Advances in Mathematics: Scientific Journal **9** (2020), no.10, 8559–8573
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.9.10.81

DYNAMICAL ANALYSIS OF THE RICCATI DIFFERENTIAL EQUATION WITH DELAY

A. M. A. EL-SAYED, S. M. SALMAN, AND S. RAMADAN[1]

ABSTRACT. In this paper, we consider the delay Riccati differential equation. Local stability analysis of equilibria is investigated. The equation exhibits a Hopf bifurcation at a critical parameter value. Numerical simulations are carried out to insure our theoretical findings.

1. INTRODUCTION

Delay differential equations (DDEs) are a type of differential equation in which the derivative of the unknown function at a certain time is given in terms of the values of the function at previous times [1, 5, 6, 16, 20]. Stability, bifurcations and chaos, a striking and complicated nonlinear phenomenon in dynamic systems, has received increasing importance during the last two decades. The delay differential equation was prepared as adequately describing the dynamic of electrochemical intercalation and of physiological systems, etc [4, 7, 9, 10, 14, 18, 19].
Consider the initial-value problem of the logistic delay equation [11].

$$\frac{dx}{dt} = -ax(t) + \rho x(t-\tau)(1 - x(t-\tau)),\ t \in [0, T], \qquad x(t) = x_0,\ t \le \tau. \tag{1.1}$$

[1]*corresponding author*

2010 *Mathematics Subject Classification.* 37E10, 37C75, 34C28.
Key words and phrases. Riccati delay, Time delay, Fixed points, Local stability, Bifurcation, Chaos.

In [11], the authors studied stability, bifurcation and chaos of equation (1.1). In this paper we taken $a = 1$, here we are concerned of the delay Riccati differential equations with two delays of the form

$$\frac{dx}{dt} = -x(t) + 1 - \rho x(t-\tau_1)x(t-\tau_2),\ t \in [0,T],$$
$$x(t) = x_0,\ t \leq \tau_1, \tau_2.$$

Here we consider the two different cases

(1) $\tau_1 = \tau_2 = 1$,
(2) $\tau_1 = 1$, $\tau_2 = 2$.

The paper is organized as follows.

(1) In Section 2, we will discuss the dynamic behavior of equation (2.1) such as local stability of fixed points, bifurcation, the discretized system, bifurcation diagrame and phase plane.
(2) In Section 3, we will discuss the dynamic behavior of equation (3.1) such as Local stability and Hopf bifurcation, the discretized system, Local stability and bifurcation analysis of the discretized system.
(3) Finally in Section 4, we will preform some numerical simulations to confirm all the previous analytical with the help of numerical simulations performed via Matlab.

2. DIFFERENTIAL EQUATION WITH ONE DELAY

Consider the initial value problem

$$\frac{dx}{dt} = -x(t) + 1 - \rho x^2(t-1),\ t \in (0,T],\ x(t) = x_0,\ t \leq 0. \tag{2.1}$$

2.1. **Local stability of fixed points and existence of bifurcation.** In this section, we consider the local stability of fixed points of the delay equation (2.1) [3]. The system has two fixed points which are the solution of the equation $-x + 1 - \rho x^2 = 0$ which has two fixed points

$$(x_{1,2})^* = (\frac{-1}{2\rho})(1 \pm \sqrt{1+4\rho}).$$

Now by checking the eigenvalues of the linearized system at the fixed points. In this problem, it is easy to check the eigenvalues of the linearized equations about the fixed points. At the neighborhood of

$$(x_{1,2})^* = (\frac{-1}{2\rho})(1 \pm \sqrt{1+4\rho}).$$

The linearized equation is

$$\frac{dy}{dt} = -y(t) + (1 \pm \sqrt{1+4\rho})y(t-1),$$
$$where,\ y(t) = x(t) - (\frac{-1}{2\rho})(1 \pm \sqrt{1+4\rho}).$$

The characteristic equation is of the form

$$\lambda + 1 - (1 \pm \sqrt{1+4\rho})e^{-\lambda} = 0. \tag{2.2}$$

Lemma 2.1. *All roots of the characteristic equation*

$$\lambda + c + be^{-\lambda} = 0,$$

where c and b are real, have negative real parts if and only if

$$c > -1,\ c+b > 0,\ b < \sqrt{c^2 + \xi^2}$$

where ξ is the root of

$$\xi = -c\tan\xi,\ 0 < \xi < \pi.\ If\ c \neq 0,\ \xi = \frac{\pi}{2},\ if c = 0.$$

Applying lemma 2. 1 to equation (2.2) with $c = 1$, and $b = -(1 \pm \sqrt{1+4\rho})$ we have the following Theorem.

