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DIMESIONAL FORMULÆ FOR THE RELATIVE INVARIANTS OF PRE-MATHIEU AND MATHIEU GROUPS

P. VANCHINATHAN, S. RADHA[1], AND SANJIT DAS

ABSTRACT. In this paper, we have computed the dimensions of the relative symmetric polynomials associated with all the irreducible characters of Pre Mathieu groups M_9, M_{10} and Small Mathieu groups M_{11}, M_{12}.

1. INTRODUCTION

The theory of symmetric polynomials is one of the most classical part of algebra and it plays an important role in the branches of algebra like Representation theory, Galois theory and Algebraic Combinatorics. A symmetric polynomial on n variables is a function that is unchanged by any permutation of its variables (ie) a polynomial $f(x_1, x_2, \ldots, x_n) = f(x_{\sigma(1)}, x_{(\sigma(2)}, \ldots, x_{\sigma(n)})$ for any $\sigma \in S_n$. Analogous to symmetric polynomials, anti-symmetric polynomials are not invariant under the action of S_n but they change sign according to the sign of the permutation (ie) $f(x_1, x_2, \ldots, x_n) = \xi(\sigma) f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$ where $\xi(\sigma)$ denotes the sign of the permutation. M. Shahryari [3] introduced the notion of Relative Symmetric polynomials as a generalization of symmetric and anti-symmetric polynomials.

Let G be a subgroup of the symmetric group S_n and χ be an irreducible character of G. Let $H_d(x_1, x_2, \ldots, x_n)$ denote the complex vector space of homogeneous polynomial of degree d in n variables. The image of the Reynold's operator on

[1]*corresponding author*

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$H_d(x_1, x_2, \ldots, x_n)$ will be the vector space of all symmetric polynomials of degree d. Let χ_0 denote the trivial irreducible character of G. Then the Reynolds operator can alternatively expressed as

$$\frac{\chi_0(1)}{|G|} \sum_{g \in G} \chi_0(g) g$$

This reformulation allows straight forward generalization to other characters.

Definition 1.1. *For an irreducible character χ of $G(\subset S_n)$, the following operator $T(G, \chi) = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g) g$ is a projection on $H_d(x_1, x_2, \ldots, x_n)$ for every d. The image of this projection operator denoted by $H_d(G, \chi)$, is defined to be the vector space of relative symmetric polynomials, or relative invariants, of G with respect to its irreducible character χ.*

Dimensions of the vector space of relative symmetric polynomials for various subgroups of the symmetric group S_n like S_n, A_n, Young Subgroups [4] were found by Babaei, Zamani and Shahryari. Later in a series of papers Babaei and Zamani gave the formulæ for the cyclic group [5], dicyclic group [6] and dihedral group [7]. Radha S and Vanchinathan P [2] gave an alternative dimensional formulæ for the relative invariants of the dihedral group by using theorems from number theory and combinatorial arguments. They have also constructed generating functions for the dimensional formulæ. As an application, a specific Supercharacter theory for the dihedral group was also constructed.

In this paper we focus on Mathieu groups M_{11}, M_{12}. They are the first two of five sporadic simple groups discovered by Mathieu in 1884. They are multiply transitive permutation groups of degree 11 and 12 respectively. The groups M_9, M_{10} which act transitively on sets 9 and 10 elements respectively are subgroups of the symmetric groups S_9, S_{10}. They are not sporadic simple groups but they are used to construct the Mathieu groups, by a process known as one-point extension. In this paper, we have determined the dimensions of the vector spaces of relative invariants for Pre-Mathieu groups M_9, M_{10} and also for the Small Mathieu groups M_{11}, M_{12}. A notable feature is that all the dimensional formulæ appear as summations of binomial coefficients.

2. Dimensional Formulæ for the Pre-Mathieu groups M_9 and M_{10}

2.1. **Pre-Mathieu group M_9.** There are 6 conjugacy classes and hence 6 irreducible characters for M_9. The character table for M_9 [1] is given below:

M_9	1^9	2^4	3^3	4^2_A	4^2_B	4^2_C
χ_0	1	1	1	1	1	1
χ_1	1	1	1	-1	1	-1
χ_2	1	1	1	1	-1	-1
χ_3	1	1	1	-1	-1	1
χ_4	2	-2	2	0	0	0
χ_5	8	0	-1	0	0	0

Here we state the first result of our paper. The dimensional formulæ for the relative invariants of the Pre-Mathieu group M_9 is given in the theorem 2.1 below.

