

ON SOME PROPERTIES OF CAYLEY GRAPHS OF RECTANGULAR BANDS

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ABSTRACT. A rectangular band is a direct product of left zero semigroup and right zero semigroup. Arthur Cayley introduced Cayley graphs of groups in 1878 and Cayley graphs of semigroups are its generalizations. In this paper we describe some properties of Cayley graphs of rectangular bands. We see that, for any arbitrary non-empty finite sets I and Λ , the Cayley graph of the rectangular band $S = I \times \Lambda$ relative to any Green's equivalence \mathcal{L} -class of S has a Hamiltonian decomposition consisting of $|\Lambda|$ components. Furthermore we also shows, when $|I|$ is odd, the decomposition is Eulerian.

1. INTRODUCTION

A rectangular band is a band S which satisfies $aba = a$ for all $a, b \in S$, equivalently $a^2 = a$ for all $a \in S$ and $abc = ac$ for all $a, b, c \in S$ [5]. In 1964, Bosak [1] studied certain graphs over semigroups. Idempotent semigroups was first studied by David McLean in [2]. Naoki Kimura [6] studied some special idempotent semigroups. In [3] S. H. Fan and Y.S. Zeng obtain a complete description of all vertex transitive Cayley graphs of bands.

Let I and Λ be two arbitrary non empty sets, we can define a semigroup operation on $I \times \Lambda$ by $(i, \lambda)(j, \mu) = (i, \mu)$ for $(i, \lambda), (j, \mu) \in I \times \Lambda$. The resulting semigroup is a rectangular band and is always regular. Throughout this paper I and Λ are considered as finite sets.

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2. PRELIMINARIES

In this section we describes some basic definitions and results of semigroup theory and graph theory which are needed in the sequel.

Definition 2.1. [5] Let S be a semigroup. We define $a\mathcal{L}b$ ($a, b \in S$) if and only if a and b generates the same principal left ideal, that is, if and only if $S^1a = S^1b$. Similarly we define $a\mathcal{R}b$ if and only if a and b generates the same principal right ideal, that is, if and only if $aS^1 = bS^1$.

Lemma 2.1. [5] Let a, b be elements of a semigroup S . Then $a\mathcal{L}b$ if and only if there exist $x, y \in S^1$ such that $xa = b$, $yb = a$ and $a\mathcal{R}b$ if and only if there exist $u, v \in S^1$ such that $au = b$, $bv = a$.

Definition 2.2. [4] If for a pair of vertices u, v both (u, v) and (v, u) are arcs of D , then (u, v) and (v, u) are symmetric pair of arcs and denoted it by (uv) . A symmetric digraph is one which every arc occurs as a symmetric pair. A complete symmetric digraph is the digraph with vertex set V and every symmetric pair (uv) for $u, v \in V$. It will be denoted by K_n^* , where $n = |V|$.

Definition 2.3. [4] A directed graph or a digraph D is a pair (V, A) , where V is a non-empty set whose elements are called the vertices of D and A is a subset $V^{(2)}$ (the set of ordered pairs of distinct elements of V), whose elements are called the directed edges or arcs of D .

Definition 2.4. [4] For any $v \in V$, the number of arcs adjacent to v is the in valence of v and the number of arcs adjacent from v is the out valence of v . We denote these by $d^-(v)$ and $d^+(v)$ respectively. The total valence or, simply, the valence of v is $d(v) = d^-(v) + d^+(v)$. If $d(v) = k$ for every $v \in V$, then D is said to be a k -regular digraph. If for every $v \in V$, $d^-(v) = d^+(v)$, the digraph is said to be an isograph.

Definition 2.5. [4] Two vertices v and u of D are strongly connected if they are mutually reachable: there is a path from v to u and a path from u to v .

Definition 2.6. [4] A spanning path of a digraph is a path that visits each vertex exactly once and a spanning cycle is a cycle that visits each vertex exactly once. A spanning path of a digraph is called a hamiltonian path and a spanning cycle, a hamiltonian cycle. A digraph with a hamiltonian cycle is called Hamiltonian. A

graph is Hamiltonian connected if for every pair of vertices there is a Hamiltonian path between them.

Definition 2.7. A sub digraph $H = (U, B)$ of a digraph $D = (V, A)$ is said to be vertex induced subgraph or induced subgraph if B consists of all the arcs of D joining pairs of vertices of U . A decomposition of a graph $D = (V, A)$ is a set of subgraphs H_1, H_2, \dots, H_K that partition the the arcs of D .

Definition 2.8. [4] A tour traversing each arc of a digraph exactly once is called an euler tour and a digraph with euler tour is called eulerian.

Theorem 2.1. A directed graph G has an Eulerian circuit if and only if it is connected and its vertices all have even in valence.

Definition 2.9. [7] Let S be a finite semigroup and let H be a non-empty subset of S . The Cayley graph $Cay(S, H)$ of S relative to H is defined as the graph with vertex set S and arc set $\{(x, y) : hx = y \text{ for some } h \in H\}$.

3. RESULTS

Proposition 3.1. Let $S = I \times \Lambda$ be a rectangular band and $s_1 = (i_1, \lambda_1), s_2 = (i_2, \lambda_2) \in S$ then (i) $s_1 \mathcal{L} s_2$ if and only if $\lambda_1 = \lambda_2$ (ii) $s_1 \mathcal{R} s_2$ if and only if $i_1 = i_2$

Remark 3.1. The \mathcal{L} -class of a rectangular band $S = I \times \Lambda$ are of the form $L_{\lambda_k} = \{(i_j, \lambda_k) \in S : i_j \in I\}$ for some $\lambda_k \in \Lambda$. Similarly the \mathcal{R} -class of a rectangular band S are of the form $R_{i_j} = \{(i_j, \lambda_k) : \lambda_k \in \Lambda\}$ for some $i_j \in I$.