Theorem 2.1. *The fixed point*

$$(x_{1,2})^* = (\frac{-1}{2\rho})(1 \pm \sqrt{1+4\rho})\ is\ stable\ if$$

$$-1 < -(1 \pm \sqrt{1+4\rho}) < \sqrt{1+\xi^2},$$

and unstable if

$$-1 > -(1 \pm \sqrt{1+4\rho}),\ -(1 \pm \sqrt{1+4\rho}) > \sqrt{1+\xi^2}.$$

2.2. **Hopf bifurcation.** Here we discuss the Hopf bifurcation. We have the following theorem.

Theorem 2.2. *When* $-(1 \pm \sqrt{1+4\rho})$ *passes through the critical value* $-(1 \pm \sqrt{1+4\rho}) = \sqrt{1+\xi^2}$*, there is a Hopf bifurcation from the equilibrium* $(x_{1,2})^* = (\frac{-1}{2\rho})(1 \pm \sqrt{1+4\rho})$ *to a periodic orbit.*

Proof. Let $(1+\sqrt{1+4\rho}) = K$, then, assume that $\lambda = i\omega_0$, $\omega_0 \in R^+$ is a pure imaginary solution of equation (2.2) for some parameter value $K = K_*$. This leads to the following equation

$i\omega_0 + 1 - K_* e^{-i\omega_0} = 0$, then , $1 - K_*\cos(\omega_0) = 0$, $\omega_0 - K_*\sin(\omega_0) = 0$, and $1 = K_*\cos(\omega_0)$.

Also,

$$\omega_0 = K_*\sin(\omega_0),\ \omega_0^2 + 1 = K_*^2[\cos(\omega_0)^2 + \sin(\omega_0)^2] = K_*^2,$$
$$K_* = \pm\sqrt{1+\omega_0^2} \text{ and } \omega_0 = -\tan(\omega_0).$$

By Theorem 2. 1 we have $K_* = -\sqrt{1+\omega_0^2}$ is the critical value of K where ω_0 is the root of $\omega_0 = -\tan(\omega_0)$, $0 < \omega_0 < \pi$.
The condition $\frac{d(Re(\lambda))}{dK}|_{K=K_*}$ is the last condition for occurrence of a Hopf bifurcation.
To show that this condition is satisfied, let $\lambda = Z(K) + i\omega(K)$ and using (2.2), we can get $Z + i\omega + 1 - Ke^{-z-i\omega} = 0$ and

$$then,\ Z + 1 - Ke^{-z}\cos(\omega) = 0, \tag{2.3}$$

$$\omega + Ke^{-z}\sin(\omega) = 0. \tag{2.4}$$

Differentiate (2.3) and (2.4) with respect to K, we obtain

$$\frac{dZ}{dK} - e^{-z}\cos(\omega) + Ke^{-z}\cos(\omega)\frac{dz}{dk} + Ke^{-z}\sin(\omega)\frac{d\omega}{dK} = 0, \tag{2.5}$$

$$\frac{d\omega}{dK} + e^{-z}\sin(\omega) + Ke^{-z}\cos(\omega)\frac{d\omega}{dK} - Ke^{-z}\sin(\omega)\frac{dZ}{dK} = 0. \tag{2.6}$$

Solving equation (2.5) and equation (2.6) for $\frac{dZ}{dK}$, we obtain

$$\frac{d(Re(\lambda))}{dK}|_{k=k_*} = \frac{d(Re(\lambda))}{dK}|_{z=0,\omega=\omega_0,k=k_*}$$
$$= \frac{\cos(\omega_0 + K_*}{(1+K_*\cos(\omega_0))^2 + (K_*\sin(\omega_0))^2}$$

$$= \frac{K_* \cos(\omega_0) + K_*^2}{K_*^2[(1 + K_* \cos(\omega_0))^2 + (K_* \sin(\omega_0))^2]}$$

$$= \frac{1 + K_*^2}{K_*[(1 + K_* \cos(\omega_0))^2 + (K_* \sin(\omega_0))^2]} \neq 0.$$

Similarly, we can prove that there is a Hopf bifurcation from the equilibrium $(x_2)^* = (\frac{-1}{2\rho})(1 - \sqrt{1 + 4\rho})$ to a periodic orbit. □