Theorem 2.1. *Let $\chi_0, \chi_1, \chi_2, \chi_3, \chi_4, \chi_5$ be the irreducible characters of M_9 with degrees $1, 1, 1, 1, 2, 8$ respectively. Then the dimensions of $H_d(M_9, \chi_i)$ for $i = 0, 1, 2, 3,$ $4, 5$, the vector spaces of relative symmetric polynomials with respect to all irreducible characters of the Pre–Mathieu group M_9 is given below:*

(1) $\dim(H_d(M_9, \chi_0)) = f_{11(d)} + 9f_{12}(d) + 8f_{13}(d) + 18f_{14}(d) + 18f_{14}(d) + 18f_{14}(d)$ $= f_{11}(d) + 9f_{12}(d) + 8f_{13}(d) + 54f_{14}(d)$
(2) $\dim(H_d(M_9, \chi_1)) = f_{11}(d) + 9f_{12}(d) + 8f_{13}(d) - 18f_{14}(d) + 18f_{14}(d) - 18f_{14}(d) = f_{11}(d) + 9f_{12}(d) + 8f_{13}(d) - 18f_{14}(d)$
(3) $\dim(H_d(M_9, \chi_2)) = f_{11}(d) + 9f_{12}(d) + 8f_{13}(d) - 18f_{14}(d)$
(4) $\dim(H_d(M_9, \chi_3)) = f_{11}(d) + 9f_{12}(d) + 8f_{13}(d) - 18f_{14}(d)$
(5) $\dim(H_d(M_9, \chi_4)) = 2f_{11}(d) - 18f_{12}(d) + 16f_{13}(d)$
(6) $\dim(H_d(M_9, \chi_5)) = 8f_{11}(d) - 8f_{13}(d)$

where $f_{11}(d), f_{12}(d), f_{13}(d)$ and $f_{14}(d)$ respectively denote the number of monomials fixed by the identity permutation, permutations of the type $2^4, 3^3$ and 4^2.

- $f_{11}(d) = \binom{d+8}{8}$
- $f_{12}(d) = \sum_{l=0}^{\lfloor d/2 \rfloor} \binom{l+3}{3} = \frac{1}{4}(\lfloor d/2 \rfloor^4 + 10\lfloor d/2 \rfloor^3 + 35\lfloor d/2 \rfloor^2 + 50\lfloor d/2 \rfloor + 24)$
- $f_{13}(d) = \binom{2+\frac{d}{3}}{2} = \frac{1}{18}(d+3)(d+6)$
- $f_{14}(d) = \sum_{l=0}^{\lfloor d/4 \rfloor}(l+1) = \frac{1}{2}(\lfloor d/4 \rfloor + 1)(\lfloor d/4 \rfloor + 2)$

Now derivation of $f_{12}(d)$ *is given here. For a monomial to be invariant under the action of the permutation of type* 2^4+1 *say* $(12)(34)(56)(78)(9)$*, the variables* $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9$ *should assume degrees* $d_1, d_1, d_2, d_2, d_3, d_3, d_4, d_4, d_5$ *respectively. But sum of all the degrees is d. Hence we have* $d_1+d_1+d_2+d_2+d_3+d_3+d_4+d_4+d_5 = d$. *(ie)* $2(d_1+d_2+d_3+d_4)+d_5 = d$. *Hence,* $d_1+d_2+d_3+d_4 = \frac{d-d_5}{2}$. *Put* $\frac{d-d_5}{2} = l$. *Now* $d_1+d_2+d_3+d_4 = l$ *denotes the number of partitions of* l *into 4 parts with l ranging from 0 to* $\lfloor d/2 \rfloor$. *Hence the number of monomials to be invariant under the action of* 2^4+1 *is given by* $f_{12}(d) = \sum_{l=0}^{\lfloor d/2 \rfloor} \binom{l+3}{3}$.

The derivations of other formulæ are easy and hence omitted.

