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THE UPPER PATH INDUCED MONOPHONIC NUMBER OF GRAPHS

I. ANNALIN SELCY[1] AND P. ARUL PAUL SUDHAHAR

ABSTRACT. A monophonic set is called a path induced monophonic set of G if $\langle M \rangle$ has a Hamiltonian path. The minimum cardinality of a path induced monophonic set is called path induced monophonic number of G and is denoted by $pim(G)$. A path induced monophonic set with $|M| = pim(G)$ is called a minimum path induced monophonic set of G or pim-set of G. A path induced monophonic set M in a connected graph G is called a minimal path induced monophonic set if no proper subset of M is a path induced monophonic set of G. The upper path induced monophonic number $pim^+(G)$ is the maximum cardinality of a minimal path induced monophonic set of G. Some general properties satisfied by this concept are studied. For any integers $3 < a \leq b$ $(b > a + 2)$, there exists a connected graph G such that $pim(G) = a$ and $pim^+(G) = b$.

1. INTRODUCTION

By a graph $G = (V,E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. We consider connected graph with at least three vertices. For basic theoretic terminology we refer to Harary [2]. For two vertices u and v in a connected graph G, the *distance* $d(u, v)$ is the length of a shortest u-v path in G. An u-v path of length $d(u, v)$ is called an u-v *geodesic*. For a vertex v of G,

[1]*corresponding author*

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the *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices is the *radius*, $radG$ and the maximum eccentricity is the *diameter*, $diamG$ of G. For subsets A and B of $V(G)$, the distance $d(A, B)$ is defined as $d(A, B) = min\{d(x, y) : x \in A, y \in B\}$. An u–v path of length $d(A, B)$ is called an A-B geodesic joining the sets A, B, where $u \in A$ and $v \in B$. A vertex x is said to lie on an A - B geodesic if x is a vertex of an A-B geodesic. For $A = (u, v)$ and $B = (z, w)$ with uv and zw edges, we write an A-B geodesic as uv - zw geodesic and $d(A, B)$ as $d(uv, zw)$. The *maximum degree* of G, denoted by $\triangle(G)$, is given by $\triangle(G) = max\{deg_G(v) : v \in V(G)\}, N(v) = \{u \in V(G) : uv \in E(G)\}$ is called the *neighborhood* of the vertex v in G. A vertex v is an *extreme vertex* of a graph G if the subgraph induced by its neighbors is complete. An edge e of a graph G is called an *extreme edge* of G, if one of its ends is an extreme vertex of G. A chord of a path $u_0, u_1, u_2, ..., u_h$ is, an edge u_iu_j with $j \geq i + 2$. An u – v path is called a monophonic path if it is a chordless path. A monophonic set of G is a set $M \subseteq V$ such that every vertex of G lies on a monophonic path joining some pair of vertices in M. A monophonic set $M \subseteq V$ is called a *path induced monophonic set* of G if $< M >$ has a Hamiltonian path. The minimum cardinality of a path induced monophonic set is called *path induced monophonic number* of G, denoted by $pim(G)$. A path induced monophonic set with $| M |= pim(G)$ is called a minimum path induced monophonic number of G or pim-set of G.

The following theorems are used in sequel.

Theorem 1.1. *[1]* Each extreme vertex of G belongs to every path induced monophonic set.

Theorem 1.2. *[1]* Let G be a path induced monophonic graph with a cut-vertex x. Then x belongs to every path induced monophonic graph set of G.

Theorem 1.3. *[1]* For a path $G = P_p$ $(p \geq 2)$, $pim(G) = p$.

2. THE UPPER PATH INDUCED MONOPHONIC NUMBER OF SOME GRAPHS

Definition 2.1. *Let G be a connected graph. A path induced monophonic set M in a connected graph G is called a minimal path induced monophonic set if no proper subset of M is a path induced monophonic set of G. The upper path*

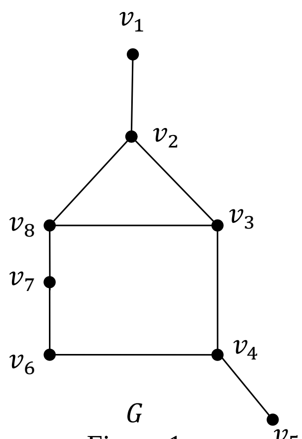

Figure 1

induced monophonic number $pim^+(G)$ *is the maximum cardinality of a minimal path induced monophonic set of* G*.*