2.3. **The discretized system.** In this section, the discretized analogue of the system (2.1) is obtained via the method of steps as follows. By applying the method of steps then the equation Let t $\in (0, 1]$, then

$$\begin{aligned} then,\ x_1 &= e^{-t}x_0 + \int_0^t e^{-(t-s))}(1 - \rho x^2)ds \\ &= e^{-t}x_0 + (1 - \rho x_0^2)(1 - e^{-t}) \end{aligned}$$

and

$$x_1(1) = e^{-1}x_0 + (1 - \rho x_0^2)(1 - e^{-1}).$$

Let t $\in (1, 2]$, then

$$\begin{aligned} x_2 &= e^{-(t-1)}x_0 + \int_1^t e^{-(t-s)}(1 - \rho x^2)ds \\ &= e^{-(t-1)}x_0 + (1 - \rho x_0^2)(1 - e^{-(t-1)}) \end{aligned}$$

and

$$x_2(2) = e^{-1}x_0 + (1 - \rho x_0^2)(1 - e^{-1}).$$

Let t $\in (2, 3]$, then

$$then,\ x_3(3) = e^{-1}x_0 + (1 - \rho x_0^2)(1 - e^{-1}).$$

Repeating the process we can easily deduce that the solution of is given by

$$x_{n+1}(t) = e^{-(t-n)}x_n + (1 - \rho x_n^2)(1 - e^{-(t-n)}),$$

Let $t \longrightarrow n + 1$, then

$$x_{n+1} = x_n e^{-1} + (1 - \rho x_n^2)(1 - e^{-1}).$$

3. DIFFERENTIAL EQUATION WITH TWO DIFFERENT DELAYS

Consider the differential-difference equation with two different delays [8, 13, 17].

$$\frac{dx}{dt} = -x(t) + 1 - \rho x(t-1)x(t-2),\ x(t) = x_0,\ t \leq 0 \tag{3.1}$$

where ρ is a positive parameter.

3.1. **Local stability of equation (3.1) and Hopf bifurcation.** In this section, we will consider the local stability of fixed points of the delay equation (3.1) [12]. The system has the two fixed points

$$(x_{1,2})^* = (\frac{-1}{2\rho})(1 \pm \sqrt{1+4\rho}).$$

At the neighborhood of $(x_1)^*$ the linearized equation is

$$\frac{dy}{dt} = -y(t) + \frac{1}{2}(1+\sqrt{1+4\rho})y(t-1) + \frac{1}{2}(1+\sqrt{1+4\rho})y(t-2)$$

where $y(t) = y(t) - ((\frac{-1}{2\rho})(1+\sqrt{1+4\rho}))$.
Then the characteristic equation is of the form

$$\lambda + 1 - \frac{1}{2}(1+\sqrt{1+4\rho})e^{-\lambda} - \frac{1}{2}(1+\sqrt{1+4\rho})e^{-2\lambda} = 0. \tag{3.2}$$

We notice that is so difficult to discuss the stability at $(x_1)^* = (\frac{-1}{2\rho})(1+\sqrt{1+4\rho})$, so we can discuss the Hopf bifurcation [15].

3.2. **Hopf bifurcation.** Here, we discuss the Hopf bifurcation. We have the following theorem

Theorem 3.1. *When the parameter ρ passes through the critical value*

$$\rho = \rho_* = \frac{1}{4}[(\frac{1+\omega_0^2 - (\cos(\omega_0)+\omega_0\sin(\omega_0))}{\cos(\omega_0)+\omega_0\sin(\omega_0)})^2 - 1],\ \omega_0 = \tan(2\omega_0)(1-s\cos(\omega_0)) + s\sin(\omega_0),$$

then there is Hopf bifurcation from the equilibrium $(x_1)^* = (\frac{-1}{2\rho})(1+\sqrt{1+4\rho})$ to a periodic orbit.

Proof. Let $\lambda = i\omega_0,\ \omega_0 \in R^+$ is a pure imaginary solution for (3.3) for some parameter value $\rho = \rho_*$. Now we can get

$$i\omega_0 + 1 - \frac{1}{2}(1+\sqrt{1+4\rho_*})e^{-i\omega_0} - \frac{1}{2}(1+\sqrt{1+4\rho_*})e^{-2i\omega_0} = 0,$$

$$1 - \frac{1}{2}(1+\sqrt{1+4\rho_*})\cos(\omega_0) - \frac{1}{2}(1+\sqrt{1+4\rho_*})\cos(2\omega_0) = 0$$

and

$$\omega_0 - \frac{1}{2}(1+\sqrt{1+4\rho_*})\sin(\omega_0) - \frac{1}{2}(1+\sqrt{1+4\rho_*})\sin(2\omega_0) = 0.$$