2.2. **Pre Mathieu group** M_{10}. There are 8 conjugacy classes for M_{10} and hence 8 irreducible characters. The character table for M_{10} [1] is given below:

M_{10}	1^{10}	2^4	3^3	4^2_A	4^2_B	5^2	$2^1 8^1_A$	$2^1 8^1_B$
χ_0	1	1	1	1	1	1	1	1
χ_1	1	1	1	1	-1	1	-1	-1
χ_2	9	1	0	1	1	-1	-1	-1
χ_3	9	1	0	1	-1	-1	1	1
χ_4	10	2	1	-2	0	0	0	0
χ_5	10	-2	1	0	0	0	ω	$\overline{\omega}$
χ_6	10	-2	1	0	0	0	$\overline{\omega}$	ω
χ_7	16	0	-2	0	0	1	0	0

Here we state the second result of our paper. The dimensional formulæ for the relative invariants of the Pre-Mathieu group M_{10} is given in the theorem 2.2 below.

Theorem 2.2. *Let* $\chi_0, \chi_1, \chi_2, \chi_3, \chi_4, \chi_5, \chi_6, \chi_7$ *denote the irreducible characters of* M_{10} *with degrees* $1, 1, 9, 9, 10, 10, 10, 16$ *respectively. Then the dimensions of* $H_d(M_{10}, \chi_i)$ *for* $i = 0, 1, 2, 3, 4, 5, 6, 7$*, the vector spaces of relative symmetric polynomials with respect to all irreducible characters of the Pre-Mathieu group* M_{10} *are given below:*

(1) $\dim(H_d(M_{10}, \chi_0)) = f_{21}(d) + 45f_{22}(d) + 80f_{23}(d) + 90f_{24}(d) + 180f_{24}(d) + 144f_{25}(d) + 90f_{26}(d) + 90f_{26}(d)$
$= f_{21}(d) + 45f_{22}(d) + 80f_{23}(d) + 270f_{24}(d) + 144f_{25}(d) + 180f_{26}(d)$

(2) $\dim(H_d(M_{10},\chi_1)) = f_{21}(d)+45f_{22}(d)+80f_{23}(d)+90f_{24}(d)-180f_{24}(d)+144f_{25}(d)-90f_{26}(d)-90f_{26}(d)$
$= f_{21}(d)+45f_{22}(d)+80f_{23}(d)-90f_{24}(d)+144f_{25}(d)-180f_{26}(d)$

(3) $\dim(H_d(M_{10},\chi_2)) = 9f_{21}(d)+45f_{22}(d)+90f_{24}(d)+180f_{24}(d)-144f_{25}(d)-90f_{26}(d)-90f_{26}(d)$
$= 9f_{21}(d)+45f_{22}(d)+270f_{24}(d)-144f_{25}(d)-180f_{26}(d)$

(4) $\dim(H_d(M_{10},\chi_3)) = 9f_{21}(d)+45f_{22}(d)+90f_{24}(d)-180f_{24}(d)-144f_{25}(d)+90f_{26}(d)+90f_{26}(d)$
$= 9f_{21}(d)+45f_{22}(d)-90f_{24}(d)-144f_{25}(d)+180f_{26}(d)$

(5) $\dim(H_d(M_{10},\chi_4)) = 10f_{21}(d)+90f_{22}(d)+80f_{23}(d)-180f_{24}(d)$

(6) $\dim(H_d(M_{10},\chi_5)) = 10f_{21}(d)-90f_{22}(d)+80f_{23}(d)+90\omega f_{26}(d)+90\overline{\omega}f_{26}(d)$
$= 10f_{21}(d)-90f_{22}(d)+80f_{23}(d)$

(7) $\dim(H_d(M_{10},\chi_6)) = 10f_{21}(d)-90f_{22}(d)+80f_{23}(d)+90\overline{\omega}f_{26}(d)+90\omega f_{26}(d)$
$= 10f_{21}(d)-90f_{22}(d)+80f_{23}(d)$

