties of nonlinear renewal equations. The convergence of the method when increasing the number of approximating ODE is proved in for the steady states and their stability. There, the authors also conjecture the same convergence properties for periodic solutions and more complicated dynamical objects together with their stability and relevant bifurcations. The proof of convergence for periodic solutions and their stability is currently ongoing work. For this reason, we shall validate the numerical outcome presented in this section in two ways: first, by exploiting the analytical results guaranteed by the symmetry property; second, by comparing it with the numerical results obtained through a more sophisticated and specific approach, which is based on the principle of linearized stability

3. THEORETICAL MODELING

Lemma 3.1. *Let z be a solution of governing differential equation on* $(-\infty, +\infty)$. *If z is even, it has period of 5.*

Proof. From the fundamental equations it follows that

$$
\begin{aligned}
z\,(t+5) &= \int_0^1 A\,(\tau)\,h\,(z\,(t+5^{\smile}\,\tau))\,d\tau \\
&= \int_0^5 \,A\,(5-\sigma)\,h\,(z\,(t+\sigma))\,d\sigma \\
&= \int_0^5 A\,(\tau)\,h\,(z\,(t+\tau))\,d\tau = z\,(-t)\,.
\end{aligned}
$$

$\square$

Lemma 3.2. *Let z be a solution of (2.1) of period 3. Then there exists $\theta \in R$ such that*

$$z(t) - \theta = \theta - z(t+2)$$

$$\begin{aligned} z(t) + z(t+2) &= \frac{1}{2}\left(\int_1^2 h(z(t-\tau))\,d\tau + \int_1^2 h(z(t+2-\tau))\,d\tau\right) \\ &= \frac{1}{2}\left(\int_1^2 h(z(t-\tau))\,d\tau + \int_{-1}^1 h(z(t-\tau))\,d\tau\right) \\ &= \frac{1}{2}\left(\int_{-1}^3 h(z(t-\tau))\,d\tau\right) \\ &= \frac{1}{2}\left(\int_{t-1}^{t+1} h(z(\sigma))\,d\sigma\right) \end{aligned}$$

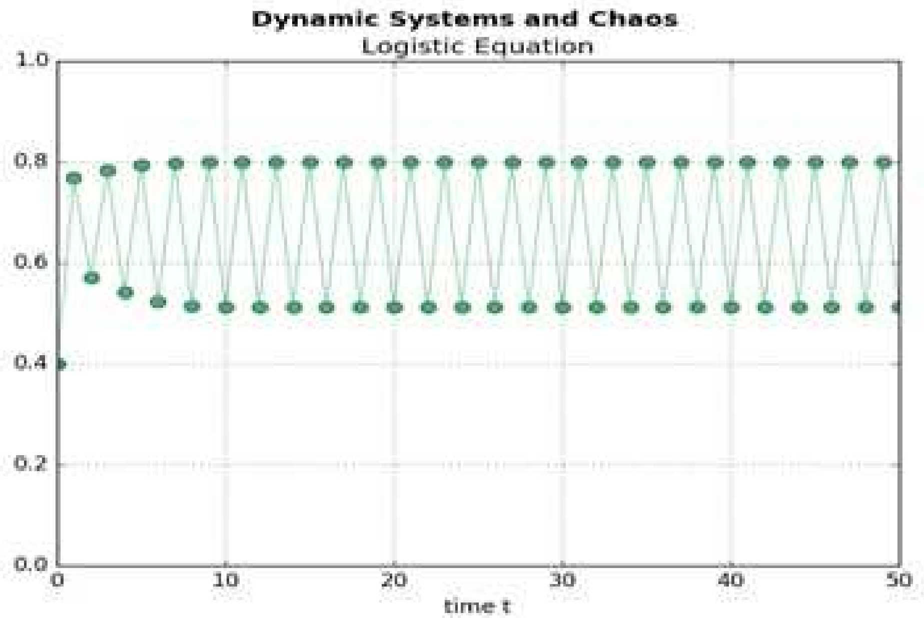

FIGURE 2. SYSTEM LEADING TO CHAOS

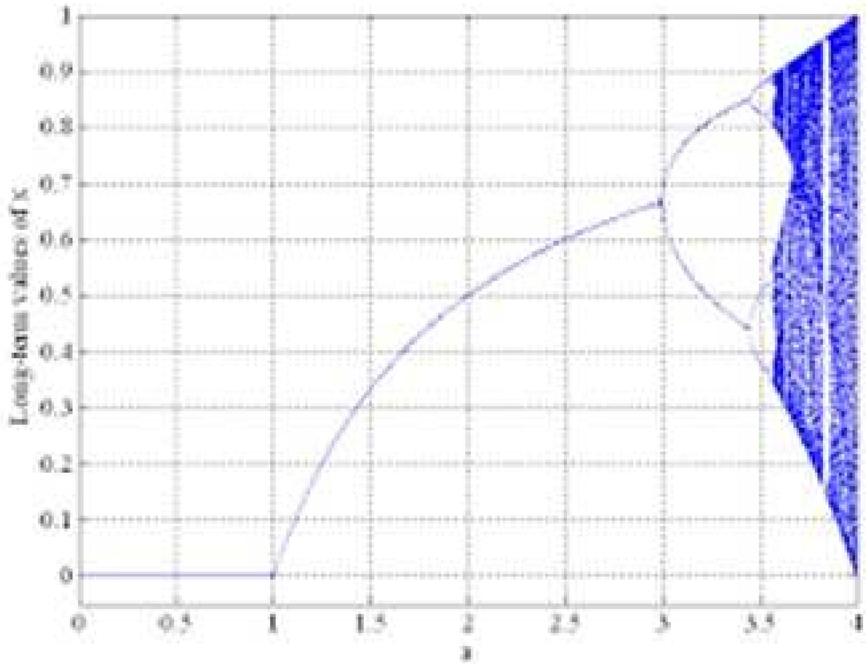

FIGURE 3. BIFURCATION DIAGRAM

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