Let $\frac{1}{2}(1+\sqrt{1+4\rho_*}) = s,$ then

$$1 - s\cos(\omega_0) - s\cos(2\omega_0) = 0, \tag{3.3}$$

$$\omega_0 - s\sin(\omega_0) - s\sin(2\omega_0) = 0. \tag{3.4}$$

Solving equation (3.3) and equation (3.4), we can get

$$s = \frac{1+\omega_0^2}{2(\cos(\omega_0)+\omega_0\sin(\omega_0))},$$

$$\rho_* = \frac{1}{4}[(\frac{1+\omega_0^2-(\cos(\omega_0)+\omega_0\sin(\omega_0))}{\cos(\omega_0)+\omega_0\sin(\omega_0)})^2 - 1],$$

$$\frac{\omega_0 - s\sin(\omega_0)}{1-s\cos(\omega_0)} = \frac{\sin(2\omega_0)}{\cos(2\omega_0)},$$
$$\omega_0 = \tan(2\omega_0)(1-s\cos(\omega_0)) + s\sin(\omega_0).$$

To show that this condition $\frac{d(Re(\lambda))}{d\rho}\ |_\rho \neq 0$ is satisfied, let $\lambda = k(\rho) + i\omega(\rho)$ and using equation (3.2), we can get

$$k + i\omega + 1 - \frac{1}{2}(1+\sqrt{1+4\rho})e^{-k-i\omega} - \frac{1}{2}(1+\sqrt{1+4\rho})e^{-2(k+i\omega)} = 0,$$

then, we have

$$k + 1 - \frac{1}{2}(1+\sqrt{1+4\rho})e^{-k}\cos(\omega) - \frac{1}{2}(1+\sqrt{1+4\rho})e^{-2k}\cos(2\omega) = 0 \tag{3.5}$$

and

$$\omega + \frac{1}{2}(1+\sqrt{1+4\rho})e^{-k}\sin(\omega) + \frac{1}{2}(1+\sqrt{1+4\rho})e^{-2k}\sin(2\omega) = 0. \tag{3.6}$$

Differentiate equation(3.5) and equation (3.6) with respect to ρ, we obtain

$$\begin{aligned}
&\frac{dk}{d\rho} + \frac{1}{2}e^{-k}\cos(\omega)\frac{dk}{d\rho} + \frac{1}{2}e^{-k}\sin(\omega)\frac{d\omega}{d\rho} \\
&\quad - \frac{1}{2}e^{-k}\cos(\omega)\frac{4}{2\sqrt{1+4\rho}} + \frac{1}{2}(\sqrt{1+4\rho})e^{-k}\cos(\omega)\frac{dk}{d\rho} \\
&\quad + \frac{1}{2}(\sqrt{1+4\rho})e^{-k}\sin(\omega)\frac{d\omega}{d\rho} + e^{-2k}\cos(2\omega)\frac{dk}{d\rho} \\
&\quad + e^{-2k}\sin(2\omega)\frac{d\omega}{d\rho} + (\sqrt{1+4\rho})e^{-2k}\cos(2\omega)\frac{dk}{d\rho} \\
&\quad + (\sqrt{1+4\rho})e^{-2k}\sin(2\omega)\frac{d\omega}{d\rho} - \frac{1}{2}e^{-2k}\frac{4}{2\sqrt{1+4\rho}}\cos(2\omega) = 0,
\end{aligned}$$

(3.7)

$$\begin{aligned}
&= \frac{dk}{d\rho}(1 + \frac{1}{2}e^{-k}\cos(\omega) + \frac{1}{2}(\sqrt{1+4\rho})e^{-k}\cos(\omega) + e^{-2k}\cos(2\omega) + (\sqrt{1+4\rho})e^{-2k}\cos(2\omega)) \\
&+ \frac{d\omega}{d\rho}(\frac{1}{2}e^{-k}\sin(\omega) + \frac{1}{2}(\sqrt{1+4\rho})e^{-k}\sin(\omega) + e^{-2k}\sin(2\omega) + (\sqrt{1+4\rho})e^{-2k}\sin(2\omega)) \\
&- \frac{e^{-k}\cos(\omega)}{\sqrt{1+4\rho}} - \frac{e^{-2k}\cos(2\omega)}{\sqrt{1+4\rho}} = 0.
\end{aligned}$$