(8) $\dim(H_d(M_{10},\chi_7)) = 16f_{21}(d)-160f_{23}(d)+144f_{25}(d)$

where

- $f_{21}(d) = \binom{d+9}{9}$
- $f_{22}(d) = \sum_{l=0}^{\lfloor d/2\rfloor}\binom{l+3}{3}(d-2l+1)$
- $f_{23}(d) = \sum_{l=0}^{\lfloor d/3\rfloor}\binom{l+2}{2} = \frac{1}{6}\left[(\lfloor d/3\rfloor+1)(\lfloor d/3\rfloor+2)(\lfloor d/3\rfloor+3)\right]$
- $f_{24}(d) = \sum_{l=0}^{\lfloor d/4\rfloor}(l+1)(d-4l+1) = \frac{1}{6}\left[(\lfloor d/4\rfloor+1)(3d\lfloor d/4\rfloor-8\lfloor d/4\rfloor^2-13\lfloor d/4\rfloor+6d+6)\right]$
- $f_{25}(d) = \binom{\frac{d}{5}+1}{1} = \begin{cases}\frac{d}{5}+1 & \textit{when d is a multiple of 5}\\ 0 & \textit{otherwise}\end{cases}$
- $f_{26}(d) = \begin{cases}\lfloor\frac{d}{8}\rfloor+1 & \textit{when d is even}\\ 0 & \textit{otherwise}\end{cases}$

The derivation of the formula $f_{22}(d)$ is given here. For a monomial to be invariant under the action of the permutation of type 2^4+1^2 say $(12)(34)(56)(78)(9)(10)$, the variables $x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10}$ should assume degrees $d_1,d_1,d_2,d_2,\ d_3,d_3,d_4,d_4,d_5,d_6$ respectively. But sum of all the degrees is d. Hence we have $d_1+d_1+d_2+d_2+d_3+d_3+d_4+d_4+d_5+d_6 = d$. (ie) $2(d_1+d_2+d_3+d_4)+d_5+d_6 = d$. Hence, $d_1+d_2+d_3+d_4 = \frac{d-d_5-d_6}{2}$. Put $\frac{d-d_5-d_6}{2} = l$. Now $d_1+d_2+d_3+d_4 = l$ denotes the number of partitions of l into 4 parts with l ranging from 0 to $\lfloor d/2\rfloor$ and simultaneously dividing $d_5+d_6 = d-2l$

into two parts. Hence the number of monomials to be invariant under the action of 2^4+1 is given by $f_{22}(d)=\sum_{l=0}^{\lfloor d/2\rfloor}\binom{l+3}{3}(d-2l+1)$

The derivations of other formulæ are easy and hence omitted.

3. Dimensional Formulæ for the Small Mathieu groups M_{11} and M_{12}

3.1. Small Mathieu group M_{11}. There are 10 conjugacy classes and hence 10 irreducible characters for M_{11}. The character table for M_{11} [1] is given below:

M_{11}	1^{11}	2^4	3^3	4^2	5^2	$2^1 3^1 6^1$	$2^1 8^1_A$	$2^1 8^1_B$	11^1_A	11^1_B
χ_0	1	1	1	1	1	1	1	1	1	1
χ_1	10	2	1	2	0	-1	0	0	-1	-1
χ_2	10	-2	1	0	0	1	α	$\overline{\alpha}$	-1	-1
χ_3	10	-2	1	0	0	1	$\overline{\alpha}$	α	-1	-1
χ_4	11	3	2	-1	1	0	-1	-1	0	0
χ_5	16	0	-2	0	1	0	0	0	β	$\overline{\beta}$
χ_6	16	0	-2	0	1	0	0	0	$\overline{\beta}$	β
χ_7	44	4	-1	0	-1	1	0	0	0	0
χ_8	45	-3	0	1	0	0	-1	-1	1	1
χ_9	55	-1	1	-1	0	-1	1	1	0	0

where $\alpha=\sqrt{-2}$ and $\beta=\frac{1}{2}(-1+\sqrt{-11})$.

Here we state the third result of our paper. The dimensional formulæ for the relative invariants of the Small Mathieu group M_{11} is given in the theorem 3.1 below.