Proposition 3.2. Let S be a rectangular band and L_{λ_k} be any \mathcal{L} -class of S . Then for any subset S' of S containing L_{λ_k} , the Cayley graph $Cay(S, S')$ is symmetric.

Proof. Let $x \in S$ and $y \in S'$. Then by the group operations on S , $y' L_{\lambda_k} x$ where $y' = yx$. Since $y' L_{\lambda_k} x$, there exist $u, v \in L_{\lambda_k}$ such that $uy' = x$ and $vx = y'$. Then by Definition 2.9, there exists a bidirected arc between x and y' in $Cay(S, S')$. Since y is arbitrary, it follows that there is a bidirectd arc between x to every $y' \in S$. Thus for any arc in $Cay(S, S')$ occurs as a symmetric pair and hence the graph is symmetric. \square

Proposition 3.3. Let S be a rectangular band and L_{λ_k} be any \mathcal{L} -class of S . Then for any subset S' of S containing L_{λ_k} , the Cayley graph $Cay(S, S')$ is i) an isograph ii) $2(p-1)$ regular, $p = |I|$.

Proof. Since every symmetric graph is an isograph, the proof of (i) is obvious. In proof of Proposition 3.2, we see that for any $x \in S$ in $Cay(S, S')$ is connected to every $y' \in S$, where $xL_{\lambda_k}y'$. Also by Remark 3.1, we get $|L_{\lambda_k}| = |I| = p$. Hence the proof of (ii). \square

Corollary 3.1. *Let S be a rectangular band and L_{λ_k} be any \mathcal{L} -class of S . Then the graph induced by the vertex set $L_{\lambda_k'}$ of $Cay(S, L_{\lambda_k})$ is i) symmetric ii) an isograph iii) $2(p-1)$ regular, $p = |I|$.*

Proposition 3.4. *Let S be a rectangular band and L_{λ_k} be any \mathcal{L} -class of S . Then the graph induced by the vertex set $L_{\lambda_k'}$ of $Cay(S, L_{\lambda_k})$ is i) strongly connected ii) hamiltonian connected iii) eulerian, when $p = |I|$ is odd.*

Proof. By Corollary 3.1, the graph induced by the vertex set $L_{\lambda_k'}$ of $Cay(S, L_{\lambda_k})$ is $2(p-1)$ regular. Since $|L_{\lambda_k}| = p$, it is K_p^* . Hence (i) and (ii) are trivial. When p is odd every vertex in this graph have even in valence. Hence by Theorem 2.1, we have the proof of (iii). \square

Proposition 3.5. *Let S be a rectangular band and L_{λ_k} be any \mathcal{L} -class of S . Then $Cay(S, L_{\lambda_k})$ has a hamiltonian decomposition consisting of q subgraphs of p vertices where $p = |I|$ and $q = |\Lambda|$.*

Proof. Let $x, y \in S$ with $x = (i_1, \lambda_1)$ and $y = (i_2, \lambda_2)$. Suppose there is an arc from x to y in $Cay(S, L_{\lambda_k})$. Then by Definition 2.9, there exists an $(i_j, \lambda_k) \in L_{\lambda_k}$ such that $(i_j, \lambda_k)x = y$, which implies $\lambda_1 = \lambda_2$ and so $xL_{\lambda_k}y$. Thus $x, y \in L_{\lambda_k'}$ for some fixed $\lambda_k' \in \Lambda$. Since the graph induced by the vertex set $L_{\lambda_k'}$ of $Cay(S, L_{\lambda_k})$ is K_p^* , the occurrence of arc from x to y in the graph induced by the vertex set $L_{\lambda_k'}$ of $Cay(S, L_{\lambda_k})$ ensures the occurrence of the arc from x to y in the union of such induced subgraphs with vertex set $L_{\lambda_k} = \cup_{\lambda_k' \in \Lambda} L_{\lambda_k'}$. Since $L_{\lambda_k'} \cap L_{\lambda_k''} = \phi$, each of such induced subgraphs are disjoint.

On the other hand there is an arc from x to y in the disjoint union of all such induced subgraphs. Hence $x, y \in L_{\lambda_k'}$ for some $\lambda_k' \in \Lambda$ and so $xL_{\lambda_k}y$. Since $xL_{\lambda_k}y$, by Proposition 3.1 we have $\lambda_1 = \lambda_2 = \lambda$. Thus $x = (i_1, \lambda)$ and $y = (i_2, \lambda)$. Also there exist $l_1 = (i_1, \lambda_k), l_2 = (i_2, \lambda_k) \in L_{\lambda_k}$. Which implies $x = l_1y$ and $y = l_2x$. Then by Definition 2.9, there is a bidirected arc between x and y in $Cay(S, L_{\lambda_k})$. Hence $Cay(S, L_{\lambda_k})$ is the disjoint union of q subgraphs of p vertices, each of which is hamiltonian. \square

Corollary 3.2. *Let S be a rectangular band and L_{λ_k} be any \mathcal{L} -class of S . Then $\text{Cay}(S, L_{\lambda_k})$ has an eulerian decomposition when p is odd.*

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