$$\begin{aligned}
&\frac{d\omega}{d\rho} - \frac{1}{2}e^{-k}\sin(\omega)\frac{dk}{d\rho} + \frac{1}{2}e^{-k}\cos(\omega)\frac{d\omega}{d\rho} \\
&\quad + \frac{1}{2}e^{-k}\sin(\omega)\frac{4}{2\sqrt{1+4\rho}} - \frac{1}{2}(\sqrt{1+4\rho})e^{-k}\sin(\omega)\frac{dk}{d\rho} \\
&\quad + \frac{1}{2}(\sqrt{1+4\rho})e^{-k}\cos(\omega)\frac{d\omega}{d\rho} - e^{-2k}\sin(2\omega)\frac{dk}{d\rho} \\
&\quad + e^{-2k}\cos(2\omega)\frac{d\omega}{d\rho} - (\sqrt{1+4\rho})e^{-2k}\sin(2\omega)\frac{dk}{d\rho} \\
&\quad + (\sqrt{1+4\rho})e^{-2k}\cos(2\omega)\frac{d\omega}{d\rho} + \frac{1}{2}e^{-2k}\frac{4}{2\sqrt{1+4\rho}}\sin(2\omega) = 0,
\end{aligned}$$

(3.8)

$$\begin{aligned}
&= \frac{dk}{d\rho}(-\frac{1}{2}e^{-k}\sin(\omega) - \frac{1}{2}(\sqrt{1+4\rho})e^{-k}\sin(\omega) - e^{-2k}\sin(2\omega) - (\sqrt{1+4\rho})e^{-2k}\cos(2\omega)) \\
&+ \frac{d\omega}{d\rho}(1 + \frac{1}{2}e^{-k}\cos(\omega) + \frac{1}{2}(\sqrt{1+4\rho})e^{-k}\cos(\omega) + e^{-2k}\cos(2\omega) + (\sqrt{1+4\rho})e^{-2k}\cos(2\omega)) \\
&+ \frac{e^{-k}\sin(\omega)}{\sqrt{1+4\rho}} + \frac{e^{-2k}\sin(2\omega)}{\sqrt{1+4\rho}} = 0.
\end{aligned}$$

Solving equation (3.7) and equation (3.8) for $\frac{dk}{d\rho}$, we obtain

$$\frac{d(Re(\lambda))}{d\rho}\ |_{\rho=\rho_*}= \frac{dk}{d\rho}\ |_{k=0,\omega=\omega_0,\rho=\rho_*}\ .$$

□

3.3. **The discretized system.** In this section we will study the discrete-time version of The system (3.1) by the following steps, the system can be written as

$$\begin{aligned} \frac{dx}{dt} &= -x(t) + 1 - \rho x(t-1)y(t-1), \\ y(t) &= x(t-1), \\ x(t) &= x_0, t \leq 0. \end{aligned}$$

The discretized model of the system (3.1) is obtained via the method of steps as

$$\begin{aligned} x_{n+1} &= x_n e^{-1} + (1 - \rho x_n y_n)(1 - e^{-1}), \\ y_n &= x_n. \end{aligned} \tag{3.9}$$

3.4. **Local stability and bifurcation analysis of the discretized system.** The system (3.9) has two fixed points $(x^*_{1,2}, y^*_{1,2}) = (\frac{-1\pm\sqrt{1+4\rho}}{2\rho}, \frac{-1\pm\sqrt{1+4\rho}}{2\rho})$. Next, we calculate the Jacobian matrix at the first fixed point (x^*_1, y^*_1)

$$J(x^*, y^*) = \begin{pmatrix} e^{-1} - \rho y^*(1-e^{-1}) & -\rho x^*(1-e^{-1}) \\ 1 & 0 \end{pmatrix}.$$

Let us rename $-\rho x^*(1-e^{-1}) = z$, and $e^{-1} - \rho y^*(1-e^{-1}) = m$. The characteristic equation

$$\lambda^2 - m\lambda - z = 0,$$

has two roots

$$\lambda_{1,2} = \frac{m \pm \sqrt{m^2 + 4z}}{2}.$$

Lemma 3.1. *[2] Let $F(\lambda) = \lambda^2 + P\lambda + Q$. Suppose that $F(1) > 0$, and $F(\lambda) = 0$ has two roots λ_1 and λ_2. Then*

(1) $F(-1) > 0$ *and* $Q < 1$ *if and only if* $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$ *;*
(2) $F(-1) < 0$ *if and only if* $|\lambda_1| < 1$ *and* $|\lambda_2| > 1$ *(or* $|\lambda_1| > 1$ *and* $|\lambda_2| < 1$*);*
(3) $F(-1) > 0$ *and* $Q > 1$ *if and only if* $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$*;*
(4) $F(-1) = 0$ *and* $P \neq 0, 2$ *if and only if* $\lambda_1 = -1$ *and* $|\lambda_2| \neq 1$*;*
(5) $P^2 - 4Q < 0$ *and* $Q = 1$ *if and only if* λ_1 *and* λ_2 *are complex and* $|\lambda_{1,2}| = 1$.