Theorem 3.1. *Let $\chi_0,\chi_1,\chi_2,\chi_3,\chi_4,\chi_5,\chi_6,\chi_7,\chi_8,\chi_9$ denote the irreducible characters of M_{11} with degrees $1,10,10,10,11,16,16,44,45,55$ respectively. Then the dimensions of $H_d(M_{11},\chi_i)$ for $i=0,1,2,3,4,5,6,7,8,9$, the vector spaces of relative symmetric polynomials with respect to all irreducible characters of the Pre Mathieu group M_{11} are given below:*

(1) $\dim(H_d(M_{11},\chi_0))=f_{31}(d)+165f_{32}(d)+440f_{33}(d)+990f_{34}(d)+1584f_{35}(d)+1320f_{36}(d)+1980f_{37}(d)+1440f_{38}(d)$

(2) $\dim(H_d(M_{11},\chi_1)) = 10f_{31}(d) + 330f_{32}(d) + 440f_{33}(d) + 1980f_{34}(d) - 1320f_{36}(d) - 1440f_{38}(d)$

(3) $\dim(H_d(M_{11},\chi_2)) = 10f_{31}(d) - 330f_{32}(d) + 440f_{33}(d) + 1320f_{36}(d) + 990\alpha f_{37}(d) + 990\overline{\alpha} f_{37}(d) - 1440f_{38}(d)$
$= 10f_{31}(d) - 330f_{32}(d) + 440f_{33}(d) + 1320f_{36}(d) - 1440f_{38}(d)$

(4) $\dim(H_d(M_{11},\chi_3)) = 10f_{31}(d) - 330f_{32}(d) + 440f_{33}(d) + 1320f_{36}(d) + 990\overline{\alpha} f_{37}(d) + 990\alpha f_{37}(d) - 1440f_{38}(d)$
$= 10f_{31}(d) - 330f_{32}(d) + 440f_{33}(d) + 1320f_{36}(d) - 1440f_{38}(d)$

(5) $\dim(H_d(M_{11},\chi_4)) = 11f_{31}(d) + 495f_{32}(d) + 880f_{33}(d) - 990f_{34}(d) + 1584f_{35}(d) - 1980f_{37}(d)$

(6) $\dim(H_d(M_{11},\chi_5)) = 16f_{31}(d) - 880f_{33}(d) + 1584f_{35}(d) + 720\beta f_{38}(d) + 720\overline{\beta} f_{38}(d) = 16f_{31}(d) - 880f_{33}(d) + 1584f_{35}(d) - 720f_{38}(d)$

(7) $\dim(H_d(M_{11},\chi_6)) = 16f_{31}(d) - 880f_{33}(d) + 1584f_{35}(d) + 720\overline{\beta} f_{38}(d) + 720\beta f_{38}(d) = 16f_{31}(d) - 880f_{33}(d) + 1584f_{35}(d) - 720f_{38}(d)$

(8) $\dim(H_d(M_{11},\chi_7)) = 44f_{31}(d)+660f_{32}(d)-440f_{33}(d)-1584f_{35}(d)+1320f_{36}(d)$

(9) $\dim(H_d(M_{11},\chi_8)) = 45f_{31}(d)-495f_{32}(d)+990f_{34}(d)-1980f_{37}(d)+1440f_{38}(d)$

(10) $\dim(H_d(M_{11},\chi_9)) = 55f_{31}(d)-165f_{32}(d)+440f_{33}(d)-990f_{34}(d)-1320f_{36}(d)+1980f_{37}(d)$

where

- $f_{31}(d) = \binom{d+10}{10}$
- $f_{32}(d) = \sum_{l=0}^{\lfloor d/2 \rfloor} \binom{l+3}{3}\binom{d-2l+2}{2}$
- $f_{33}(d) = \sum_{l=0}^{\lfloor d/3 \rfloor} \binom{l+2}{2}(d-3l+1)$
- $f_{34}(d) = \sum_{l=0}^{\lfloor d/4 \rfloor} (l+1)\binom{d-4l+2}{2}$
- $f_{35}(d) = \sum_{l=0}^{\lfloor d/5 \rfloor} (l+1) = \frac{1}{2}\left[\lfloor \frac{d}{5} \rfloor + 1\right]\left[\lfloor \frac{d}{5} \rfloor + 2\right]$
- $f_{36}(d) = \begin{cases} \sum_{l=0}^{\lfloor d/6 \rfloor} \left[\lfloor \frac{1}{2}\lfloor \frac{d-6l}{3} \rfloor \rfloor + 1\right] \textit{when d is even} \\ \sum_{l=0}^{\lfloor d/6 \rfloor} \left\lceil \frac{1}{2}\lfloor \frac{d-6l}{3} \rfloor \right\rceil \textit{when d is odd} \end{cases}$
- $f_{37}(d) = \sum_{l=0}^{\lfloor d/8 \rfloor} \left[\lfloor \frac{d-8l}{2} \rfloor + 1\right]$
- $f_{38}(d) = \begin{cases} 1 & \textit{if d is a multiple of 11} \\ 0 & \textit{elsewhere} \end{cases}$