Example 1. *For the graph* G *given in Figure 1,* $M_1 = \{v_1, v_2, v_3, v_4, v_5\}$ *and* $M_2 = \{v_1, v_2, v_8, v_7, v_6, v_4, v_5\}$ *are the only two minimal path induced monophonic sets of* G*. Therefore* $pim^+(G) = 7$*.*

Theorem 2.1. *For a connected graph* G*,* $2 \leq pim(G) \leq pim^+(G) \leq p$*.*

Proof. Any path induced monophonic set needs at least two vertices and so $pim(G) \geq 2$. Since every minimum path induced monophonic set is a minimal path induced monophonic set, $pim(G) \leq pim^+(G)$. Also, since $V(G)$ induces a minimal path induced monophonic set of G, it is clear that $pim^+(G) \leq p$. Thus $2 \leq pim(G) \leq pim^+(G) \leq p$. □

Remark 2.1. *For the graph* $G = P_3$*,* $pim(G) = 3$*. For the path* $G = P_p$*,* $pim^+(G) = p$*. Also, all the inequalities in Theorem 2.1 are strict. For the graph* G *given in Figure 1,* $pim(G) = 5$*,* $pim^+(G) = 7$ *and* $p = 8$ *so that* $2 < pim(G) < pim^+(G) < p$*.*

Theorem 2.2. Let G be a path induced monophonic graph. Then $pim(G) = p$ if and only if $pim^+(G) = p$.

Proof. Let $pim^+(G) = p$. Then $M = V(G)$ is the unique minimal path induced monophonic set of G. Since no proper subset of M is a path induced monophonic set, it is clear that M is the unique minimum path induced monophonic set of G and so $pim(G) = p$. The converse follows from Theorem 2.1. □

Theorem 2.3. Let G be a path induced monophonic graph. Then every extreme vertex of a connected graph G belongs to every minimal path induced monophonic set of G.

Proof. Since every minimal path induced monophonic set is a path induced monophonic set, the result follows from Theorem 1.1. □

Corollary 2.1. *For the complete graph $G = K_p$, $pim^+(G) = p$.*

Proof. This follows from Theorem 2.2. □

Corollary 2.2. For any path $G = P_p$, $pim^+(G) = pim(G) = p$.

Proof. This follows from Theorem 1.2 and Theorem 2.2. □

Theorem 2.4. For the complete bipartite graph $G = K_{m,n}$ $(2 \leq m \leq n)$, $pim^+(G) = 4$.

Proof. Without loss of generality, let $m \leq n$. Let $U = \{u_1, u_2, ..., u_m\}$ and $V = \{v_1, v_2, ..., v_n\}$ be a bipartition of G. Let M be any path induced monophonic set of G. We prove that M contains at least two vertices from U and at least two vertices from V. Suppose that M contains at most one vertex from U and at most one vertex from V. Then M is not a path induced monophonic set of G, which is a contradiction. Therefore M contains at least two vertices from U and at least two vertices from V. We prove that $pim^+(G) = 4$. Suppose $pim^+(G) \geq 4$. Then there exists a minimal path induced monophonic set M_1 of G with $| M_1 \geq 5$. Since M_1 contains at least two vertices from U and at least two vertices from V, without loss of generality, let $u_1, u_2, u_3, v_1, v_2 \in M_1$. Then $M_2 = M_1 - \{u_3\}$ is a path induced monophonic set of G with $M_2 \subseteq M_1$, which is a contradiction to M_1 a miniminal path induced monophonic set of G. Therefore $pim^+(G) = 4$. □

Theorem 2.5. For the cycle $G = C_p$, $pim^+(G) = 3$.

Proof. Let $C_p : v_1, v_2, v_3, ..., v_p, v_1$. Let $M = \{x, y, z\}$ be a set of vertices of G such that $xy, yz \in E(G)$. Then M is a minimal path induced monophonic set of G

so that $pim^+(G) \geq 3$. We show that $pim^+(G) = 3$. On the contrary suppose that $pim^+(G) \geq 4$. Then there exists a minimal path induced monophonic set M' of G such that $\mid M' \mid \geq 4$. If $< M' >$ is connected, then $M \subset M'$. If $< M' >$ is not connected then $< M' >$ has no Hamiltonian path, which is a contradiction. Therefore M' is not a minimal path induced monophonic set of G. Hence $pim^+(G) = 3$. □

Theorem 2.6. For the wheel $G = K_1 + C_{p-1}$ $(p \geq 4)$, $pim^+(G) = 3$.