Applying Lemma 3.1, we get

$$F(\lambda) = \lambda^2 - m\lambda - z = \lambda^2 + P\lambda + Q = 0,$$

$P = -m$ and $Q = -z$. Now, we have

$$F(1) = 1 - m - z > 0,\ 1 > m + z.$$

Applying condition 1 of Lemma 3.1 we obtain

$$F(-1) = 1 + m - z > 0,\ 1 + m > z, \tag{3.10}$$

$$Q < 1 \Rightarrow -z < 1,\ z > -1 \text{ where } -\rho x^*(1 - e^{-1}) = z.$$

Substitute the value of x^*, we get

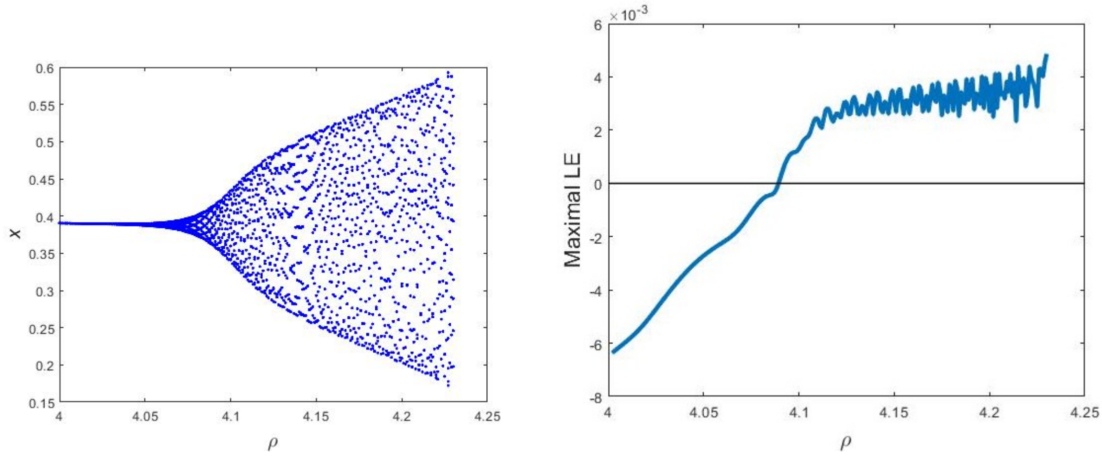
$$-\rho[\frac{-1 + \sqrt{1 + 4\rho}}{2\rho}](1 - e^{-1})$$

$$= (\frac{1 - \sqrt{1 + 4\rho}}{2})(1 - e^{-1}) > -1. \tag{3.11}$$

If (3.10) and (3.11) satisfied, then (x_1^*, y_1^*) is stable.
The same can be done for the seconde fixed point.

4. NUMERICAL SIMULATIONS

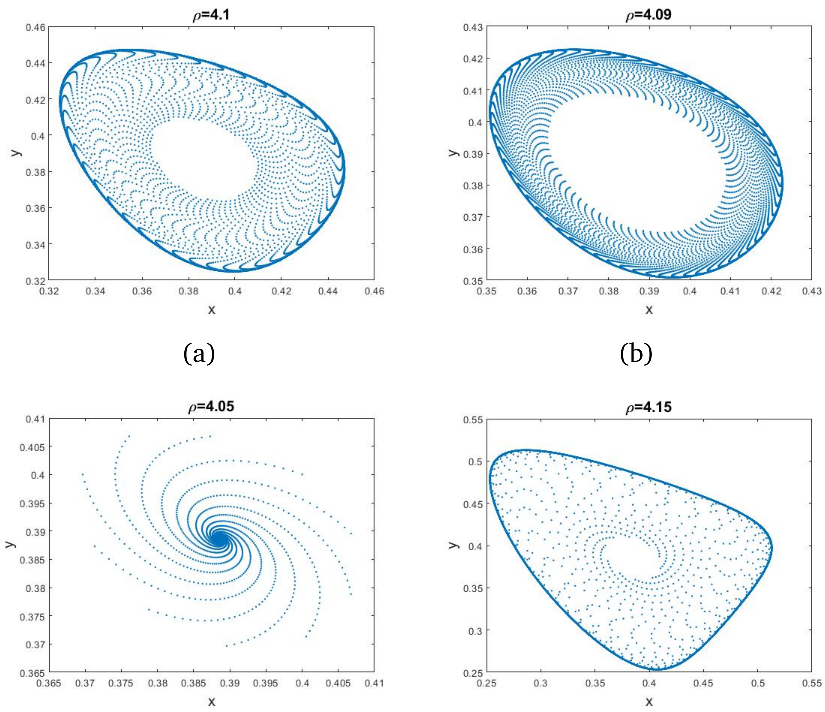
We confirm all the previous analytical findings with the help of numerical simulations performed via Matlab. In all numerical simulations the initial condition is taken as $(x_0, y_0) = (0.4, 0.4)$ and the bifurcation parameter is taken as ρ where $4 < \rho < 5$.
Figure 1 confirms the analysis of Section 3.4 by the bifurcation diagram and the graph of Lyapunov exponent.