The derivation of the formula $f_{33}(d)$ is given here. For a monomial to be invariant under the action of the permutation of type 3^3+1^2 say $(123)(456)(789)(10)(11)$, the variables $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}$ should assume degrees d_1, d_1, d_1, $d_2, d_2, d_2, d_3, d_3, d_3, d_4, d_5$ respectively. But sum of all the degrees is d.

Hence we have $d_1 + d_1 + d_1 + d_2 + d_2 + d_2 + d_3 + d_3 + d_3 + d_4 + d_5 = d$. (ie) $3(d_1 + d_2 + d_3) + d_4 + d_5 = d$. Hence, $d_1 + d_2 + d_3 = \frac{d-d_4-d_5}{3}$. Put $\frac{d-d_4-d_5}{3} = l$. Now $d_1 + d_2 + d_3 = l$ denotes the number of partitions of l into 3 parts with l ranging from 0 to $\lfloor d/3 \rfloor$ and simultaneously dividing $d_4 + d_5 = d - 3l$ into two parts. Hence the number of monomials to be invariant under the action of $3^3 + 1^2$ is given by $f_{33}(d) = \sum_{l=0}^{\lfloor d/3 \rfloor} \binom{l+2}{2}(d - 3l + 1)$

3.2. **Small Mathieu group M_{12}.** There are 15 conjugacy classes and hence 15 irreducible characters for M_{12}. The character table for M_{12} [1] is given below:

M_{12}	1^{12}	2^4	2^6	3^3	3^4	4^2	$2^2 4^2$	5^2	$2^1 3^1 6^1$	6^2	$2^1 8^1$	$4^1 8^1$	$2^1 10^1$	11^1_A	11^1_B
χ_0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_1	11	3	-1	2	-1	3	-1	1	0	-1	1	-1	-1	0	0
χ_2	11	3	-1	2	-1	-1	3	1	0	-1	-1	1	-1	0	0
χ_3	16	0	4	-2	1	0	0	1	0	1	0	0	-1	ω	$\overline{\omega}$
χ_4	16	0	4	-2	1	0	0	1	0	1	0	0	-1	$\overline{\omega}$	ω
χ_5	45	-3	5	0	3	1	1	0	0	-1	-1	-1	0	1	1
χ_6	54	6	6	0	0	2	2	-1	0	0	0	0	1	-1	-1
χ_7	55	7	-5	1	1	-1	-1	0	1	1	-1	-1	0	0	0
χ_8	55	-1	-5	1	1	-1	3	0	-1	1	-1	1	0	0	0
χ_9	55	-1	-5	1	1	3	-1	0	-1	1	1	-1	0	0	0
χ_{10}	66	2	6	3	0	-2	-2	1	-1	0	0	0	1	0	0
χ_{11}	99	3	-1	0	3	-1	-1	-1	0	-1	1	1	-1	0	0
χ_{12}	120	-8	0	3	0	0	0	0	1	0	0	0	0	-1	-1
χ_{13}	144	0	4	0	-3	0	0	-1	0	1	0	0	-1	1	1
χ_{14}	176	0	-4	-4	-1	0	0	1	0	-1	0	0	1	0	0

where $\omega = \frac{1}{2}(-1 + \sqrt{-11})$.

Here we state the fourth result of our paper. The dimensional formulæ for the relative invariants of the Small Mathieu group M_{12} is given in the theorem 3.2 below.