Proof. The pim-set of G are $M = \{x, v, w\}$ where $uv, vw \in E(G)$ and $S = \{u, x, v\}$ where $ux, vx \in E(G)$ and $uv \notin E(G)$ so that $pim^+(G) = 3$. We prove that $pim^+(G) = 3$. Suppose that $pim^+(G) \geq 4$. Then there exists a minimal path induced monophonic set M_1 of G such that $\mid M_1 \mid \geq 4$. Since $< M_1 >$ contains a Hamiltonian path, then either $M \subset M_1$ or $S \subset M_1$, which is a contradiction. Therefore $pim^+(G) = 3$. □

Theorem 2.7. For any integers $3 < a \leq b$ $(b > a + 2)$, there exists a connected graph G such that $pim(G) = a$ and $pim^+(G) = b$.

Proof. If $a = b$, let $G = P_p$. Then by Theorem 1.3 and Corollary 2.1, $pim(G) = a = pim^+(G)$.

Let $3 < a < b$. Let $P_{a-2} : v_1, v_2, ..., v_{a-2}$ be a path on $a - 2$ vertices. Let $Q_{b-a} : u_1, u_2, ..., u_{b-4}$ be a path on $b - 4$ vertices. Let G be the graph obtained from P_{a-2} and Q_{b-a-4} by adding the vertices x, y and introducing the edges $xv_1, yv_{a-2}, u_{b-4}v_1$ and u_1v_{a-2}. Let $M = \{x, y, v_1, v_2, ..., v_{a-2}\}$. Then M is a path induced monophonic set of G so that $pim(G) \leq a$. We prove that $pim(G) = a$. Suppose that $pim(G) \leq a - 1$. Then there exists a path induced monophonic set M' such that $\mid M' \mid \leq a - 1$. By Theorem 1.1, $x, y \in M'$. Since $b > 2a - 2$, $< M' >$ has no Hamiltonian path, which is a contradiction. Then $pim(G) = a$.

Next, we show that $pim^+(G) = b$. Let $M_2 = \{x, y, u_1, u_2, ..., u_{b-4}, , v_1, v_{a-2}\}$. Then M_2 is a path induced monophonic set of G. We prove that M_2 is a minimal path induced monophonic set of G. If not suppose that there exists a path induced monophonic set M_3 such that $M_3 = M_2$. Since $x, y \in M_3$ and $< M_3 >$ has a Hamiltonian path, we have $M_3 \subset M_2$, which is a contradiction. Therefore M_3 is a minimal path induced monophonic set of G so that $pim^+(G) \geq b$. We prove that $pim^+(G) = b$. Suppose that $pim^+(G) \geq b + 1$. Then there exists a minimal path induced monophonic set M such that $\mid M \mid \geq b + 1$. Since

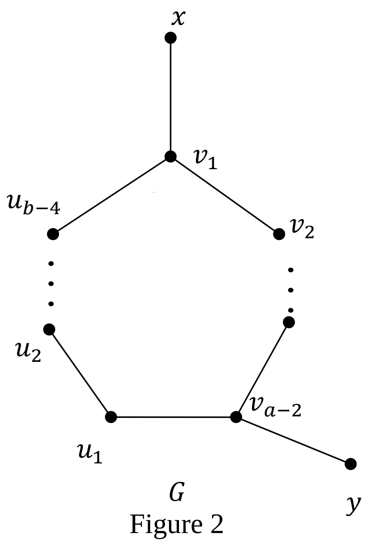

Figure 2

$x, y \in M$ then M has no Hamiltonian path, which is a contradiction. Therefore $pim^+(G) = b$. □

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DEPARTMENT OF MATHEMATICS
PONJESLY COLLEGE OF ENGINEERING
NAGERCOIL-629 003
Email address: iaselcy@mail.com

DEPARTMENT OF MATHEMATICS
RANI ANNA GOVERNMENT ARTS COLLEGE (W)
TIRUNELVELI-627 008
Email address: arulpaulsudhar@gmail.com