(a) (b)

FIGURE 1

Figure 2 represents Phase portraits of system (3.9) for different values of ρ.



(a) (b)

(c) (d)

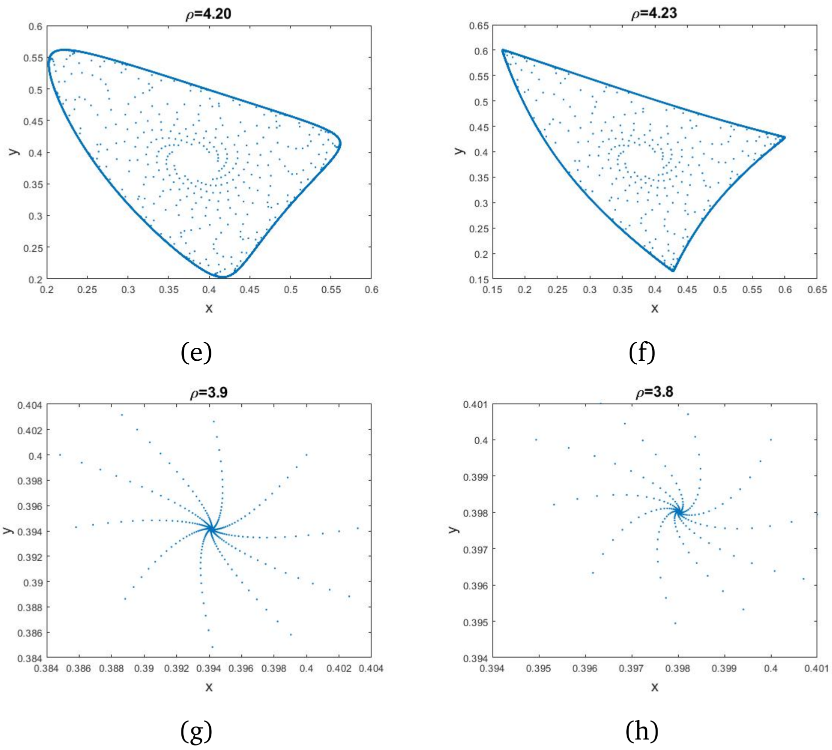
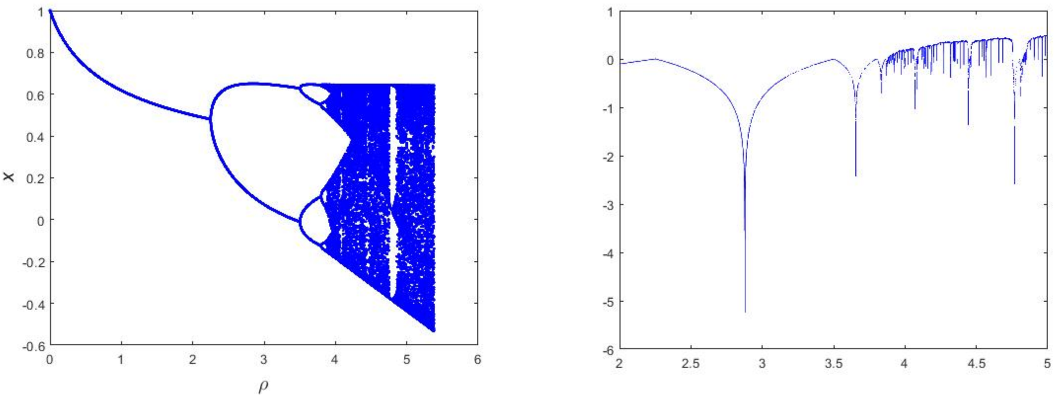


FIGURE 2

Figure 3 confirms the analysis of Section 2.3 by the bifurcation diagram and the graph of Lyapunov exponent where ρ is the bifurcation parameter.



(a) (b)

FIGURE 3. Bifurcation diagram and the graph of Lyapunov exponent of system (3.9) .