Theorem 3.2. *Let* $\chi_0, \chi_1, \chi_2, \chi_3, \chi_4, \chi_5, \chi_6, \chi_7, \chi_8, \chi_9, \chi_{10}, \chi_{11}, \chi_{12}, \chi_{13}, \chi_{14}$ *denote the irreducible characters of* M_{12} *with degrees 1, 11, 11, 16, 16, 45, 54, 55, 55, 55, 66, 99, 120, 144,176 respectively. Then the dimensions of* $H_d(M_{12}, \chi_i)$ *for* $i = 0, 1, 2, \ldots, 3, 14$*, the vector spaces of relative symmetric polynomials with respect to all irreducible characters of the Pre Mathieu group* M_{12} *are given below:*

(1) $\dim(H_d(M_{12}, \chi_0)) = g_1(d) + 495g_2(d) + 396g_3(d) + 1760g_4(d) + 2640g_5(d) + 2970[g_6(d) + g_7(d)] + 9504[g_8(d) + g_{13}(d)] + 15840g_9(d) + 7920g_{10}(d) + 11880[g_{11}(d) + g_{12}(d)] + 17280g_{14}(d)$

(2) $\dim(H_d(M_{12},\chi_1)) = 11g_1(d)+1485g_2(d)-396g_3(d)+3520g_4(d)-2640g_5(d)+8910g_6(d)-2970g_7(d)+9504(g_8(d)-g_{13}(d))-7920g_{10}(d)+11880[g_{11}(d)-g_{12}(d)]$

(3) $\dim(H_d(M_{12},\chi_2)) = 11g_1(d)+1485g_2(d)-396g_3(d)+3520g_4(d)-2640g_5(d)-2970g_6(d)+8910g_7(d)+9504[g_8(d)-g_{13}(d)]-7920g_{10}(d)+11880[g_{12}(d)-g_{11}(d)]$

(4) $\dim(H_d(M_{12},\chi_3)) = 16g_1(d)+1584g_3(d)-3520g_4(d)+2640g_5(d)+9504[g_8(d)-g_{13}(d)]+7920g_{10}(d)-8640g_{14}(d)$

(5) $\dim(H_d(M_{12},\chi_4)) = 16g_1(d)+1584g_3(d)-3520g_4(d)+2640g_5(d)+9504[g_8(d)-g_{13}(d)]+7920g_{10}(d)-8640g_{14}(d)$

(6) $\dim(H_d(M_{12},\chi_5)) = 45g_1(d)-1485g_2(d)+1980g_3(d)+7920g_5(d)+2970[g_6(d)+g_7(d)]-7920g_{10}(d)-11880[g_{11}(d)+g_{12}(d)]+17280g_{14}(d)$

(7) $\dim(H_d(M_{12},\chi_6)) = 54g_1(d)+2970g_2(d)+2376g_3(d)+5940[g_6(d)+g_7(d)]+9504[g_{13}(d)-g_8(d)]-17280g_{14}(d)$

(8) $\dim(H_d(M_{12},\chi_7)) = 55g_1(d)+3465g_2(d)-1980g_3(d)+1760g_4(d)+2640g_5(d)-2970[g_6(d)+g_7(d)]+15840g_9(d)+7920g_{10}(d)-11880[g_{11}(d)+g_{12}(d)]$

(9) $\dim(H_d(M_{12},\chi_8)) = 55g_1(d)-495g_2(d)-1980g_3(d)+1760g_4(d)+2640g_5(d)-2970g_6(d)+8910g_7(d)-15840g_9(d)+7920g_{10}(d)+11880[g_{12}(d)-g_{11}(d)]$

(10) $\dim(H_d(M_{12},\chi_9)) = 55g_1(d)-495g_2(d)-1980g_3(d)+1760g_4(d)+2640g_5(d)+8910g_6(d)-2970g_7(d)-15840g_9(d)+7920g_{10}(d)+11880[g_{11}(d)-g_{12}(d)]$

(11) $\dim(H_d(M_{12},\chi_{10})) = 66g_1(d)+990g_2(d)+2376g_3(d)+5280g_4(d)-5940[g_6(d)+g_7(d)]+9504[g_8(d)+g_{13}(d)]-15840g_9(d)$

(12) $\dim(H_d(M_{12},\chi_{11})) = 99g_1(d)+1485g_2(d)-396g_3(d)+7920g_5(d)-2970[g_6(d)+g_7(d)]-9504[g_8(d)+g_{13}(d)]-7920g_{10}(d)+11880[g_{11}(d)+g_{12}(d)]$