5. CONCLUSION

In this work, we have considered the Riccati differential equation with delay in view of its dynamical analysis. At first, we discussed the dynamic behavior of differential equation of delay, we get out its fixed points then we studied their local stability and existence of bifurcation by checking the eigenvalues of the linearized equations about the fixed points and its related characteristic equation. At second, we show that there is Hopf bifurcation with restricted condition for occurrence. Then, we applied the method of steps to get the discretized system. Local stability and bifurcation analysis of the discretized system. Finally, we have to confirm our analytical findings by numerical simulations, which including phase portraits, bifurcation diagram and its corresponding Lyapunov exponent.

ACKNOWLEDGMENT

The authors would like to thank the anonymous reviewers for their careful reading of the manuscript.

REFERENCES

[1] A. H. ALMUSHARRF: *Delay differential equations and the logistic equation with two delays*, ProQuest LLC, 2017.

[2] C. J. L. ALBERT: *Regularity and complexity in dynamical systems*, New York: Springer, 2012.

[3] N. P. BHATIA, G. P. ŠZEGO: *Stability theory of dynamical systems*, 2002.

[4] T. BANERJEE, D. B̌ISWAS: *Time-delayed chaotic dynamical systems from theory to electronic experiment*, Springer, New York, 2018.

[5] R. D. DRIVER: *Ordinary and delay differential equations*, in: Applied mathematical sciences, vol. 20, Springer-Verlag, New York, 1977.

[6] A. M. A. EL-SAYED, S. M. SALMAN: *Dynamic behavior and chaos control in a complex Riccati-type map*, Quaestions Mathematicae, **39**(5) (2016), 665–681. doI:10.2989/16073606.2015.1115441.

[7] A. A. ELSADANY, S. M. SALMAN: *On the bifurcation of Marotto's map and its application in image encryption*, Journal of Computational and Applied Mathematics, **328** (2018), 177-196.

[8] A. M. A. EL-SAYED, M. E. NASR: *Discontinuous dynamical system represents the Logistic retarded functional equation with two different delays*, Malaya Journal of Matematik, **1**(1) (2013), 50-56.

[9] K. GOPALSAMY: *Stability and oscillations in delay differential equations of population dynamics*, Kluwer academic publishers, 1992.

[10] L. CADARIU, V. ŘADU: *Fixed points and the stability of Jensen's functional equation*, J. Inequal. Pure Appl. Math, **4**(1) (2003), ID4.

[11] M. JIANG, Y. SHEN, J. JIAN, X. LIAO: *Stability, bifurcation and a new chaos in the logistic differential equation with delay*, physics Letters A, **350** (2006) 221-227.

[12] X. LIN, H. W̌ANG: *Stability analysis of delay differential equations with two discrete delays*, Canadian applied mathematics quarterly, **20**(4) 2012, 519-533.

[13] N. MACDONALD: *Two delays may not destabilize although either can delay*, Mathe-matical Biosciences, **82**(2) (2006), 127-140.

[14] E. OTT: *Chaos in dynamical systems*, Cambridge University Press,Cambridgel, 1993.

[15] C. G. RAGAZZO, C. P. M̌ALTA: *Singularity structure of the Hopf bifurcation sur-face of a differential equation with two delays*, J. Dynamics Differential Equations, **4** (1992), 617-650.

[16] J. P. RICHARD: *Time delay systems: An overview of some recent advances and open problems*, Automatica, **39**(10) (2003), 1667-1694.

[17] S. RUAN, J. W̌EI: *On the zeros of transcendental functions with appli-cation to stability of differential equations with two delays*, Dynamics of continuous,discrete and impulsive Systems Series A: Mathematical Analysis, **10** (2003), 863-874.

[18] S. M. SALMAN, A. M. YOUSEF, A. A. ELSADANY: *Stability, bifurcation analysis and chaos control of a discrete predator-prey system with square root functional response*, Chaos, Solitons and Fractals, **93** (2016), 20-31.

[19] M. C. MACKEY, L. ǦLASS: *Oscillations and Chaos in Physiological Control Systems*, Science, **197** (1997), 287-289.

[20] E. I. VERRIEST, S. I. ŇICULESCU: "*Delay-independent stability of linear neutral systems: A Riccati equation approach.*" *Stability and control of time-delay systems*, Springer, Berlin, Heidelberg, 1998. 92-100.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE
UNIVERSITY OF ALEXANDRIA
Email address: amasayed@alexu.edu.eg

DEPARTMENT OF MATHEMATICS, FACULTY OF EDUCATION
UNIVERSITY OF ALEXANDRIA
Email address: samastars9@gmail.com

DEPARTMENT OF MATHEMATICS, FACULTY OF EDUCATION
UNIVERSITY OF ALEXANDRIA
Email address: sousarah56@gmail.com