(13) $\dim(H_d(M_{12},\chi_{12})) = 120g_1(d)-3960g_2(d)+5280g_4(d)+15840g_9(d)-17280g_{14}(d)$

(14) $\dim(H_d(M_{12},\chi_{13})) = 144g_1(d)+1584g_3(d)-7920g_5(d)-9504[g_8(d)+g_{13}(d)]+7920g_{10}(d)+17280g_{14}(d)$

(15) $\dim(H_d(M_{12},\chi_{14})) = 176g_1(d)-1584g_3(d)-7040g_4(d)-2640g_5(d)+9504[g_8(d)+g_{13}(d)]-7920g_{10}(d)$

where

- $g_1(d) = \binom{d+11}{11}$
- $g_2(d) = \sum_{l=0}^{\lfloor d/2\rfloor} \binom{l+3}{3}\binom{d-2l+3}{3}$

- $g_3(d) = \begin{cases} \binom{5+d/2}{5} & \textit{when } d \textit{ is even} \\ 0 & \textit{otherwise} \end{cases}$
- $g_4(d) = \sum_{l=0}^{\lfloor d/3 \rfloor} \binom{l+2}{2}\binom{d-3l+2}{2}$
- $g_5(d) = \begin{cases} \binom{3+d/3}{3} & \textit{when } d \textit{ is a multiple of 3} \\ 0 & \textit{otherwise} \end{cases}$
- $g_6(d) = \sum_{l=0}^{\lfloor d/4 \rfloor} (l+1)\binom{d-4l+3}{3}$
- $g_7(d) = \begin{cases} \sum_{l=0}^{\lfloor d/4 \rfloor} \left[\lfloor \frac{d-4l}{2} \rfloor + 1\right](l+1) & \textit{when } d \textit{ is even} \\ 0 & \textit{otherwise} \end{cases}$
- $g_8(d) = \sum_{l=0}^{\lfloor d/5 \rfloor} (l+1)(d-5l+1)$
- $g_9(d) = \sum_{l=0}^{\lfloor d/6 \rfloor} \sum_{k=0}^{\lfloor \frac{d-6l}{3} \rfloor} \left[\lfloor \frac{d-6l-3k}{2} \rfloor + 1\right]$
- $g_{10}(d) = \begin{cases} \frac{d}{6} + 1 & \textit{when } d \textit{ is a multiple of 6} \\ 0 & \textit{otherwise} \end{cases}$
- $g_{11}(d) = \sum_{l=0}^{\lfloor d/8 \rfloor} \sum_{k=0}^{d-8l} \left[\lfloor \frac{d-8l-k}{2} \rfloor + 1\right]$
- $g_{12}(d) = \begin{cases} \lfloor \frac{d}{8} \rfloor + 1 & \textit{when } d \textit{ is a multiple of 4} \\ 0 & \textit{otherwise} \end{cases}$
- $g_{13}(d) = \begin{cases} \lfloor \frac{d}{10} \rfloor + 1 & d \textit{ is even} \\ 0 & \textit{otherwise} \end{cases}$
- $g_{14}(d) = \lfloor \frac{d}{11} \rfloor + 1$

Proof for all the above formulæ $g_i(d), i = 0, 1, \ldots, 14$ are calculated using the same procedure and derivations followed in M_9, M_{10} and M_{11} respectively.

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DIVISION OF MATHEMATICS
SCHOOL OF ADVANCED SCIENCES
VELLORE INSTITUTE OF TECHNOLOGY
CHENNAI, TAMILNADU, INDIA
Email address: vanchinathan.p@vit.ac.in

DIVISION OF MATHEMATICS
SCHOOL OF ADVANCED SCIENCES
VELLORE INSTITUTE OF TECHNOLOGY
CHENNAI, TAMILNADU, INDIA
Email address: radha.s@vit.ac.in

DIVISION OF PHYSICS
SCHOOL OF ADVANCED SCIENCES
VELLORE INSTITUTE OF TECHNOLOGY
CHENNAI, TAMILNADU, INDIA
Email address: sanjit.das@vit.ac.in